PRAGMATIC LECTURES

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Abstract. These notes are extended versions of my lectures given at *PRAGMATIC 2017*, a summer workshop held at the Università di Catania, Catania, Italy held from June 19th to July 7th, 2017.

Contents

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1. Lecture 1: The associated primes of powers of ideals

The primary decomposition of ideals in Noetherian rings is a fundamental result in commutative algebra and algebraic geometry. It is a far reaching generalization of the fact that every positive integer has a unique factorization into primes. We recall one version of this result.

Theorem 1.1. *Every ideal I in a Noetherian ring R has a minimal primary decomposition*

$$I = Q_1 \cap \cdots \cap Q_s$$

where each Q_i is a primary ideal and $Q_1 \cap \cdots \cap \widehat{Q}_j \cap \cdots Q_s \not\subset Q_j$ for all $j = 1, \ldots, s$. Furthermore, the set of associated primes *of I, that is,*

$$\operatorname{ass}(I) = \left\{ \sqrt{Q_1} = P_1, \ldots, \sqrt{Q_s} = P_s \right\}$$

is uniquely determined by I.

The primary decomposition of an ideal is a standard topic in most introductory commutative algebra books; one reference is Chapter 5 of Atiyah and Macdonald's classic book [2].

Starting in the 1970's, (and related to the theme of this lecture series) the following problem was investigated:

Question 1.2. *Given an ideal I in a Noetherian ring R, describe the sets* $\operatorname{ass}(I^s)$ *as s varies.*

At first glance, it might be surprising that the set of associated primes of an ideal changes when you take its power. However, as the next example shows (and the many examples of my second lecture on this topic), new associated primes can appear in higher powers, and in fact, all sorts of pathological behaviour can occur.

Example 1.3. In the ring $R = k[x, y, z]$, consider the monomial ideal $I = \langle xy, xz, yz \rangle$. Then this ideal has the primary decomposition

$$I = \langle x, y \rangle \cap \langle x, z \rangle \cap \langle y, z \rangle = P_1 \cap P_2 \cap P_3.$$

On the other hand, the primary decomposition of I^2 is given by

$$\begin{aligned} I^2 &= \langle x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2 \rangle \\ &= \langle x, y \rangle^2 \cap \langle x, z \rangle^2 \cap \langle y, z \rangle^2 \cap \langle x^2, y^2, z^2 \rangle. \end{aligned}$$

We thus have

$$\operatorname{ass}(I) = \{P_1, P_2, P_3\} \subsetneq \operatorname{ass}(I^2) = \operatorname{ass}(I) \cup \{\langle x, y, z \rangle\}.$$

In 1979, Brodmann [7] proved the following result which gives an asymptotic answer to Question 1.2.

Theorem 1.4 ([7]). *For any ideal $I \subseteq R$ in a Noetherian ring, there exists an integer s_0 such that*

$$\operatorname{ass}(I^s) = \operatorname{ass}(I^{s_0}) \text{ for all } s \geq s_0.$$

As we shall see in the next lecture, Theorem 1.4 inspires a number of new questions, many of which we only have partial solutions. Given the importance of Brodmann's result, the goal of this lecture is to sketch out the main ideas behind the proof of Theorem 1.4. A by-product of our approach is to learn some techniques related to associated primes that will hopefully be useful in your own research. As we move forward, R will always denote a Noetherian ring.

As a final comment, these lecture notes are greatly indebted not only to Brodmann's original paper, but to the monograph of McAdam [32] and the lecture notes of Swanson [41].

1.1. **Associated primes of modules.** We begin with a review/introduction to associated primes of modules. As we shall see, this is the correct point-of-view to take when studying the associated primes of I^s. Much of this material is standard. We use [43] as a reference, although other books contain this material.

Definition 1.5. Let $N \subseteq M$ be modules over R. A prime ideal $P \subseteq R$ is an *associated prime* of the R-module M/N if there exists some $m \in M$ such that

$$(N :_R m) = \{r \in R \mid rm \in N\} = P.$$

Note that if $N = (0_M)$ is the zero submodule of M, then P is an associated prime of $M \cong M/(0_M)$ if there exists an $m \in M$ such that $(0_M :_R m) = P$. This observation is related to the next definition.

Definition 1.6. Let M be an R-module. For any $m \in M$, the *annihilator* of m is

$$\operatorname{ann}(m) = (0_M :_R m) = \{r \in R \mid rm = 0_M\}.$$

It is a straightforward exercise to show that $\operatorname{ann}(m)$ is an ideal of R. It follows from what we have said that P is an associated prime of the module M if and only if $P = \operatorname{ann}(m)$ for some $m \in M$.

Definition 1.7. Let $N \subseteq M$ be modules over R. The *set of associated primes* of M/N is

$$\operatorname{Ass}_R(M/N) = \{P \subseteq R \mid P \text{ a prime ideal associated to } M/N\}.$$

We now state a number of useful facts about $\operatorname{Ass}_R(M/N)$.

Theorem 1.8 ([43, Corollary 2.1.18]). *Let $N \subseteq M$ be modules over a Noetherian ring R. Then*

$$|\operatorname{Ass}_R(M/N)| < \infty.$$

Notice that we are using a slightly different notation for the set of associated primes for modules versa the set of associated primes of an ideal (as in Theorem 1.1). However, the relationship is explained in the next theorem.

Theorem 1.9 ([43, Corollary 2.1.28])**.** *For any ideal $J \subseteq R$,*

$$\mathrm{Ass}_R(R/J) = \mathrm{ass}(J).$$

1.2. **Reducing the problem.** The strategy behind the proof of Theorem 1.4 is to focus on the set of associated primes of the R-module I^s/I^{s+1}. We now explain why this is the case.

The reduction of the problem comes from the fact that we have the following containments of sets.

Lemma 1.10. *For any ideal $I \subseteq R$ and any integers $s \geq 1$, we have*

$$\mathrm{Ass}_R(I^s/I^{s+1}) \subseteq \mathrm{Ass}_R(R/I^{s+1}) \subseteq \mathrm{Ass}_R(I^s/I^{s+1}) \cup \mathrm{Ass}_R(R/I^s).$$

Proof. The proof of this fact exploits the natural short exact sequence

$$0 \longrightarrow I^s/I^{s+1} \longrightarrow R/I^{s+1} \longrightarrow R/I^s \longrightarrow 0.$$

We then know how the sets of associated primes behave on short exact sequences; for example, see [43, Lemma 2.1.17]. □

Brodmann's proof reduces to proving the following theorem.

Theorem 1.11. *For any ideal $I \subseteq R$, there exists an integers s_0 such that*

$$\mathrm{Ass}_R(I^s/I^{s+1}) = \mathrm{Ass}_R(I^{s_0}/I^{s_0+1}) \text{ for all } s \geq s_0.$$

Indeed, we can use the above statement to prove Brodmann's result.

Proof. (of Theorem 1.4.) We want to show that there exists an integer $s^\star$ such that for all $s \geq s^\star$,

$$\mathrm{ass}(I^s) = \mathrm{Ass}_R(R/I^s) = \mathrm{Ass}_R(R/I^{s+1}) = \mathrm{ass}(I^{s+1}).$$

By Theorem 1.11, there exists an integer s_0 such that if $s \geq s_0$,

$$\mathrm{Ass}_R(I^{s+1}/I^{s+2}) = \mathrm{Ass}_R(I^s/I^{s+1}) \subseteq \mathrm{Ass}_R(R/I^{s+1})$$

where the last containment follows from Lemma 1.10. Using this inclusion, and again using Lemma 1.10, we have the following inclusions:

$$\mathrm{Ass}_R(R/I^{s+2}) \subseteq \mathrm{Ass}_R(R/I^{s+1}) \cup \mathrm{Ass}_R(I^{s+1}/I^{s+2}) \subseteq \mathrm{Ass}_R(R/I^{s+1}).$$

It then follows that for all $t \geq 1$,

$$\cdots \subseteq \mathrm{Ass}_R(R/I^{s+t}) \subseteq \mathrm{Ass}_R(R/I^{s+t-1}) \subseteq \cdots \subseteq \mathrm{Ass}_R(R/I^{s+1}).$$

By Theorem 1.8 we know that $|\mathrm{Ass}_R(R/I^{s+1})| < \infty$, so this sequence must eventually stabilize. That is, there exists some $s^\star \geq s_0$ such that for all $s \geq s^\star$, we have

$$\mathrm{Ass}_R(R/I^{s^\star}) = \mathrm{Ass}_R(R/I^{s^\star+1}) = \cdots$$

In other words, the sets $\operatorname{ass}(I^s)$ stabilize for $s \geq s^\star$. □

Remark 1.12. Note that the s_0 of Theorem 1.11 is not necessarily the same s_0 of Theorem 1.4. However, as we will see in the next lecture (see Lemma 2.6) that when I is a monomial ideal, these values are the same.

1.3. **Sketch of the missing details.** What I have said so far seems to imply that Brodmann's proof is not overly complicated. However, the "nitty-gritty" details of Theorem 1.4 are embedded in the proof of Theorem 1.11. I will now attempt to explain the main steps one would use to prove Theorem 1.11.

The first step is to change the "point-of-view" again. Instead of viewing I^s/I^{s+1} as an R-module, you want to view it as an R/I-module. In particular, you need to show:

Lemma 1.13. *Let $I \subseteq R$ be an ideal and s a positive integer. Then*

$$\operatorname{Ass}_R(I^s/I^{s+1}) = \operatorname{Ass}_R(I^{s+1}/I^{s+2}) \text{ if and only if } \operatorname{Ass}_{R/I}(I^s/I^{s+1}) = \operatorname{Ass}_{R/I}(I^{s+1}/I^{s+2}).$$

Note that Lemma 1.13 implies that to prove Theorem 1.11, we need to prove the statement for $\operatorname{Ass}_{R/I}(I^s/I^{s+1})$.

The second step is to work in a new ring constructed from I and R.

Definition 1.14. Given an ideal I in the ring R, the *associated graded ring* is the ring

$$G_I(R) = \bigoplus_{s=0}^{\infty} \frac{I^s}{I^{s+1}} \text{ where } I^0 = R.$$

The ring $G_I(R)$ is a graded ring where the d-th graded piece is $[G_I(R)]_d = I^d/I^{d+1}$. In particular, the 0-th graded piece is $[G_I(R)]_0 = R/I$. In this ring the multiplication of homogeneous elements is defined by

$$\begin{array}{ccc} I^d/I^{d+1} \times I^e/I^{e+1} & \stackrel{\times}{\longrightarrow} & I^{d+e}/I^{d+e+1} \\ (F + I^{d+1}, G + I^{e+1}) & \mapsto & (FG + I^{d+e+1}). \end{array}$$

Note that you need to verify that this map is well-defined, i.e., it is independent of your coset representatives F and G.

The following result about $G_I(R)$ is then required.

Theorem 1.15 ([2, Proposition 10.22])**.** *If R is a Noetherian ring, and $I \subseteq R$ is an ideal, then $G_I(R)$ is a Noetherian ring.*

The third step is to relate the prime ideals that appear in $\operatorname{Ass}_{R/I}(I^s/I^{s+1})$ to the prime ideals of the ring $G_I(R)$. The desired relationship is described in the next theorem.

Theorem 1.16. *Let $I \subseteq R$ be an ideal, and suppose that*

$$\mathcal{P} \in \bigcup_{s \geq 0} \operatorname{Ass}_{R/I}(I^s/I^{s+1}).$$

Then there exists a prime ideal $\mathcal{P}^\star \subseteq G_I(R)$ such that

(*i*) $\mathcal{P}^\star \cap [G_I(R)]_0 = \mathcal{P}^\star \cap (R/I) = \mathcal{P}$.
(*ii*) $\mathcal{P}^\star \in \operatorname{Ass}_{G_I(R)}(G_I(R))$.

Theorem 1.16 then gives the following corollary.

Corollary 1.17. *Let $I \subseteq R$ be an ideal. Then*

$$\left| \bigcup_{s\geq 0} \operatorname{Ass}_{R/I}(I^s/I^{s+1}) \right| < \infty.$$

Proof. Theorem 1.16 (*i*) implies that each distinct prime $\mathcal{P} \in \bigcup_{s\geq 0} \operatorname{Ass}_{R/I}(I^s/I^{s+1})$ gives rise to a distinct prime $\mathcal{P}^\star$. So, if $\left|\bigcup_{s\geq 0} \operatorname{Ass}_{R/I}(I^s/I^{s+1})\right| = \infty$, then Theorem 1.16 (*ii*) would imply that $|\operatorname{Ass}_{G_I(R)}(G_I(R))| = \infty$. But because $G_I(R)$ is Noetherian by Theorem 1.15, it follows from Theorem 1.8 that $|\operatorname{Ass}_{G_I(R)}(G_I(R))| < \infty$, thus giving a contradiction. □

There are two technical arguments that need to be made:

Lemma 1.18. *For any ideal $I \subseteq R$, there exists an integer ℓ such that for all $s \geq \ell$,*

$$(0_{G_I(R)} :_{G_I(R)} [G_I(R)]_1) \cap [G_I(R)]_s = (0_{G_I(R)}).$$

In other words, all the elements in $G_I(R)$ that annihilate the degree one piece $[G_I(R)]_1 = I/I^2$ have degree less than ℓ. This lemma is then used to prove the next lemma.

Lemma 1.19. *For any ideal $I \subseteq R$, there exists an integer ℓ such that for all $s \geq \ell$,*

$$\operatorname{Ass}_{R/I}(I^s/I^{s+1}) \subseteq \operatorname{Ass}_{R/I}(I^{s+1}/I^{s+2}).$$

Proof. (Sketch) The idea is to work in the ring $G_I(R)$, and then use Lemma 1.18 to justify why a prime $\mathcal{P} \in \operatorname{Ass}_{R/I}(I^s/I^{s+1})$ is also contained in $\operatorname{Ass}_{R/I}(I^{s+1}/I^{s+2})$. Lemma 1.18 is used to construct the required annihilator. □

We can use these pieces to prove Theorem 1.11.

Proof. (of Theorem 1.11). By Lemma 1.13, it is enough to show that there exists an integer s_0 such that

$$\operatorname{Ass}_{R/I}(I^s/I^{s+1}) = \operatorname{Ass}_{R/I}(I^{s_0}/I^{s_0+1}) \text{ for all } s \geq s_0.$$

It follows from Lemma 1.19 that there exists an integer ℓ such that

$$\operatorname{Ass}_{R/I}(I^\ell/I^{\ell+1}) \subseteq \operatorname{Ass}_{R/I}(I^{\ell+1}/I^{\ell+2}) \subseteq \cdots \subseteq \bigcup_{s\geq 0} \operatorname{Ass}_{R/I}(I^s/I^{s+1}).$$

But by Corollary 1.17 $\left|\bigcup_{s\geq 0} \operatorname{Ass}_{R/I}(I^s/I^{s+1})\right| < \infty$. We thus have a sequence of subsets in a finite set, where the i-th set is contained in the $i+1$-th set. So, there must exist some s_0 such that

$$\operatorname{Ass}_{R/I}(I^{s_0}/I^{s_0+1}) = \operatorname{Ass}_{R/I}(I^{s_0+1}/I^{s_0+2}) = \cdots$$

thus completing the proof. □

1.4. **Final comments.** Brodmann's Theorem (Theorem 1.4) is a good example of the idea in commutative algebra that ideals behave "nicely" asymptotically (see also Tai's lectures on the powers of ideals and regularity for more on this idea).

Of course, Brodmann's Theorem also inspires a number of natural questions (e.g., given an ideal I, can we determine the value of s_0). In the next lecture we will explore some of these problems in the case the I is a monomial ideal.

We end with a very recent result of Hà, Nguyen, Trung, and Trung that shows if $s < s_0$, the sets $\mathrm{ass}(I^s)$ need not be related to each other. Moreover, we can make examples where s_0 is arbitrarily large (although we may need to work in a very large polynomial ring!).

Theorem 1.20 ([23, Corollary 6.8])**.** *Let Γ be any finite subset of $\mathbb{N}^+$. Then there exists a monomial ideal I in a polynomial ring R such that*

$$\mathfrak{m} \in \mathrm{ass}(I^s) \text{ if and only if } s \in \Gamma.$$

Here, $\mathfrak{m}$ is the unique maximal monomial ideal of R.

Remark 1.21. The above result answers an old question first raised by Ratliff [35].

2. Lecture 2: The associated primes of powers of square-free monomial ideals

In the last lecture, we looked at a result of Brodmann concerning the associated primes of ideals. In particular, we worked through the proof of the following theorem:

Theorem 2.1 ([7]). *Let I be any ideal in a Noetherian ring R. Then there exists an integers s_0 such that*

$$\operatorname{ass}(I^s) = \operatorname{ass}(I^{s_0}) \text{ for all } s \geq s_0.$$

This theorem inspires a number of natural questions. To state these questions, we introduce some suitable terminology.

Definition 2.2. The *index of stability* of an ideal I in a Noetherian ring R, denoted $\operatorname{astab}(I)$, is defined to be

$$\operatorname{astab}(I) := \min\{s_0 \mid \operatorname{ass}(I^s) = \operatorname{ass}(I^{s_0}) \text{ for all } s \geq s_0\}.$$

Definition 2.3. An ideal I in a Noetherian ring R is said to have the *persistence property* if $\operatorname{ass}(I^i) \subseteq \operatorname{ass}(I^{i+1})$ for all $i \geq 1$.

Brodmann's result is the inspiration for the following questions:

Question 2.4. *Let I be an ideal of a Noetherian ring R.*

- (*i*) *What is* $\operatorname{astab}(I)$*?*
- (*ii*) *Does I have the persistence property?*
- (*iii*) *What are the elements of* $\operatorname{ass}(I^s)$ *with* $s \geq \operatorname{astab}(I)$*?*

In general, these questions appear to be quite difficult. (Note that Theorem 1.20 implies the existence of ideals that fail the persistence property.) In this lecture, we want to focus on the case that I is a (square-free) monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$. In this context, we have a much better understanding of the problems raised in Question 2.4.

2.1. General (useful) facts.

As mentioned above, we are going to focus on the case of monomial ideals. This restriction imposes restrictions on what primes can be associated primes, and it gives us some information about the annihilator.

Lemma 2.5. *Let I be any monomial ideal of $R = k[x_1, \ldots, x_n]$.*

- (*i*) *If $P \in \operatorname{ass}(I)$, then P is also a monomial ideal. Consequently, $P = \langle x_{i_1}, \ldots, x_{i_r}\rangle$ for some $\{x_{i_1}, \ldots, x_{i_r}\} \subseteq \{x_1, \ldots, x_n\}$.*
- (*ii*) *If $P \in \operatorname{ass}(I)$, then there exists a monomial $m \in R \setminus I$ such that $I : \langle m\rangle = P$.*

Proof. Statement (*i*) follows from the irreducible decomposition of monomial ideals (see, e.g., [25, Chapter 1]). For (*ii*), since $P \in \operatorname{ass}(I)$, there exists an element $f \in R$ such that

$I : \langle f \rangle = P$. If f is not a monomial, we can write it as $f = c_1m_1 + \cdots + c_sm_s$ with $c_i \in k$ and m_i a monomial. By (i), we know that $P = \langle x_{i_1}, \ldots, x_{i_r} \rangle$. So, for any $x_j \in P$,

$$fx_j = c_1m_1x_j + \cdots c_sm_sx_j \in I \Rightarrow m_kx_j \in I \text{ for each } k \in \{1, \ldots, s\}$$

since I is a monomial ideal. But this means that $x_j \in I : \langle m_k \rangle$ for all $k \in \{1, \ldots, s\}$. Since this is true for each $x_j \in P$, we have

$$P \subseteq \bigcap_{i=1}^{s} I : \langle m_i \rangle.$$

If $g \in \bigcap_{i=1}^{s} I : \langle m_i \rangle$, then $fg = c_1m_1g + \cdots + c_sm_sg \in I$. This means that $g \in I : \langle f \rangle = P$.

We have thus shown that $P = \bigcap_{i=1}^{s} I : \langle m_i \rangle$. But because a prime ideal is an irreducible ideal, we must have $P = I : \langle m_i \rangle$ for some $i \in \{1, \ldots, s\}$. □

In Brodmann's proof of the asymptotic stability of $\operatorname{ass}(I^s)$, he used the stability of $\operatorname{Ass}_R(I^s/I^{s+1})$ to prove the stability of $\operatorname{Ass}_R(R/I^{s+1})$. In general, these sets are not equal. However, in the case of monomial ideals, these sets are the same.

Lemma 2.6. *For any monomial ideal $I \subseteq R$,*

$$\operatorname{Ass}_R(I^s/I^{s+1}) = \operatorname{Ass}_R(R/I^{s+1}) \textit{ for all } s \geq 0.$$

Proof. As we observed in the last lecture, we always have $\operatorname{Ass}_R(I^s/I^{s+1}) \subseteq \operatorname{Ass}_R(R/I^{s+1})$ for any ideal in a Noetherian ring. It suffices to prove the reverse containment for monomial ideals. Let $P \in \operatorname{Ass}_R(R/I^{s+1}) = \operatorname{ass}(I^{s+1})$. By Lemma 2.5 (i) and (ii) there exists a monomial $m \in R$ such that

$$P = \langle x_{i_1}, \ldots, x_{i_r} \rangle = I^{s+1} : \langle m \rangle \text{ with } m \in R \setminus I^{s+1}.$$

So, for any $x_j \in P$,

$$mx_j = m_1 \cdots m_{s+1}M \in I^{s+1}$$

with m_i a monomial generator of I and M a monomial. After relabeling, we can assume that $x_j \mid (m_{s+1}M)$. So, $m_1 \cdots m_s \mid m$, which implies that $m \in I^s$.

We thus have $m \in I^s \setminus I^{s+1}$ and $P = I^{s+1} : \langle m \rangle$. But this is precisely the condition for $P \in \operatorname{Ass}_R(I^s/I^{s+1})$. □

Corollary 2.7. *For a monomial ideal $I \subseteq R = k[x_0, \ldots, x_n]$,*

$$\operatorname{astab}(I) = \min \left\{ s_0 \;\middle|\; \operatorname{Ass}_R(I^s/I^{s+1}) = \operatorname{Ass}_R(I^{s_0}/I^{s_0+1}) \textit{ for all } s \geq s_0 \right\}.$$

For any monomial ideal I, we let $G(I)$ denote the unique set of minimal generators of I. For any monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, we define the support of m to be

$$\operatorname{supp}(m) = \{x_i \mid a_i > 0\}.$$

We end this section with a useful localization "trick". This theorem is useful because it sometimes allows us to reduce to the case that $P = \langle x_1, \ldots, x_n \rangle$ is the unique monomial ideal that is also a maximal ideal.

Theorem 2.8. *Let I be a monomial ideal of $R = k[x_1, \dots, x_n]$ with $G(I) = \langle m_1, \dots, m_s\rangle$. Then*

$$P = \langle x_{i_1}, \dots, x_{i_r}\rangle \in \mathrm{Ass}_R(k[x_1, \dots, x_n]/I^s)$$

if and only if

$$P = \langle x_{i_1}, \dots, x_{i_r}\rangle \in \mathrm{Ass}_S(k[x_{i_1}, \dots, x_{i_r}]/(I_P)^s)$$

where $I_P = \langle m \in G(I) \mid \mathrm{supp}(m) \subseteq \{x_{i_1}, \dots, x_{i_r}\}\rangle$ and $S = k[x_{i_1}, \dots, x_{i_r}]$.

Proof. See [17, Lemma 2.11]. Note that this lemma is expressed in the language of hypergraphs, but it is equivalent to the theorem given above. □

2.2. **The index of stability.** We now look at Question 2.4 (i) for monomial ideals. In general, we know very little about $\mathrm{astab}(I)$ for monomial ideals. One of the few results in this direction is the following result of Hoa:

Theorem 2.9 ([29])**.** *Let I be a monomial ideal with n variables, s generators, and d the largest degree of a minimal generator. Then*

$$\mathrm{astab}(I) \leq \max\left\{d(ns+s+d))(\sqrt{n})^{n+1}(\sqrt{2}d)^{(n+1)(s-1)}, s(s+n)^4 s^{n+2} d^2 (2d^2)^{s^2-s+1}\right\}.$$

Example 2.10. The bound of Theorem 2.9 is very far from optimal. For example, for the ideal $I = \langle x_1x_2, x_2x_3\rangle \subseteq k[x_1, x_2, x_3]$, Theorem 2.9 gives the bound $\mathrm{astab}(I) \leq 81,920,000$, but $\mathrm{astab}(I) = 1$.

It would be nice to have better uniform bounds for all square-free monomial ideals. In a personal conversation with J. Herzog, he suggested that perhaps $\mathrm{astab}(I) \leq n-1$, where n is the number of variables of R. In all known cases, this bounds appears to hold.

If we restrict to some families of monomial ideals related to finite simple graphs, we can significantly improve these bounds. We recall these constructions. We write $G = (V, E)$ to denote the *finite simple graph* with vertex set $V = \{x_1, \dots, x_n\}$ and edge set E, i.e., E is a collection of subsets $e \subseteq V$ with $|e| = 2$. We may sometimes write $(V(G), E(G))$ if we wish to highlight the vertex set and edge set of G. By identifying the vertices of V with the variables of $R = k[x_1, \dots, x_n]$, we can construct two monomial ideals.

Definition 2.11. The *edge ideal* of G is the ideal

$$I(G) = \langle x_ix_j \mid \{x_i, x_j\} \in E\rangle$$

where the monomial generators correspond to the edges of G. The *cover ideal of* G is the ideal

$$J(G) = \bigcap_{\{x_i,x_j\}\in E} \langle x_i, x_j\rangle.$$

(The cover ideal name comes from the fact that the minimal generators correspond to the minimal vertex covers of G.)

To state our results about $\operatorname{astab}(I)$, we first recall some of the relevant definitions from graph theory. The *complement* of G is the graph $G^c = (V(G^c), E(G^c))$ where $V(G^c) = V(G)$, but $E(G^c) = \{\{x_i, x_j\} \subseteq V(G) \mid \{x_i, x_j\} \notin E(G)\}$. In other words, it is the graph consisting of the non-edges of G. The *induced graph* of G on $W \subseteq V$ is the graph

$$G_W = (W, E(G_W)) = \{\{x_i, x_j\} \in E(G) \mid \{x_i, x_j\} \subseteq W\}.$$

A *cycle* on n vertices, denoted C_n, is the graph

$$C_n = (\{x_1, \ldots, x_n\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}).$$

A graph G is a *perfect graph* if both G and G^c have no induced cycles with n odd and $n \geq 5$.

A *colouring* of G is an assignment of colours to each vertex of G so that adjacent vertices receive distinct colours. The *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colours needed in a colouring.

Example 2.12. We illustrate some of these ideas with an example. The graph $G = C_5$ is given in the figure below. Then $\chi(G) = 3$ since we can colour vertices x_1, x_3 RED,

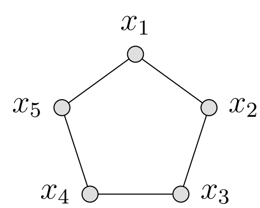


Figure 1. The five cycle graph

vertices x_2, x_4 BLUE, and x_5 PURPLE. When we consider $W = \{x_2, x_3, x_5\}$, the induced graph $G_W = (C_5)_W$ is the the graph:

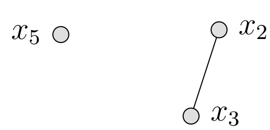


Figure 2. The induced graph $(C_5)_W$ with $W = \{x_2, x_3, x_5\}$

We can use the combinatorial information about G to place some bounds on $\operatorname{astab}(I(G))$ and $\operatorname{astab}(J(G))$.

Theorem 2.13. *Let G be a finite simple graph.*

(*i*) *If G has no induced odd cycle, then* $\operatorname{astab}(I(G)) = 1$.
(*ii*) *If the smallest induced odd cycle of G has size $2k+1$, then* $\operatorname{astab}(I(G)) \leq n - k$.
(*iii*) *If G is a perfect graph, then* $\operatorname{astab}(J(G)) = \chi(G) - 1$.

Proof. If G has no induced odd cycle, then G is said to be bipartite. Statement (i) then follows from work of Simis, Vasconcelos, and Villarreal [37, Theorem 5.9]. Note that they show that $I = I(G)$ is normally torsion free, but this implies that $I^m = I^{(m)}$ for all $m \geq 1$. One can then show that $\mathrm{ass}(I^m) = rmass(I^{(m)}) = \mathrm{ass}(I)$ for all $m \geq 1$. Statement (ii) follows from a more general result of Chen, Morey, and Sung [8, Corollary 4.3]. For (iii), see [17, Corollary 5.11]. □

2.3. **Persistence of primes.** We now turn to Question 2.4 (ii), i.e., when does a monomial ideal have the persistence property. Persistence for monomial ideals fails in general. We do, however, have the following necessary condition for persistence which is found in Martinez-Bernal, Morey, and Villarreal [31].

Theorem 2.14. *Suppose that I is a monomial ideal such that $I^{k+1} : I = I^k$ for all $k \geq 1$. Then I has the persistence property.*

Proof. Let $P \in \mathrm{ass}(I^k)$. By Theorem 2.8, we can assume that $P = \langle x_1, \ldots, x_n \rangle$; furthermore, it is enough to show that P persists.

Since $P \in \mathrm{ass}(I^k)$, there exists a monomial $m \in R \setminus I^k$ such that $I^k : \langle m \rangle = P$. Since $m \in R \setminus I^k$, $m \notin I^k = I^{k+1} : I$. So, there exists a monomial $q \in I$ such that $mq \notin I^{k+1}$. Now for each $i = 1, \ldots, n$, the variable x_i satisfies

$$(mq)x_i \in (mx_i)q \in I^k I = I^{k+1} \text{ because } mx_i \in I^k.$$

This implies that $P \subseteq I^{k+1} : \langle mq \rangle$. Because $mq \notin I^{k+1}$, $I^{k+1} : \langle mq \rangle \subsetneq \langle 1 \rangle$, i.e., $I^{k+1} : \langle mq \rangle$ is a proper monomial ideal of R, and in particular, it must be a subset of P. So $P = I^{k+1} : \langle mq \rangle$. But since $mq \in R \setminus I^{k+1}$, this implies that $P \in \mathrm{ass}(I^{k+1})$. □

Martinez-Bernal, Morey, and Villarreal used the above result to show that all edge ideals have the persistence properties. To prove the following theorem, they needed to use a classical result about matchings in a graph. This result is a nice example of using graph theory results to prove a result in commutative algebra.

Theorem 2.15 ([31, Corollary 2.17])**.** *For any graph G, the edge ideal $I(G)$ satisfies $I(G)^{k+1} : I(G) = I(G)^k$ for all $k \geq 1$. In particular, $I(G)$ has the persistence property.*

Remark 2.16. Herzog-Qureshi [26] called an ideal I *Ratliff* if $I^{k+1} : I = I^k$ for all $k \geq 1$. They show that if I is any ideal (not just a monomial ideal) that is Ratliff, then I has the persistence property.

Example 2.17. Theorem 2.14 does not classify ideals with the persistence property. As an example, consider the Stanley-Reisner ideal of the triangulation of the projective plane, i.e.,

$$I = \langle x_1x_2x_5, x_1x_3x_4, x_1x_2x_6, x_1x_3x_6, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_6, x_3x_5x_6, x_4x_5x_6 \rangle.$$

Then a computer algebra program can show that $I^2 : I = I$, but $I^3 : I \neq I^2$, so I is not Ratliff. However, I has the persistence property.

Theorem 2.15 shows that *all* edge ideals have the persistence property. This leads to the natural question of whether cover ideals have this property. For many large classes of graphs, this is indeed the case.

Theorem 2.18 ([17])**.** *If G is a perfect graph, then $J(G)$ has the persistence property.*

In fact, there are a number of graphs G that are not perfect, but $J(G)$ has the persistence property (e.g., the cover ideals of cycles). For a while, it was thought that cover ideals of all graphs (and in fact, all square-free monomial ideals) satisfied the persistence property. Francisco, Hà, and myself formulated a graph theory conjecture, that if true, would have implied the persistence property (see [16]). Interestingly, T. Kaiser, M. Stehlík, R. Škrekovski [30], all graph theorists, disproved our graph theory conjecture. The example, which is given below, is another nice example of the intersection between graph theory and commutative algebra.

Example 2.19 ([30])**.** The cover ideal of the graph G in Figure 3 fails to have the

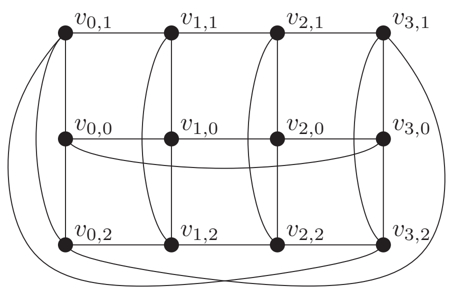


Figure 3. A graph whose cover ideals fails the persistence property; illustration source [30].

persistence property. In particular, the maximal ideal is an associated prime of $J(G)^3$, but it is not an associated prime of $J(G)^4$. This graph G can be extended to an infinite family of graphs that fail to have the persistence property. The details are worked out in a paper of Hà and Sun [24].

One obvious open question is the following:

Question 2.20. *Classify all finite simple graphs G whose cover ideal fails to have the persistence property.*

It should be noted that there are some other families of square-free monomial ideals (that are neither edge ideals or cover ideals) that are known to satisfy the persistence property. This includes polymatroidal ideals [27] and some generalized cover ideals [4].

2.4. **Elements of** $\operatorname{ass}(I^s)$**.** We now turn to the problem of determining the elements of $\operatorname{ass}(I^s)$, i.e., the focus of Question 2.4 (*iii*). We will focus on the on cover ideals of graphs, although many of these ideals extend to all square-free monomial ideals using the language of hypergrahps (see [16] for more details). Some of the material of this section can also be found in [42].

The key idea that that you should take away is that if $P = \langle x_{i_1}, \ldots, x_{i_r} \rangle \in \operatorname{ass}(J(G)^s)$, then something "interesting" is happening on the induced graph G_P, where we view P as both an ideal generated by the variables and as a subset $P \subseteq V(G)$. Specifically, associated primes are related to colourings of the graph.

The following theorem, which is interesting in its own right, shows that the chromatic number is related to powers of ideals.

Theorem 2.21 ([16])**.** *For any graph G on n vertices,*

$$\chi(G) = \min\{d \mid (x_1 \cdots x_n)^{d-1} \in J(G)^d\}.$$

We now take a detour to explain the significance of the name cover ideal.

Definition 2.22. A subset $W \subseteq V(G)$ is a *vertex cover* if $W \cap e \neq \emptyset$ for all $e \in E(G)$. A vertex cover W is a *minimal vertex cover* if no proper subset of W is a vertex cover.

Lemma 2.23. *Let G be a graph with cover ideal $J(G)$. Then*

$$J(G) = \langle x_W \mid W \subseteq V(G) \text{ is a minimal vertex cover of } G \rangle$$

where $x_W := \prod_{x_i \in W} x_i$ if $W \subseteq V(G)$.

Proof. Let $L = \langle x_W \mid W \subseteq V(G)$ is a minimal vertex cover of $G\rangle$.

Let x_W be a generator of L with W a minimal vertex cover. Then, for every edge $e = \{x_i, x_j\} \in E(G)$, we have $W \cap e \neq \emptyset$. So, either $x_i \in W$ or $x_j \in W$. Consequently, either $x_i | x_W$ or $x_j | x_W$, whence $x_W \in \langle x_i, x_j \rangle$. Since e is arbitrary, we have

$$x_W \in \bigcap_{\{x_i, x_j\} \in E(G)} \langle x_i, x_j \rangle = J(G).$$

Conversely, let $m \in J(G)$ be any minimal generator. Note that m must be square-free since $J(G)$ is the intersection of finitely many square-free monomial ideals. So $m = x_{i_1} \cdots x_{i_r}$. Let $W = \{x_{i_1}, \ldots, x_{i_r}\}$. Since $m \in \langle x_i, x_j \rangle$ for each each $\{x_i, x_j\} \in E(G)$. either $x_i | m$ or $x_j | m$, and thus, $x_i \in W$ or $x_j \in W$. Thus W is a vertex cover. Let $W' \subseteq W$ be a minimal vertex cover. Because $x_{W'} \in L$ and $x_{W'}$ divides $m = x_W$, we have $m \in L$. □

We need to recall some graph theory.

Definition 2.24. A graph G is *critically s-chromatic* if $\chi(G) = s$, and for every $x \in V(G)$, $\chi(G \setminus \{x\}) < s$.

Example 2.25. Let $G = C_n$ be the n-cycle with n odd. Then G is a critically 3-chromatic graph since $\chi(G) = 3$, but if we remove any vertex x, $\chi(G \setminus \{x\}) = 2$.

Example 2.26. Let $G = K_n$ be the clique of size n. Then G is a critically n-chromatic graph since $\chi(G) = n$, but if we remove any vertex x, $G \setminus \{x\} = K_{n-1}$, and thus $\chi(G \setminus \{x\}) = n - 1$.

Remark 2.27. You should be able to convince yourself that the only critically 1-chromatic graph is the graph of an isolated vertex, and the only critically 2-chromatic graph is K_2. The only critically 3-chromatic graphs are precisely the graphs $G = C_n$ with n odd. However, for $s \geq 4$, there is no known classification of critically s-chromatic graphs.

As the next theorem shows, some of the associated primes of $J(G)^s$ are actually detecting induced subgraphs that are critically $(s+1)$-chromatic.

Theorem 2.28. *Let G be a graph and suppose $P \subseteq V(G)$ is such that G_P is critically $(s+1)$-chromatic. Then*

(1) $P \notin \operatorname{ass}(J(G)^d)$ *for* $1 \leq d < s$.
(2) $P \in \operatorname{ass}(J(G)^s)$.

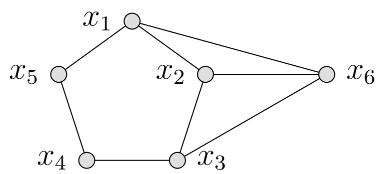
Proof. By Lemma 2.8, we can assume that $G = G_P$. We will only prove (2). To prove (1), one shows that if $P \in \operatorname{ass}(J(G)^d)$ with $d < s$, then we would be able to show that $\chi(G_P) < s + 1$, a contradiction.

We are given
$$\chi(G) = \min\{d \mid (x_1 \cdots x_n)^{d-1} \in J(G)^d\} = s + 1$$
so $m = (x_1 \cdots x_n)^{s-1} \notin J(G)^s$. In other words, we have, $J(G)^s : \langle m \rangle \subsetneq \langle 1 \rangle$, and hence $J(G)^s : \langle m \rangle \subseteq \langle x_1, \ldots, x_n \rangle$. We will now show that $J(G)^s : \langle m \rangle \supseteq \langle x_1, \ldots, x_n \rangle$; the conclusion will then follow from this fact.

Since $\chi(G)$ is critically $(s+1)$-chromatic, for each $x_i \in V(G)$, $\chi(G \setminus \{x_i\}) = s$. Let
$$V(G \setminus \{x_i\}) = C_1 \cup \cdots \cup C_s$$
be the s colouring of $V(G \setminus \{x_i\})$ where C_i denotes all the vertices coloured i. Then
$$V(G) = C_1 \cup \cdots \cup C_s \cup \{x_i\}$$
is an $(s+1)$-colouring of G.

For $j = 1, \ldots, s$, set
$$W_j = C_1 \cup \cdots \cup \widehat{C_j} \cup \cdots \cup C_s \cup \{x_i\}.$$
Each W_j is a vertex cover, so $x_{W_j} \in J(G)$. Thus
$$\prod_{j=1}^{s} x_{W_j} \in J(G)^s.$$
But $\prod_{j=1}^{s} x_{W_j} = (x_1 \cdots x_n)^{s-1} x_i$. Thus, $x_i \in J(G)^s : \langle m \rangle$. This is true for each $x_i \in V(G)$, whence $\langle x_1, \ldots, x_n \rangle \subseteq J(G)^s : \langle m \rangle \subseteq \langle x_1, \ldots, x_n \rangle$, as desired. □

Example 2.29. We consider the following graph



Note that the induced graph on $\{x_1, x_2, x_6\}$ is a K_3 (and C_3), a critically 3-chromatic graph. So $P = \langle x_1, x_2, x_6 \rangle$ is in $\mathrm{Ass}(J(G)^2)$, but not in $\mathrm{Ass}(J(G))$. Similarly, since the induced graph on $\{x_1, x_2, x_3, x_4, x_5\}$ is a C_5, we will have $\langle x_1, x_2, x_3, x_4, x_5 \rangle \in \mathrm{Ass}(J(G)^2)$.

When $s = 2$, we can find a converse of Theorem 2.28. A complete characterization of the associated primes of $J(G)^2$ was first given in the paper [18].

Theorem 2.30. *For any graph G, $P \in \mathrm{ass}(J(G)^2)$ if and only if*

(i) $P = \langle x_i, x_j \rangle$ and $\{x_i, x_j\} \in E(G)$, and
(ii) $P = \langle x_{i_1}, \ldots, x_{i_r} \rangle$ where r is odd and $G_P = C_r$, an odd cycle.

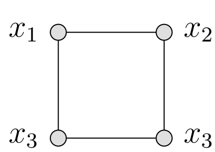
Unfortunately, the converse of Theorem 2.28 is false in general; that is, if $P \in \mathrm{ass}(J(G)^s)$, but $P \notin \mathrm{ass}(J(G)^d)$ with $1 \leq d < s$, then the graph G_P is not necessarily a critically $(s+1)$-chromatic graph.

Example 2.31. If we consider the graph of Example 2.29, then $P = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle \in \mathrm{ass}(J(G)^3)$ but not in $\mathrm{ass}(J(G))$ or $\mathrm{ass}(J(G)^2)$. However, the graph $G = G_P$ is not critically 4-chromatic. In fact, $\chi(G) = 3$.

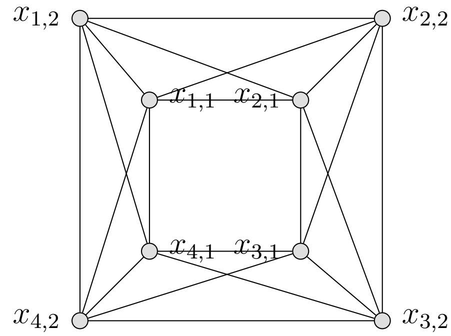
What is happening here is that we are looking in the "wrong" graph.

Definition 2.32. Given a graph $G = (V(G), E(G))$ and integer $s \geq 1$, the *s-th expansion of G*, denoted G^s, is the graph constructed from G as follows: (a) replace each $x_i \in V(G)$ with a clique of size s on the vertices $\{x_{i,1}, \ldots, x_{i,s}\}$, and (b) two vertices $x_{i,a}$ and $x_{j,b}$ are adjacent in G^s if and only if x_i and x_j were adjacent in G.

Example 2.33. We illustrate this example when $G = C_4$, and we construct G^2. Recall that C_4 is the graph:



Then the second expansion of G is the graph:



We now come to our main result which gives a combinatorial interpretation for the elements of $\operatorname{ass}(J(G)^s)$.

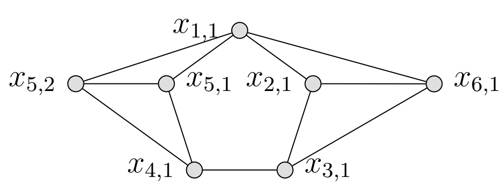
Theorem 2.34 ([16]). *Let G be a graph with cover ideal $J(G)$. Then $\langle x_{i_1}, \ldots, x_{i_r}\rangle \in \operatorname{ass}(J(G)^s)$ if and only if there exists a set $T \subseteq V(G^s)$ with*

$$\{x_{i_1,1}, x_{i_2,1}, \ldots, x_{i_r,1}\} \subseteq T \subseteq \{x_{i_1,1}, \ldots, x_{i_1,s}, \ldots, x_{i_r,1}, \ldots, x_{i_r,s}\}$$

such that the induced graph $(G^s)_T$ is a critically $(s+1)$-chromatic graph.

The proof is a mixture of a number of ingredients. First, instead of looking at the primary decomposition, one considers the irreducible decomposition of $J(G)^s$. Then one uses tools such as generalized Alexander duality, polarization and depolarization of monomial ideals, and a result of Sturmfels and Sullivant [39]. We have only stated it for cover ideals of graphs, but it works also for cover ideals of hypergraphs, i.e., any square-free monomial ideal.

Example 2.35. Let us return to Example 2.29 and explain why $\langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ appears in $\operatorname{ass}(J(G)^3)$. We form G^3, the 3-rd expansion of G. If we consider the induced subgraph on $T = \{x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}, x_{5,1}, x_{5,2}, x_{6,1}\} \subseteq V(G^3)$, we find the graph



You can now convince yourself that this graph is critically 4-chromatic. Consequently, $\langle x_1, x_2, x_3, x_4, x_5, x_6\rangle \in \operatorname{ass}(J(G)^3)$.

Remark 2.36. In my work with Francisco and Hà on the associated primes of I^s when I is a square-free monomial ideal, we took the point of view that I was the cover ideal of a hypergraph, and consequently, the generators correspond to minimal vertex covers. Hien, Lam, and Trung [28] took an alternative point-of-view. They viewed the generators of I as the edges of a (hyper)graph, and described the associated primes in terms of this (hyper)graph.

Remark 2.37. Bayati, Herzog, and Rinaldo [3] have shown that for any monomial ideal I, there is an algorithm to find all the primes in $\operatorname{ass}(I^{\operatorname{astab}(I)})$ using Koszul homology.

We end with a question that computer experiments have suggested that is true.

Question 2.38. *Let I be any square-free monomial ideal. If $P \in \operatorname{ass}(I^2)$, is $P \in \operatorname{ass}(I^s)$ for all $s \geq 2$?*

3. Lecture 3: The Waldschmidt constant of square-free monomial ideal

In this lecture we will introduce the Waldschmidt constant of a homogeneous ideal. The main result of this lecture is to explain how to compute this invariant in the case of square-free monomial ideals. In the case of edge ideals, I will also give a combinatorial interpretation of this invariant. In this lecture, $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k, where k has characteristic zero and is algebraically closed. All ideals $I \subseteq R$ will be assumed to be homogeneous. Furthermore, we set

$$\alpha(I) = \min\{i \mid \text{exist } 0 \neq F \in I \text{ with } \deg F = i\}.$$

That is, $\alpha(I)$ is the smallest degree of a minimal generator of I.

3.1. Origin story: points in $\mathbb{P}^2$.

Before defining the Waldschmidt constant, we describe some of the historical background that led to the development of this constant. In particular, we start our lecture by discussing points in the projective plane.

Recall that

$$\mathbb{P}^2 = \mathbb{P}^2_k := \{[a : b : c] \mid (a, b, c) \in k^3, (a, b, c) \neq (0, 0, 0)\}/\sim$$

where $\sim$ denotes the equivalence relation defined by

$$[a : b : c] \sim [d : e : f] \text{ if and only if } (a, b, c) = (\lambda d, \lambda e, \lambda f) \text{ for some } 0 \neq \lambda \in k.$$

Via the Algebra-Geometry Dictionary (see, for example, Chapter 4 of [12]) there is a correspondence between homogeneous ideals of $R = k[x_0, x_1, x_2]$ and varieties of $\mathbb{P}^2$.

We are only interested in the case of points, which we recall here. For any point $P = [a : b : c] \in \mathbb{P}^2$, the homogeneous ideal associated with P is the ideal

$$I(P) = \{F \in R \mid F \text{ homogeneous and } F(P) = 0\}.$$

Given any finite set of points $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$, the homogeneous ideal associated with $\mathbb{X}$ is the ideal

$$\begin{aligned} I_{\mathbb{X}} &= I(P_1) \cap \cdots \cap I(P_s) \\ &= \{F \in R \mid F \text{ homogeneous and } F(P) = 0 \text{ for all } P \in \mathbb{X}\}. \end{aligned}$$

For a finite set of points $\mathbb{X}$, we then have

$$\begin{aligned} \alpha(I_{\mathbb{X}}) &= \min\{i \mid 0 \neq F \in (I_{\mathbb{X}})_i\} \\ &= \text{the smallest degree of a curve that passes through all points of } \mathbb{X}. \end{aligned}$$

The following questions is then of interest:

Question 3.1. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ and fix an integer $m \geq 1$. What is the smallest degree of a curve that passes through all the points of $\mathbb{X}$ with multiplicity m?*

One way to answer this question is to use the m-th differential power. We recall this definition for $R = k[x_0, x_1, x_2]$, although this definition generalizes in the natural way (see [40]).

Definition 3.2. Let I be a homogeneous ideal of $R = k[x_0, x_1, x_2]$ and $m \geq 1$ an integer. The *m-th differential power* of I is the ideal

$$I^{\langle m \rangle} = \left\langle F \in R \;\middle|\; \frac{\partial^{a_0+a_1+a_2} F}{\partial x_0^{a_0} \partial x_1^{a_1} \partial x_2^{a_2}} \in I \text{ with } a_0 + a_1 + a_2 \leq m - 1 \right\rangle.$$

In other words, $I^{\langle m \rangle}$ contains all the homogeneous elements whose partial derivatives of order $\leq m - 1$ are all contained in I.

Using this language, an answer to Question 3.1 is straightforward. Namely, if $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ and $m \geq 1$ is an integer, then

$$\begin{aligned} \alpha((I_{\mathbb{X}})^{\langle m \rangle}) \;&=\; \text{the smallest degree of a curve that passes all points} \\ &\;\; \text{of } \mathbb{X} \text{ with multiplicity } m. \end{aligned}$$

Before moving forward, let's take a break to do a example which illustrates some of these ideas.

Example 3.3. Consider the three coordinate points of $\mathbb{P}^2$, that is,

$$P_1 = [1:0:0],\ P_2 = [0:1:0]\ ,\ \text{and } P_3 = [0:0:1].$$

The associated ideals are

$$I(P_1) = \langle x_1, x_2 \rangle,\ I(P_2) = \langle x_0, x_2 \rangle \text{ and } I(P_3) = \langle x_0, x_1 \rangle$$

and thus

$$I_{\mathbb{X}} = \langle x_0x_1, x_0x_2, x_1x_2 \rangle.$$

Since $I_{\mathbb{X}}$ is a monomial ideal, to compute $I_{\mathbb{X}}^{\langle 2 \rangle}$, it is enough to check which monomials belong to this ideal. The minimal generators can be shown to be

$$I_{\mathbb{X}}^{\langle 2 \rangle} = \langle x_0x_1x_2, x_0^2x_1^2, x_0^2x_2^2, x_1^2x_2^2 \rangle.$$

So, we have $\alpha(I_{\mathbb{X}}) = 2$ and $\alpha(I_{\mathbb{X}}^{\langle 2 \rangle}) = 3$. Note that $x_0x_1x_2$ is a curve of degree three that passes through each point of $\mathbb{X}$ with multiplicity two.

Remark 3.4. We want to stress that although we are only presenting the case of points in $\mathbb{P}^2$, everything we have said can be extended to the case of points in $\mathbb{P}^n$.

As we have seen, to answer Question 3.1, we need to be able to determine $\alpha(I_{\mathbb{X}}^{\langle m \rangle})$. Although we have poised this problem as a question about algebraic geometry and commutative algebra, the question about finding $\alpha(I_{\mathbb{X}}^{\langle m \rangle})$ first arose in complex analysis. In particular, this problem was studied in the 1970's and 1980's by by Waldschmidt [44, 45], Skoda [38], Chudnovsky [9], Esnault and Viehweig [15], among others.

Before moving to our discussion of the Waldschmidt constant, we mention two important conjectures related to $\alpha(I_{\mathbb{X}}^{\langle m \rangle})$.

Conjecture 3.5 (Chudnovsky [9]). *For any set of points $\mathbb{X} \subseteq \mathbb{P}^n$,*

$$\frac{\alpha(I_{\mathbb{X}}^{\langle m \rangle})}{m} \geq \frac{\alpha(I_{\mathbb{X}}) + n - 1}{n}.$$

Chudnovsky was able to verify his conjecture in the case that $n = 2$. In 2012, Dumnicki [13] verified the conjecture for $n = 3$. More recently, Foulli, Mantero, and Xie [19] were able to verify the conjecture for large classes of points in $\mathbb{P}^n$ (this last reference contains a more complete list of other known cases).

The following conjecture of Nagata dates back to the 1950's, and so predates all the other work we have mentioned. The conjecture, which arose in Nagata's solution to Hilbert's 14-th problem, continues to provide motivation to study Question 3.1.

Conjecture 3.6 (Nagata [33]). *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a general set of s points. Then*

$$\frac{\alpha(I_{\mathbb{X}}^{\langle m \rangle})}{m} \geq \sqrt{s}.$$

3.2. **Symbolic powers and the Waldschmidt constant.** We now wish to make the connection to symbolic powers and introduce the Waldschmidt constant. We first recall the definition of a symbolic power of an ideal (but only in a special case).

Definition 3.7. Let I be a homogeneous radical ideal in R, i.e., $I = \sqrt{I}$ and thus $I = P_1 \cap \cdots \cap P_s$ with all P_i prime. Then the m-th symbolic power of I is the ideal

$$I^{(m)} = \bigcap_{P \in \{P_1, \ldots, P_s\}} (I^m R_P \cap R)$$

where $I^m R_P$ denotes the ideal in I^m in the ring R_P, i.e., the ring R localized at P.

The m-th symbolic power and the m-th differential power are then related, as first shown by Nagata and Zariski.

Theorem 3.8 (Nagata-Zariski). *If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous radical ideal with $I \neq \langle x_0, \ldots, x_n \rangle$ (so $I = I(V)$ for some variety $V \subseteq \mathbb{P}^n$), then*

$$I^{(m)} = I^{\langle m \rangle} \text{ for all integers } m \geq 1.$$

Note that the Nagata-Zariski theorem implies that the questions introduced in the last section can be translated into questions about the m-symbolic power of an ideal.

We now introduce the invariant of interest:

Definition 3.9. Let I be a homogeneous radical ideal of R. The *Waldschmidt constant* of I is

$$\widehat{\alpha}(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

Remark 3.10. The Waldschmidt constant $\widehat{\alpha}(I)$ was first introduced by Bocci and Harbourne [6]. They proved that this limit exists. This invariant was introduced in order to study the containment problem[1] of ideals, i.e., for a fixed r, what is the smallest integer m such that $I^{(m)} \subseteq I^r$.

3.3. **The square-free monomial case.** In general, computing the Waldschmidt constant of an ideal is quite difficult. Indeed, both Chudnosky's Conjecture and Nagata's Conjecture (Conjectures 3.5 and 3.6) can be restated as conjectures about the Waldschmidt constant about the ideal of a set of points. However, when I is a square-free monomial ideal, [5] showed that there is a procedure to compute $\widehat{\alpha}(I)$. We will describe this procedure.

Recall that we call a monomial ideal I a square-free monomial ideal if it is generated by square-free monomials. That is, each generator of I has the form $m = x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \in \{0, 1\}$ for all i. The following theorem summarizes some of the nice features of square-free monomial ideals.

Theorem 3.11. *Let I be a square-free monomial ideal in $R = k[x_1, \ldots, x_n]$.*

- (i) *There exists unique prime ideals of the form $P_i = \langle x_{i,1}, \ldots, x_{i,t_i} \rangle$ such that $I = P_1 \cap \cdots \cap P_s$.*
- (ii) *With the P_i's as above, the m-th symbolic power of I is given by $I^{(m)} = P_1^m \cap \cdots \cap P_s^m$.*
- (iii) *For all integers $m \geq 1$,*

$$\alpha(I^{(m)}) = \min\{a_1 + \cdots + a_n \mid x_1^{a_1} \cdots x_n^{a_n} \in I^{(m)}\}.$$

Proof. Statement (i) follows from the theory of primary decomposition of monomial ideals (see Chapter 1 of [25]). Statement (ii) is a special case of [10, Theorem 3.7]. Statement (iii) follows directly from the definition of $\alpha(-)$ and a monomial ideal. ☐

Example 3.12. We consider the following square-free monomial ideal which we will use as our running example. In particular, we look at

$$I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1 \rangle \subseteq k[x_1, \ldots, x_5].$$

This ideal has the following primary decomposition

$$I \;=\; \langle x_1, x_3, x_4 \rangle \cap \langle x_2, x_4, x_5 \rangle \cap \langle x_3, x_5, x_1 \rangle \cap \langle x_4, x_1, x_2 \rangle \cap \langle x_5, x_2, x_3 \rangle.$$

The next result enables us to determine if a particular monomial belongs to $I^{(m)}$.

Lemma 3.13. *Let $I \subseteq R$ be a square-free monomial ideal with minimal primary decomposition $I = P_1 \cap P_2 \cap \cdots \cap P_s$ with $P_i = \langle x_{i,1}, \ldots, x_{i,t_i} \rangle$ for $i = 1, \ldots, s$. Then $x_1^{a_1} \cdots x_n^{a_n} \in I^{(m)}$ if and only if $a_{i,1} + \cdots + a_{i,t_i} \geq m$ for $i = 1, \ldots, s$.*

Proof. See [5, Lemma 2.6]. ☐

[1]See Brian Harbourne's lectures

The above lemma is the key observation that is needed in order to determine the Waldschmidt constant of square-free monomial ideals. To make this more precise, let's return to our running example.

Example 3.14. Let I be as in Example 3.12. To determine if $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5} \in I^{(m)}$, Lemma 3.13 says we need to find integers $a_1, \ldots, a_5$ that satisfies the following inequalities

$$\begin{aligned} a_1 + a_3 + a_4 &\geq m \\ a_2 + a_4 + a_5 &\geq m \\ a_3 + a_5 + a_1 &\geq m \\ a_4 + a_1 + a_2 &\geq m \\ a_5 + a_2 + a_3 &\geq m. \end{aligned}$$

If we also want to find $\alpha(I^{(m)})$, we also need to find the tuple $(a_1, a_2, a_3, a_4, a_5)$ that not only satisfies the above inequalities, but also minimizes $a_1+a_2+a_3+a_4+a_5$ (by Theorem 3.11).

Stepping back for a moment, notice in the above example, we have described the computation of $\alpha(I^{(m)})$ as a solution to linear program. This idea can be extended to all square-free monomial ideals.

In particular, given a primary decomposition of our square-free monomial ideal $I = P_1 \cap \cdots \cap P_s$, we define an $s \times n$ matrix A where

$$A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i \end{cases}.$$

We then define our linear program (LP) constructed from I as follows:

$$\begin{aligned} &\textit{minimize} && \mathbf{1}^T\mathbf{y} \\ &\textit{subject to} && A\mathbf{y} \geq \mathbf{1} \textit{ and } \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Here $\mathbf{y}^T = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}$, and $\mathbf{1}$, respectively $\mathbf{0}$, is an appropriate sized vector of 1's, respectively 0's.

Example 3.15. Continuing with our running example, the ideal I of Example 3.12 gives us the following LP:

$$\begin{aligned} &\textit{minimize} && \mathbf{1}^T\mathbf{y} = y_1 + y_2 + y_3 + y_4 + y_5 \\ &\textit{subject to} && \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \textit{ and } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

We now come to our theorem which relates this idea of a LP to the Waldschmidt constant.

Theorem 3.16 ([5, Theorem 3.2])**.** *Let $I \subseteq R$ be a square-free monomial ideal with minimal primary decomposition $I = P_1 \cap P_2 \cap \cdots \cap P_s$ with $P_i = \langle x_{i,1}, \ldots, x_{i,t_i} \rangle$ for $i = 1, \ldots, s$. Let A be the $s \times n$ matrix where*

$$A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i. \end{cases}$$

Consider the following LP:

$$\begin{array}{rl} \text{minimize} & \mathbf{1}^T\mathbf{y} \\ \text{subject to} & A\mathbf{y} \geq \mathbf{1} \text{ and } \mathbf{y} \geq \mathbf{0} \end{array}$$

and suppose that $\mathbf{y}^*$ is a feasible solution (i.e., $\mathbf{y}^*$ is a vector that satisfies the LP) that realizes the optimal value. Then

$$\widehat{\alpha}(I) = \mathbf{1}^T\mathbf{y}^*.$$

That is, $\widehat{\alpha}(I)$ is the optimal value of the LP.

Remark 3.17. Because the Waldschmidt constant of a square-free monomial ideal can be formulated in terms of a LP, it can be solved by using the *simplex method* developed by Dantzig in the 1940's. There are a number of online calculators that will allow you to solve a LP. Here is one example:

`http://comnuan.com/cmnn03/cmnn03004/`

Remark 3.18. Note that to set up the LP to find the Waldschmidt constant of a square-free monomial ideal, we only need to know information about the primary decomposition of the monomial ideal I.

Example 3.19. For the LP in Example 3.15, the feasible solution that gives the optimal value is

$$\mathbf{y}^T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Consequently,

$$\widehat{\alpha}(I) = \frac{1}{3} + \cdots + \frac{1}{3} = \frac{5}{3}.$$

By rephrasing the Waldschmidt constant as a solution as a LP, one can prove a Chudnovsky-like result (i.e., a result similar to the statement of Conjecture 3.5).

Theorem 3.20. *Let I be a square-free monomial ideal and $e = \operatorname{bight}(I) = \max\{\operatorname{ht}(P_i) \mid I = P_1 \cap \cdots \cap P_s\}$. Then*

$$\widehat{\alpha}(I) \geq \frac{\alpha(I) + e - 1}{e}$$

Remark 3.21. For an ideal of points $I_{\mathbb{X}}$, $\operatorname{bight}(I_{\mathbb{X}}) = n$. The above inequality was conjectured to be true for all monomial ideals in [10].

3.4. **Connection to graph theory.** To wrap up this lecture, I want to make a connection between the Waldschmidt constant and graph theory. Recall that we use $G = (V, E)$ to denote a finite simple graph on the vertex set $V = \{x_1, \dots, x_n\}$ with edge set E. By identifying the vertices of G with the variables of $R = k[x_1, \dots, x_n]$, we can define the *edge ideal* of G. Specifically,

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E \rangle.$$

Example 3.22. Let $G = (V, E)$ be the graph with vertex set $V = \{x_1, \dots, x_5\}$ and edge set $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}\}$. The graph G is an example of a cycle (specifically, the five cycle) because we can represent it pictorially as in Figure 4. The edge ideal of this graph is then $I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1 \rangle$. This ideal is the

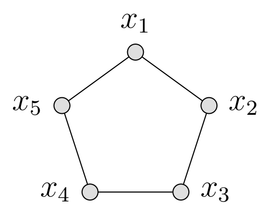


FIGURE 4. The five cycle graph

same ideal as our running example (Example 3.12) in the previous section.

We now introduce a notion that generalizes the idea of a colouring of a graph.

Definition 3.23. Let G be a graph. A *b-fold colouring* of G is an assignment of b colours to each vertex so that adjacent vertices receive different colours. The *b-fold chromatic number* of G, denoted $\chi_b(G)$, is the minimal number of colours needed to give G a b-fold colouring.

Example 3.24. Consider the graph G of Example 3.22. The 2-fold chromatic number of this graph G is $\chi_2(G) = 5$ since the graph can be coloured as in Figure 5. Here, R is

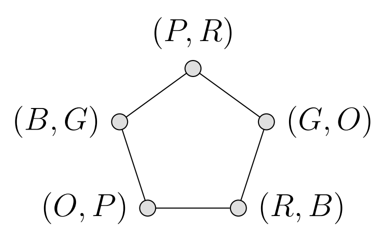


FIGURE 5. A 2-fold colouring of the five cycle graph

RED, B is BLUE, G is GREEN, O is ORANGE, and P is PURPLE.

The b-fold chromatic number allows us to define a new invariant of a graph.

Definition 3.25. The *fractional chromatic number* of G, denoted $\chi_f(G)$, is defined to be

$$\chi_f(G) := \lim_{b\to\infty} \frac{\chi_b(G)}{b}.$$

We can now connect the Waldschmidt constant to the fractional chromatic number.

Theorem 3.26. *Let G be a finite simple graph with edge ideal $I(G)$. Then*

$$\widehat{\alpha}(I(G)) = \frac{\chi_f(G)}{\chi_f(G)-1}.$$

Proof. (Sketch of main idea.) It is known that the fractional chromatic number of a graph can also be expressed as a solution to a LP (see, for example, the nice book [36]). Then one relates to this LP with the LP of Theorem 3.16. □

Remark 3.27. Although we have only stated the above result for edge ideals, the result holds more general for all square-free monomial graphs. The appropriate combinatorial object is a hypergraph. The Waldschmidt constant is then related to the fractional chromatic number of the hypergraph.

Example 3.28. In Example 3.19, we showed that $\widehat{\alpha}(I(C_5)) = \frac{5}{3}$. By Theorem 3.26, we have

$$\frac{5}{3} = \frac{\chi_f(C_5)}{\chi_f(C_5)-1} \Rightarrow \chi_f(C_5) = \frac{5}{2}.$$

This agrees with [36, Proposition 3.1.2].

Remark 3.29. Besides the Waldschmidt constant, the fractional chromatic number has also appeared in connection to the problems mentioned in the last two lectures. In particular, Francisco, Hà, and myself [16] used some conditions on the fractional chromatic number to show that a particular prime was an associated prime of a power of a cover ideal.

We have only focused on the case of square-free monomial ideals. The natural next step is still open:

Question 3.30. *Is there a similar procedure to find $\widehat{\alpha}(I)$ for nonsquare-free monomial ideals?*

4. Lecture 4: Comparing $I^{(m)}$ and I^m: the symbolic defect

The purpose of this lecture is to introduce the symbolic defect of a homogeneous ideal. This concept was introduced relatively recently by Galetto, Geramita, Shin, and myself [20]. There are a number of interesting questions one can ask about this invariant, and hopefully this lecture will inspire you to investigate the symbolic defect of your favourite family of homogeneous ideals. Throughout this lecture, we will assume that $R = k[x_1, \ldots, x_n]$ is polynomial ring over an algebraically closed field of characteristic zero, and I will be a homogeneous ideal of R.

4.1. **Introducing the symbolic defect.** We begin this lecture with some observations. For any homogeneous ideal I, we always have $I^m \subseteq I^{(m)}$. As a consequence the R-module $I^{(m)}/I^m$ is well-defined. The main idea behind the symbolic defect of an ideal is that $I^{(m)}/I^m$ is somehow a measure of the "failure" of I^m to equal $I^{(m)}$. That is, the "bigger" the module $I^{(m)}/I^m$, the more I^m fails to equal $I^{(m)}$. This suggests we may wish to study the module $I^{(m)}/I^m$ in more detail. Beside my own paper, I only know of the paper[2] of Arsie and Vatne [1] that has looked at this module.

But what do we mean by "bigger"? Note that when I is a homogeneous ideal, the R-module $I^{(m)}/I^m$ is also a graded R-module (and also an R/I^m-module). Furthermore, since R is Noetherian, the module $I^{(m)}/I^m$ is Noetherian. Consequently, the quotient $I^{(m)}/I^m$ is a finitely generated graded R-module, and furthermore, the number of minimal generators is an invariant of $I^{(m)}/I^m$. So, one way to measure "bigger" is determine the number of minimal generators of $I^{(m)}/I^m$.

For any R-module M, let $\mu(M)$ denote the number of minimal generators of M. We can then define the symbolic defect of an ideal.

Definition 4.1. Let I be a homogeneous ideal of R, and $m \geq 1$ any positive integer. The *m-th symbolic defect* of I, is

$$\operatorname{sdefect}(I, m) := \mu\left(I^{(m)}/I^m\right).$$

The *symbolic defect sequence* of I is the sequence

$$\{\operatorname{sdefect}(I, m)\}_{m \in \mathbb{N}}.$$

Note that it follows directly from the definition that $\operatorname{sdefect}(I, m) = 0$ if and only if $I^{(m)} = I^m$. From this point-of-view, it makes sense to view $\operatorname{sdefect}(I, m)$ as measuring the failure of I^m to equal $I^{(m)}$. Before going further, let's work out an example.

Example 4.2. We consider the monomial ideal[3]

$$I = \langle xy, xz, yz \rangle \subseteq R = k[x, y, z].$$

[2] A previous PRAGMATIC project!

[3] This is the third time we have used this ideal; it is a useful example.

Using either a computer algebra system, or computing by hand, we can show

$$\begin{aligned} I^2 &= \langle x^2y^2, x^2y, z, xy^2z, x^2z^2, xyz^2, y^2z^2 \rangle \\ I^{(2)} &= \langle xyz, x^2y^2, x^2z^2, y^2z^2 \rangle. \end{aligned}$$

Thus

$$\begin{aligned} I^{(2)}/I^2 &= \left\langle xyz + I^2, x^2y^2 + I^2, x^2z^2 + I^2, y^2z^2 + I^2 \right\rangle \\ &= \left\langle xyz + I^2 \right\rangle \subseteq R/I^2 \end{aligned}$$

So, $\operatorname{sdefect}(I, 2) = 1$.

We would like to make one other remark about this module since we will return to it at the end of the lecture. Note that the module $I^{(2)}/I^2$ is a graded R-module. We can actually compute the dimension of each graded piece. In particular, we have

$$\dim_k \left[I^{(2)}/I^2 \right]_t = \begin{cases} 0 & \text{if } 0 \leq t < 3 \\ 1 & \text{if } t = 3 \\ 0 & \text{if } t > 3. \end{cases}$$

To see why, note that if $t < 3$, then $[I^{(2)}]_t = (0)$, so the first case follows. As we observed above, $I^{(2)}/I^2$ has exactly one generator of degree 3. This gives the result for $t = 3$. For $t > 4$, we claim that $[I^{(2)}]_t = [I^2]_t$. We only need to check that $[I^{(2)}]_t \subseteq [I^2]_t$ since $I^2 \subseteq I^{(2)}$ takes care of the other inclusion. Take any monomial m of degree t in $[I^{(2)}]_t$. Since $I^{(2)}$ is a monomial ideal, m is divisible by one of $\{xyz, x^2y^2, x^2z^2, y^2z^2\}$. If m is divisible by one of x^2y^2, x^2z^2, or y^2z^2, then it must also be I^2 since these are generators of I^2. Suppose that m is divisible by xyz. Since m has degree $t \geq 4$, there must also be another variable that divides m. But that means that one of x^2yz, xy^2z, or xyz^2 must divide m. So, m is in I^2, as desired.

Now that we have defined the symbolic defect, a number of natural questions arise:

Question 4.3. *Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$.*

- (i) *How can we compute* $\operatorname{sdefect}(I, m)$*?*
- (ii) *When is* $\operatorname{sdefect}(I, m) = 1$*? (In this case, $I^{(m)}$ is "almost" I^m since $I^{(m)} = \langle F \rangle + I^m$ for some homogeneous form F.)*
- (iii) *Are there any application of* $\operatorname{sdefect}(I, m)$*?*
- (iv) *Are there any connects to the containment problem?*
- (v) *What can one say about the symbolic defect sequence?*

In this lecture, I will touch upon $(i) - (iv)$ in Question 4.3. I actually know very little about Question 4.3 (v).

4.2. **Some basic properties.** We quickly describe some basic properties that will be useful for our future discussion.

If $\operatorname{sdefect}(I,m) = s$, then there exists s homogeneous forms $F_1,\ldots,F_s \in I^{(m)}$ such that

$$I^{(m)}/I^m = \langle F_1 + I^m, \ldots, F_s + I^m \rangle .$$

Note that this implies that

$$I^{(m)} = \langle F_1,\ldots,F_s\rangle + I^m.$$

It is important to note that the F_i's are not unique. In particular, one can use other coset representatives. That is, for each $i = 1,\ldots,s$, let G_i be a form such that $G_i + I^m = F_i + I^m$. Then

$$I^{(m)}/I^m = \langle G_1 + I^m, \ldots, G_s + I^m\rangle$$

and also $I^{(m)} = \langle G_1,\ldots,G_s\rangle + I^m$. Note, however, that it might be the case that

$$\langle F_1,\ldots,F_s\rangle \neq \langle G_1,\ldots,G_s\rangle.$$

The only thing that is the same is the number of generators. We make this concrete in the next example:

Example 4.4. Let I be as in Example 4.2. Then

$$I^{(2)}/I^2 = \langle xyz + I^2\rangle = \langle xyz + x^2y^2 + I^2\rangle$$

but $\langle xyz\rangle \neq \langle xyz + x^2y^2\rangle$.

The following result summarizes some useful results of $\operatorname{sdefect}(I,m)$.

Theorem 4.5. *For all homogeneous radical ideals I,*

- (i) $\operatorname{sdefect}(I,1) = 0$.
- (ii) *if I is a complete intersection,* $\operatorname{sdefect}(I,m) = 0$ *for all $m \geq 1$.*
- (iii) *if $\mathbb{X} \subseteq \mathbb{P}^2$, and if $\mathbb{X}$ is not a complete intersection, then* $\operatorname{sdefect}(I,m) \neq 0$ *for all $m \geq 2$.*

Proof. Statement (i) follows from the fact that $I^{(1)} = I^1$. For (ii), this follows from a classical result of Zariski-Samuel [46] that $I^{(m)} = I^m$ for all $m \geq 1$ when I defines a complete intersection. For (iii), see [11, Remark 2.12(i)]. $\square$

4.3. **Computing** $\operatorname{sdefect}(I,m)$**.** In general, I do not know of any algorithm to compute $\operatorname{sdefect}(I,m)$ efficiently[4]. However, one can use the following strategy to compute this value:

Strategy 4.6 (Computing $\operatorname{sdefect}(I,m)$)**.** *Let I be a homogeneous ideal of R.*

- (a) *Find an ideal J such that $I^{(m)} = J + I^m$.*
- (b) *Show that all the minimal generators of J are required.*
- (c) $\operatorname{sdefect}(I,m) = \mu(J)$.

[4]This might be a good research problem

Note that if one only carries out (a), you can only shown that $\operatorname{sdefect}(I, m) \leq \mu(J)$. In [20], we used Strategy 4.6 to find $\operatorname{sdefect}(I, 2)$ when I is a star configuration. Interestingly, the ideal J that we needed for (a) turned out to be a star configuration as well. Without further ado, here is the definition of a star configuration.

Definition 4.7. Fix positive integers n, c, and s with $1 \leq c \leq \min\{n, s\}$. Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be a set of s linear homogeneous polynomials in $k[x_0, \ldots, x_n]$ such that all subsets of $\mathcal{L}$ of size $c+1$ are complete intersections. Set

$$I_{c,\mathcal{L}} = \bigcup_{1 \leq i_1 < i_2 < \cdots < i_c \leq s} \langle L_{i_1}, \ldots, L_{i_c} \rangle.$$

The vanishing locus of $I_{c,\mathcal{L}}$, i.e., $V(I_{c,\mathcal{L}}) \subseteq \mathbb{P}^n$, is a *linear star configuration.*

Remark 4.8. In the above definition, we have required all the elements of $\mathcal{L}$ to be linear forms. One can drop this requirement, and still define a star configuration. To simplify our discussion, we will only focus on the linear case. See [20, Definition 3.1]. For further details on star configurations, see [21] and [22].

Example 4.9. The name “star configuration” was suggested by A.V. Geramita. It was inspired by picture in the case that $n = 2$, $s = 5$, and $c = 2$. In this case, we take five linear forms in $k[x_0, x_1, x_2]$, say $\mathcal{L} = \{L_1, \ldots, L_5\}$. The fact that any three linear forms of $\mathcal{L}$ is a complete intersection is equivalent to the fact that no three of the associated lines meet at the same point. In this case, the star configuration $V(I_{2,\mathcal{L}}) \subseteq \mathbb{P}^2$ is the $10 = \binom{5}{2}$ points of intersections of these five lines. When we draw the five lines, as in Figure 6, we see that they make a “star” shape. Classically, linear star-configurations were sometimes

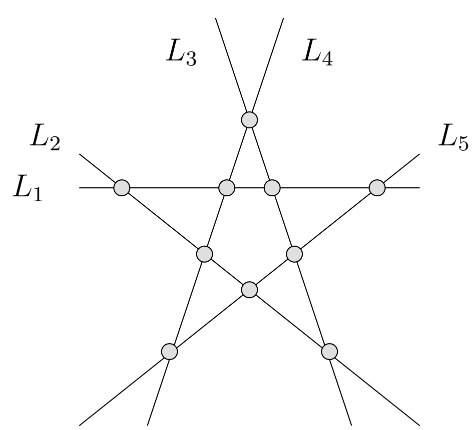


FIGURE 6. The linear star configuration of 10 points in $\mathbb{P}^2$

called ℓ-laterals (see, for example, [13]).

Example 4.10. Example 4.2 is also an example of a star configuration. In this case, $n = 2$, $s = 3$, and $c = 2$, and the linear forms are $\mathcal{L} = \{x, y, z\}$ in $R = k[x, y, z]$.

We now describe some properties of the defining ideals of star configurations.

Lemma 4.11. *Fix positive integers n, c, and s with $1 \le c \le \min\{n, s\}$ and let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be s linear forms in $R = k[x_0, \ldots, x_n]$. Then the ideal $I_{c,\mathcal{L}}$ is minimally generated by $\binom{s}{s-c+1}$ homogeneous generators of degree $s - c + 1$.*

Proof. See [34, Theorem 2.3] for generation and [34, Corollary 3.5] for minimality. □

Theorem 4.12. *Fix positive integers n, c, and s with $1 \le c \le \min\{n, s\}$ and let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be s linear forms in $R = k[x_0, \ldots, x_n]$. Then for all integers $m \ge 2$,*

$$I_{c,\mathcal{L}}^{(m)} = I_{c,\mathcal{L}}^{m} + M \text{ for all } m \ge 2$$

where

$$M = \left\langle L_1^{a_1} \cdots L_s^{a_s} \;\middle|\; \begin{array}{c} |\{a_i \mid a_i > 0\}| \ge s - c + 2, \text{ and} \\ a_{i_1} + \cdots + a_{i_c} \ge m \text{ for all } 1 \le i_i < \cdots < i_c \le s \end{array} \right\rangle.$$

Proof. This result is [20, Theorem 3.13]. The main idea is to first prove a monomial version of this result, i.e., first verify the theorem for the case that $\mathcal{L} = \{x_0, \ldots, x_n\}$. Then one applies a powerful result of Geramita, Harbourne, Migliore, and Nagel [22, Theorem 3.6] that allows one to extend the monomial case to any star configuration. □

Example 4.13. Returning to the star configuration of Example 4.2, we have $n = 2, s = 3$, and $c = 2$, with $\mathcal{L} = \{x, y, z\}$. If we consider the case $m = 2$, then the ideal M of Theorem 4.12 is

$$M = \left\langle x^{a_1} y^{a_2} z^{a_3} \;\middle|\; \begin{array}{c} |\{a_i \mid a_i > 0\}| \ge 3 - 2 + 2 = 3, \text{ and} \\ a_1 + a_2 \ge 2, a_1 + a_3 \ge 2, a_2 + a_3 \ge 2 \end{array} \right\rangle = \langle xyz \rangle$$

So,

$$I_{2,\mathcal{L}}^{(2)} = \langle xyz \rangle + I_{2,\mathcal{L}}^2.$$

Note that in the above example, the ideal $M = \langle xyz \rangle$ actually equals

$$I_{1,\mathcal{L}} = \langle x \rangle \cap \langle y \rangle \cap \langle z \rangle = \langle xyz \rangle.$$

This is an example of a much more general phenomenon, as first shown in [20, Corollary 3.14].

Corollary 4.14. *With the notation as in Theorem 4.12,*

$$I_{c,\mathcal{L}}^{(2)} = I_{c-1,\mathcal{L}} + I_{c,\mathcal{L}}^2.$$

We now have enough machinery to determine the symbolic defect for all linear star configurations when $m = 2$.

Theorem 4.15. *Fix positive integers n, c, and s with $1 \le c \le \min\{n, s\}$ and let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be s linear forms in $R = k[x_0, \ldots, x_n]$. Then*

$$\operatorname{sdefect}(I_{c,\mathcal{L}}, 2) = \binom{s}{c-2}.$$

Proof. By Corollary 4.14, we know $I_{c,\mathcal{L}}^{(2)} = I_{c-1,\mathcal{L}} + I_{c,\mathcal{L}}^2$, so we need to show that all the generators of $I_{c-1,\mathcal{L}}$ are required. By Lemma 4.11, the ideal $I_{c-1,\mathcal{L}}$ has $\binom{s}{s-(c-1)+1} = \binom{s}{c-2}$ minimal generators of degree $s-c+2$. By the same lemma, the ideal $I_{c,\mathcal{L}}$ is generated in degree $(s-c+1)$, so $I_{c,\mathcal{L}}^2$ is generated by forms of degree $2(s-c+1) > s-c+2$. So, we need all of the generators of $I_{c-1,\mathcal{L}}$ via this degree argument. □

The above result leads to the following open question:[5]

Question 4.16. *What is* $\operatorname{sdefect}(I_{c,\mathcal{L}}, m)$ *for* $m > 2$*?*

Some upper bounds $\operatorname{sdefect}(I_{c,\mathcal{L}}, m)$ can be found in [20], but there are no exact formulas.

4.4. **Two applications.** We now want to take a moment to discuss two possible applications of the symbolic defect. We begin by recalling that for any homogeneous ideal $I \subseteq R$, $\alpha(I) = \min\{d \mid (I)_d \neq 0\}$.

Our first application deals with finite sets of points in $\mathbb{P}^2$. That is, let $\mathbb{X} = \{P_1, \ldots, P_s\}$, and let $I_{\mathbb{X}}$ be the associated homogeneous ideal that contains all forms that vanish on $\mathbb{X}$. We then have the following lemma (see [20, Lemma 4.1]):

Lemma 4.17. *Let* $\mathbb{X} \subseteq \mathbb{P}^2$ *be a set of points with* $\alpha(I_{\mathbb{X}}) = \alpha$. *Then* $I_{\mathbb{X}}$ *has at most* $\alpha + 1$ *minimal generators of degree* α.

As first shown in [20, Theorem 4.5], if $\operatorname{sdefect}(I_{\mathbb{X}}, 2) = 1$, the points must lie in a special configuration.

Theorem 4.18. *Let* $\mathbb{X} \subseteq \mathbb{P}^2$ *be a set of* s *points such that* $I_{\mathbb{X}}$ *has* $\alpha+1$ *minimal generators of degree* $\alpha = \alpha(I_{\mathbb{X}})$. *If* $\operatorname{sdefect}(I_{\mathbb{X}}, 2) = 1$, *then* $\mathbb{X}$ *is a linear star configuration.*

As noted in Theorem 4.5, if $\mathbb{X}$ is a set of points in $\mathbb{P}^2$ that is not a complete intersection, then $\operatorname{sdefect}(I_{\mathbb{X}}, 2) \neq 0$. In some cases, we can now show that it is not equal to one.

We say that $\mathbb{X} \subseteq \mathbb{P}^2$ is in *generic position* if the Hilbert function of $R/I_{\mathbb{X}}$ is

$$H_{R/I_{\mathbb{X}}}(t) = \min\left\{\dim_k R_t = \binom{t+2}{2}, |\mathbb{X}|\right\} \text{ for all integers } t \geq 0.$$

Corollary 4.19. *Suppose* $\mathbb{X} \subseteq \mathbb{P}^2$ *is a set of points in generic position with* $|\mathbb{X}| = \binom{\ell}{2}$ *for some* $\ell \geq 4$. *Then* $\operatorname{sdefect}(I_{\mathbb{X}}, 2) > 1$.

Proof. Because $\mathbb{X}$ is in generic position, $\mathbb{X}$ is not a complete intersection, which means $\operatorname{sdefect}(I_{\mathbb{X}}, 2) \neq 0$. Since $|\mathbb{X}| = \binom{\ell}{2}$, we can show that $\alpha(I_{\mathbb{X}}) = \ell - 1$, and $I_{\mathbb{X}}$ has ℓ minimal generators of degree α. Since $\mathbb{X}$ is not a star configuration, Theorem 4.18 implies that $\operatorname{sdefect}(I_{\mathbb{X}}, 2) \neq 1$. So the conclusion follows. □

[5]Another good research problem.

As a second application, we make the observation that when $\operatorname{sdefect}(I,m) = 1$, then one can create a useful short exact sequence that may be exploited. Specifically, if $\operatorname{sdefect}(I,m) = 1$, this means that there exists a homogeneous form F such that $I^{(m)} = \langle F\rangle + I^m$. We can then build a short exact sequence that relates I^m and $I^{(m)}$:

$$0 \longrightarrow I^m \cap \langle F\rangle \longrightarrow I^m \oplus \langle F\rangle \longrightarrow I^m + \langle F\rangle = I^{(m)} \longrightarrow 0.$$

This short exact sequence allowed [20, Theorem 5.3] to use a mapping cone construction to determine the minimal graded free resolution of $I^{(2)}_{2,\mathcal{L}} \subseteq \mathbb{P}^2$.

4.5. **Connection to the containment problem.** We now turn our attention to the containment problem. For any homogeneous ideal $I \subseteq R$, Ein, Lazarsfeld, and Smith [14] showed that for any fixed m, there exists an integer $r \geq m$ such that $I^{(r)} \subseteq I^m$. The containment problem is to find the smallest such r.

Arsie and Vatne [1] observed that the module $I^{(m)}/I^m$ has the following submodules:

$$\frac{I^{(m)}}{I^m} \supseteq \frac{I^{(m+1)} + I^m}{I^m} \supseteq \frac{I^{(m+2)} + I^m}{I^m} \supseteq \cdots \supseteq \frac{I^{(r)} + I^m}{I^m}.$$

The containment problem is thus equivalent to finding the smallest r such that $\frac{I^{(r)}+I^m}{I^m} = 0$. One can ask the following question:

Question 4.20. *If* $\operatorname{sdefect}(I,m) = 1$, *does this give any information on the containment problem?*

I don't know much about this question. However, here is an example that shows that the containment problem is related to the longest possible chain of proper submodules in $I^{(m)}/I^m$.

Example 4.21. Return to the ideal $I = \langle xy, xz, yz\rangle$ in Example 4.2. We showed that

$$\left[\frac{I^{(2)}}{I^2}\right]_t = 0 \text{ except if } t = 3.$$

In fact, the only possible proper submodule of of $I^{(2)}/I^2$ is the zero module. Since $(I^{(3)} + I^2)/I^2$ is a proper submodule of $I^{(2)}/I^2$, this forces $(I^{(3)} + I^2)/I^2 = 0$, or equivalently, $I^{(3)} \subseteq I^2$.

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