The Zero Forcing Number of Circulant Graphs

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Abstract

This project studies the zero forcing number, utilizing the colour-change rule, a graph parameter originally introduced to help solve a minimum rank problem. Since its introduction, studies surrounding the zero forcing number have produced some interesting results. This paper begins by reviewing graph theory concepts, and discusses the family of circulant graphs. The colour-change rule, and past findings about the zero forcing number for general and circulant graphs are presented. Once the preliminary topics are covered, we focus on the family of circulant graphs and determine the zero forcing number for various circulant families.

Acknowledgements

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CHAPTER 1

Introduction

This project looks at families of circulant graphs and studies their zero forcing number. The zero forcing number of a finite simple graph G is an invariant of a graph that was first introduced in [**1**]. Although it was originally designed to investigate the minimum rank problem of graphs, it has been shown to be an interesting property on its own right. It is of interest to determine the zero forcing number (and a related invariant, the propagation time) for specific families of graphs. One family of graphs, the circulant graphs, is a k-regular family of graphs with lots of symmetry. To date, the zero forcing number has only been computed for a few members of this family. We wish to determine the zero forcing number for a few more members.

In this chapter we will first introduce the required graph theory background. We will then introduce circulant graphs as well as present some of their properties which are useful for generalizing results. Then the zero forcing number and the associated propagation time will be introduced. After the basic concepts are covered, we give a brief review of what is known about the zero forcing number of a graph. The second chapter will focus on our new results. We look at a few families of circulant graphs and calculate the zero forcing number and propagation time building on past research. The appendices contain tables listing the calculated the zero forcing numbers and propagation times for a all circulant graphs on up to 16 vertices, as well as other families of circulant graphs. In these tables, there are a few patterns which go unproven in this paper. These are presented as conjectures at the end of chapter 2.

1. Graph Theory Background

We begin with a review of the relevant background of graph theory. The textbook *Pearls in Graph Theory* by Nora Hartsfield and Gerhard Ringel [**11**] will be our primary reference for graph theory terminology.

Definition 1.1. A *graph* G is a pair of sets $(V(G), E(G))$ where $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ is nonempty, and $E(G)$ is a (possibly empty) set of unordered pairs of elements of $V(G)$. The elements of $V(G)$ are called the *vertices* of G, and the elements of $E(G)$ are called the *edges* of G.

Definition 1.2. A *simple* graph is defined as a graph which has no edges which are loops or multiple edges between a pair of vertices.

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The vertex set $V(G)$ is a collection of points or nodes in the plane, usually labeled $\{v_0, v_1, \ldots, v_{n-1}\}$ for some $n \in \mathbb{N}$. An edge in $E(G)$ is pictorially expressed as a line or connection between two of these points.

DEFINITION 1.3. The *order* of a graph G is the number of vertices of G, denoted as $|G|$.

DEFINITION 1.4. An edge is *incident* to a vertex if the edge has the vertex as one of its endpoints. For example, the edge $\{v_i, v_j\}$ is incident to the vertices $v_i, v_j \in V(G)$.

Because there is no one method for drawing a graph G, there is the possibility that G can be drawn in many different ways. We use the following definition to determine if two graph are the same.

DEFINITION 1.5. Two graphs G_1 and G_2 are said to be *isomorphic*, denoted $G_1 \simeq G_2$, if there exists a bijective function $f : V(G_1) \to V(G_2)$ such that $\{v_i, v_j\} \in E(G_1)$ if and only if $\{f(v_i), f(v_j)\} \in E(G_2)$.

DEFINITION 1.6. If v_i and v_j are vertices of $V(G)$, we say that v_i is *adjacent* to v_j if there is an edge $\{v_i, v_j\} \in E(G)$ between v_i and v_j. We say that v_j is a *neighbour* of v_i.

DEFINITION 1.7. The degree of a vertex $v \in V(G)$ is the number of edges in $E(G)$ which are incident to v. The minimum degree of a vertex in a graph G is denoted $\delta(G)$.

EXAMPLE 1.8. The graph in Figure 1 has vertices v_0 through v_5 with varying degrees:

- vertex v_6 has degree equal to 1
- vertices v_1 and v_3 have degree equal to 2
- vertices v_2, v_4, and v_5 have degree equal to 3

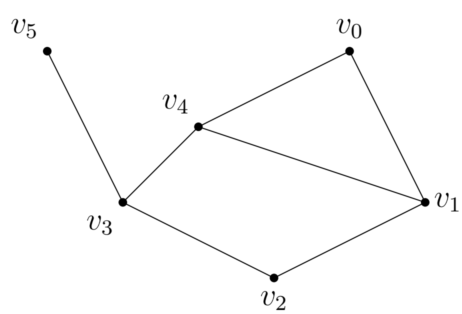

FIGURE 1. A graph with varying vertex degree

DEFINITION 1.9. A graph G is said to be *regular of degree k*, or *k-regular*, if every vertex $v \in V(G)$ has degree equal to k. A 3-regular graph is also called a *cubic* graph.

DEFINITION 1.10. A graph G is *vertex-transitive* if for any two vertices v_1, and $v_2 \in V(G)$, there exists an automorphism $f : V(G) \to V(G)$ such that $f(v_1) = v_2$.

FIGURE 2. The graph P_5

There are a few important basic families of graphs from which the family of circulant graphs are built upon. The following examples will be used as a beginning step to help understand the constructions which follow.

DEFINITION 1.11. A *path* on n vertices is the graph P_n such that $E(P_n) = \{\{v_i, v_{i+1}\} : i = 0, 1, \ldots, n-2\}$. Figure 2 shows the path graph P_5 on five vertices.

DEFINITION 1.12. A *cycle* on n vertices is the graph C_n such that $E(C_n) = \{\{v_i, v_{i+1}\} : i = 0, 1, \ldots, n-2\} \cup \{\{v_n, v_0\}\}$. See Figure 5 for an example of C_9.

An example of a regular graph is the family of cycle graphs C_n. Every vertex in a cycle will have degree equal to two, so every cycle graph is 2-regular. See Figure 3 for an example of the cycle graph C_3, which is a 2-regular graph on three vertices.

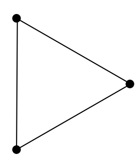

FIGURE 3. The 2-regular cycle graph C_3.

DEFINITION 1.13. A *complete graph* on n vertices is the graph K_n such that $E(G) = \{\{v_i, v_j\} : 0 \leq i < j \leq n-1\}$. See Figure 6 for an example of K_9.

DEFINITION 1.14. A graph $G' = (V(G'), E(G'))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

EXAMPLE 1.15. For any $n \in \mathbb{N}$, the graph P_n is a subgraph of C_n, which is a subgraph of K_n, since $E(P_n) \subseteq E(C_n) \subseteq E(K_n)$.

DEFINITION 1.16. A *tree* is a connected graph which contains no subgraph isomorphic to a cycle.

DEFINITION 1.17. A graph G is *bipartite* if the vertex set $V(G)$ can be partitioned into two disjoint non-empty subsets $U, W \subset V(G)$, such that every edge in $E(G)$ has one incident vertex in U and the other in W. A *complete bipartite graph* is the bipartite graph $K_{a,b} = (U \cup W, E(G))$ such that $|U| = a, |W| = b$ and $E(G) = \{\{u, v\} : u \in U$ and $v \in V\}$.

DEFINITION 1.18. A *walk* in a graph G is an alternating sequence of vertices $v_i \in V(G)$ and edges $e_i \in E(G)$, $e_1 v_1 e_2 v_2 \ldots e_{n-1} v_{n-1} e_n v_n$, such that every edge e_i is incident with vertices v_i and v_{i+1}, and $v_i \neq v_{i+1}$.

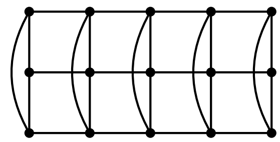

FIGURE 4. The Cartesian product of C_3 and P_5 results in the graph shown, $C_3 \Box P_5$.

DEFINITION 1.19. A graph G is said to be *connected* if for every pair of vertices $u, v \in V(G)$, there is a walk from u to v. A graph is *disconnected* if it is not connected.

DEFINITION 1.20. In a disconnected graph G, the subgraphs that are connected and not contained in any larger connected subgraph are called the *components* of G.

Using the above notion, this project will only focus on connected graphs.

DEFINITION 1.21. The *Cartesian product* of two graphs G and H, denoted $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if $u = u'$ and $\{v, v'\} \in E(H)$, or $v = v'$ and $\{u, u'\} \in E(G)$.

EXAMPLE 1.22. An example of the Cartesian product of two graphs is the product $C_3 \Box P_5$. See Figure 4.

EXAMPLE 1.23. Another example of the Cartesian product of two graphs is the product $K_n \Box K_m$. The resulting graph is referred to as the *Rook's graph* when $m = n = 8$. The Rook's graph depicts the possible legal moves of a rook chess piece on a chessboard.

2. Circulant Graphs

This section defines the construction and notation of circulant graphs. After the basic definitions are covered, a few properties will be presented.

DEFINITION 1.24. We say that the graph G on n vertices with vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ is a *circulant graph* if $E(G)$ is generated by a set $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ where $E(G) = \{\{v_i, v_{i+j}\} | i \in \{0, \ldots, n-1\}$ and $j \in S\}$ where we take the subscript addition modulo n. We denote a circulant graph on n vertices with the set S as $C_n(S)$.

The standard notation will be adjusted for simplicity purposes. For any $S = \{s_1, \ldots, s_t\} \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, we usually simplify our notation and write $C_n(s_1, \ldots, s_t)$ instead of $C_n(\{s_1, \ldots, s_t\})$. Because circulant graphs have cyclic properties, we note that all vertex subscripts and labels are taken modulo $|G|$.

EXAMPLE 1.25. An example of a circulant graph is the cycle graph C_n, which is equivalent to the circulant graph on n vertices with the set $S = \{1\}$. In other words, $C_n \simeq C_n(1)$, where $E(C_n(1)) = \{\{v_i, v_{i+1}\} | i \in \{0, \ldots, n-1\}\}$. Note that since addition is modulo n, $E(C_n(1)) = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_0\}\}$. In Figure 5 is the cycle graph on nine vertices $C_9 \simeq C_9(1)$.

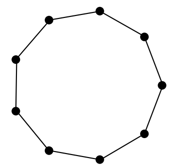

FIGURE 5. The circulant graph $C_9(1) \simeq C_9$.

EXAMPLE 1.26. The family of complete graphs K_n are also examples of circulant graphs. In particular, the set $S = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and $K_n \simeq C_n(1, \ldots, \lfloor \frac{n}{2} \rfloor)$. Figure 6 is the circulant complete graph on 9 vertices $K_9 \simeq C_9(1, 2, 3, 4)$.

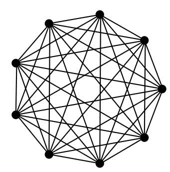

FIGURE 6. The circulant graph $C_9(1, 2, 3, 4) \simeq K_9$.

By construction, circulant graphs are all vertex-transitive and k-regular for some k based on the size of S.

LEMMA 1.27. *Consider $G = C_n(s_1, \ldots, s_a)$ such that $s_1 < s_2 < \ldots < s_a$.*

- *If n is even and $s_a = \frac{n}{2}$, then G is $(2a-1)$-regular.*
- *Otherwise, G is $2a$-regular.*

PROOF. Consider a circulant graph $G = C_n(S)$ with a set $S = \{s_1, \ldots, s_a\}$ such that $s_1 < s_2 < \ldots < s_a$. If $|G| = n$ is odd, then no element of S can be equal to $\frac{n}{2}$. So the degree of each vertex in G will be equal to $2a$. Indeed, for each $v_i \in V(G)$, v_i will be adjacent to $\{v_{i-s_a}, \ldots, v_{i-s_1}, v_{i+s_1}, \ldots, v_{i+s_a}\}$.

If $s_a < \frac{n}{2}$ and n is even, then the degree of each vertex will be equal to $2a$ and G is $2a$-regular.

If $|G| = n$ is even, the degree of the vertices will vary depending on s_a, the largest element of S. The element $s_a = \frac{n}{2}$ constructs a set of edges $\in E(G)$ such that for a vertex $v_i \in V(G)$ both $\{\{v_i, v_{i+s_a}\}, \{v_i, v_{i-s_a}\}\} \in E(G)$. However, when n is even $v_{i+\frac{n}{2}} = v_{i-\frac{n}{2}}$ when taken modulo n. Thus s_a only creates a single edge in $E(G)$. Therefore, if $s_a = \frac{n}{2}$, then the degree of a given vertex in G will be equal to $2a-1$, and G is $(2a-1)$-regular. $\square$

This notion will be very useful when determining the zero forcing number of a circulant graph G. The regularity of a graph will later on be helpful for providing a lower bound. Another important concept to remember when working with circulant graphs is the idea of connectedness. Depending on the set S and the number of vertices of a circulant graph $G_n(S)$, the graph can be disconnected, we arrive at the following theorem which was presented in [**2**].

THEOREM 1.28. [**2**] *If* $1 < s_1, s_2, \ldots, s_t < n$*, then the circulant graph* $G = C_n(s_1, s_2, \ldots, s_t)$*, is disconnected if and only if* $gcd(n, s_1, s_2, \ldots, s_t) \neq 1$*.*

When working with circulant graphs, it is possible that two graphs are isomorphic. To prove that two graphs G and H are isomorphic, we need to show there exists an bijection $f : V(G) \to V(H)$ such that $\{v_i, v_j\} \in E(G)$ if and only if $\{f(v_i), f(v_j)\} \in E(H)$.

THEOREM 1.29. [**2**] *Let* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *be two circulant graphs with vertex sets* $V(G)$ *and* $V(H)$*, where* $|V(G)| = |V(H)| = n \geq 1$*. The function* $f : \mathbb{Z}_n \to \mathbb{Z}_n$*, defined by* $f(x) = kx$*, where* $gcd(k, n) = 1$*, is a graph isomorphism between* G *and* H*.*

OBSERVATION 1.30. [**2**] *Let* $n > 1$*. When operating on circulant graphs, the graph isomorphism* $f : \mathbb{Z}_n \to \mathbb{Z}_n$*, defined by* $f(x) = kx$*, where* $gcd(k, n) = 1$*, sends* $C_n(s_1, s_2)$ *to* $C_n(ks_1, ks_2)$*.*

Although Observation 1.30 is only proven in [**2**] with two values s_1 and s_2, the bijection is defined based on the values of n and k, and so the set S can have any number of elements. Thus we have the following lemma.

LEMMA 1.31. [**2**] *If* $gcd(k, n) = 1$*, then* $C_n(s_1, s_2, \ldots, s_a) \simeq C_n(ks_1, ks_2, \ldots, ks_a)$*.*

PROOF. We verify this fact using the bijection $f : \mathbb{Z}_n \to \mathbb{Z}_n$, defined by $f(x) = kx$. □

It has been shown in [**10**] that a family of circulant graphs are bipartite graphs.

THEOREM 1.32. [**10**] *Let* $G = C_{2n}(a_1, \ldots, a_t)$ *be a connected circulant. Then* G *is bipartite if and only if* $a_1, \ldots, a_t$ *are odd.*

3. Zero Forcing and Propagation Time

For a graph G, we consider a colouring of the vertex set $V(G)$ with two colours, for example, black and white. The concept of zero forcing uses the graph propagation algorithm commonly referred to as the *colour-change rule*, defined below. The concept of the zero forcing number was introduced formally in [**1**] to provide a lower bound for a minimum rank problem.

DEFINITION 1.33. [**1**] Let $G = (V(G), E(G))$ be a finite simple graph.

- *Colour-change rule*: If G is a graph with each vertex coloured either black or white, and if v_i is a black vertex of G, and exactly one neighbour v_j of v_i is white, then change the colour of v_j to black.
- Given a colouring of G, the *derived colouring* is the result of applying the colour-change rule until no more changes are possible.

- A *zero forcing set* for a graph G is a subset of vertices $F \subseteq V(G)$ such that if initially the vertices $v_i \in F$ are coloured black, and the remaining vertices are coloured white, then the derived colouring of G is all black.
- The zero forcing number is $Z(G) = \{|F| : F \subseteq V(G) \text{ is a zero forcing set }\}$.

When a black vertex v_i is adjacent to exactly one white vertex v_j, and the vertex v_j is then coloured black, we will say that v_i *forced* v_j.

EXAMPLE 1.34. Consider the graph G of Figure 1. We wish to find a zero forcing set F for G. Take vertices $F = \{v_0, v_1\} \subseteq V(G)$ to initially be coloured black, and the remaining vertices $\{v_2, v_3, v_4, v_5\}$ to be coloured white. Vertex v_4 is the only vertex adjacent to vertex v_0 which is white, thus vertex v_4 is coloured black. Now that vertex v_4 is black, the only white vertex adjacent to vertex v_1 is vertex v_2. So we can then colour v_2 black. Similarly, vertex v_3 can be forced by vertex v_4. Finally, the only remaining white vertex is v_5 which is adjacent to v_3, so it can be forced, and coloured black. We end with a derived colouring which is all black, therefore $F = \{v_0, v_1\}$ is a zero forcing set. It is not possible to force the graph G beginning with a single vertex coloured black, thus $Z(G) = 2$.

We now give some simple bounds on $Z(G)$ which will be very important in the majority of the proofs later on.

LEMMA 1.35. *For any graph G, if $\delta(G) = min\{deg(v)|v \in V(G)\}$, then $\delta(G) \leq Z(G) \leq |V(G)| - 1$. If G is k-regular, then $Z(G) \geq k$.*

PROOF. Consider a graph G. If $|V(G)| = n$, then by the colour-change theorem, there must be at least one vertex in $V(G)$ coloured white, and thus $Z(G) \leq V(G) - 1$. If $\delta(G)$ is the minimum degree of a vertex $v_i \in V(G)$, for v_i to force another vertex, at least $\delta(G) - 1$ of the vertices must also be in the zero forcing set. If v_i is not in the zero forcing set, then there is some initial vertex v_j which has $deg(v_j) > \delta(G)$ in the initial zero forcing set. For v_j to force any other vertex, it must be adjacent to at least $\delta(G) + 1 > \delta(G)$ vertices. Thus $Z(G) \geq \delta(G)$.

If G is k-regular, then $k = \delta(G) \leq Z(G)$. □

Theorem 1.35 will be used in many cases when proving the zero forcing number of a graph, this is because the regularity of the graph provides a lower bound on the cardinality of a zero forcing set F of G. Since the colour-change rule and zero forcing number relies on the forcing set F of minimum size, we use Theorem 1.35 to reduce the proof of the zero forcing number. We do this by showing the existence of a forcing set of size $\delta(G)$.

Although the zero forcing number of a graph G is an invariant, there can be many possible zero forcing sets F of G such that $|F| = Z(G)$. The following theorem utilizing this idea is presented in [**8**].

THEOREM 1.36. [**8**] *For any graph G of order ≥ 1, no vertex is in every optimal zero forcing set of G.*

DEFINITION 1.37. The *propagation time* of the zero forcing of a graph G, denoted $P(G)$, is the minimum number of iterations of the colour-change rule to colour all of the vertices $v \in V(G)$ black over all of the zero forcing sets F such that $|F| = Z(G)$.

LEMMA 1.38. *For any graph G, $1 \leq P(G) \leq V(G) - Z(G)$.*

PROOF. Since there must be at least one vertex forced per iteration, by the colour-change rule the maximum propagation time for a given graph G is $P(G) \leq V(G) - Z(G)$. Because there must be at least one vertex not initially in the zero forcing set then $|V(G)| > Z(G)$. Thus we have that there is at least one vertex which must be forced. So $P(G) \geq 1$ for all graphs. □

EXAMPLE 1.39. Consider the graph $C_n = C_n(1)$. The zero forcing set can be taken to be any two adjacent vertices. Consider the circulant graph C_n and label the vertices v_0 to v_{n-1}. By Lemma 1.35, it is known that $Z(C_n) \geq 2$. If the two vertices are adjacent, then they will each be adjacent to one vertex not yet in the set. Once the adjacent vertices are forced then the pattern continues until all n vertices are forced. Since there exists a zero forcing set which can force every vertex, then $Z(C_n) = 2$ for all $n \geq 3$.

After each iteration of the forcing there are two more vertices which are forced. With the two initial vertices in the zero forcing set, after one iteration there are four vertices now forced. If $n = 4$, then the process stops. If not, the iteration continues. We obtain a value for $P(G)$ from the total number of vertices n minus the initial forcing set, divided by the number of vertices forced per iteration, in this case two. We thus obtain the result that $P(C_n) = \lceil \frac{n-2}{2} \rceil$.

EXAMPLE 1.40. Consider the complete graph $K_n = C_n(1, 2, 3, ..., \lfloor \frac{n}{2} \rfloor)$. By Lemma 1.35, since K_n is $(n-1)$-regular, it gives that $n - 1 \geq Z(C_n) \geq n - 1$. Therefore $Z(C_n(1, 2, 3, ..., \lfloor \frac{n}{2} \rfloor)) = n - 1$ for all $n \geq 1$.

It has been shown that $Z(K_n) = n - 1$. There is only one vertex to force, which can be forced by any of the vertices in the zero forcing set. By Lemma 1.38 there must be at least one vertex forced per iteration. Thus $P(K_n) = 1$.

From Examples 1.39 and 1.40 we obtain the following lemmas which will be used to simplify later results.

LEMMA 1.41. *If $G = C_n$, then $Z(G) = 2$ and $P(G) = \lceil \frac{n-2}{2} \rceil$.*

LEMMA 1.42. *If $G = K_n$, then $Z(G) = n - 1$ and $P(G) = 1$.*

The next example will determine the zero forcing number and propagation time of a less straight forward example.

EXAMPLE 1.43. Consider the circulant graph $G = C_6(1, 3)$ where the vertices are labeled from v_0 to v_5 (see Figure 7). From Lemma 1.27, G is 3-regular. Since G is 3-regular, Lemma 1.35 implies $Z(G) \geq 3$. It will be shown that $Z(G) = 4$, so it must be shown that it is not possible that a zero forcing set F exists such that $|F| = 3$, but it can be shown that there exists F with $|F| = 4$.

Suppose there exists a F with $|F| = 3$. From the colour-change rule, at least one of the three initial vertices must force one of the three remaining vertices. Without loss of generality, let vertex v_0 be the forcing vertex. Vertex v_0 is adjacent to vertices $\{v_1, v_3, v_5\}$. Without loss of generality again take vertices $\{v_1, v_3\}$ to also be in the initial forcing set, so vertex v_5 will be forced in the first iteration, and let it be forced. For any two of the three vertices are chosen to be in F, the third adjacent vertex will always be forced in the first iteration. This will always result in the same vertices being forced after one iteration. Once vertex v_5 is forced, there are four vertices in total which are forced. However, all four of the vertices are adjacent to two vertices not in the forcing set. Thus vertices $\{v_0, v_1, v_3\}$ is not a zero forcing set. If any other subset of two vertices adjacent to vertex v_0 were chosen to be in the original forcing set a similar result would occur with two vertices in G unable to be forced. Because of the structure of the graph G, it is not possible to have a zero forcing F set such that $|F| = 3$. So $Z(G) > 3$.

Since a forcing set of size three was not successful in colouring the entire graph, an additional vertex is added to the initial set of forced vertices. Maintaining the original three vertices, add vertex v_2 so the initial set is $F' = \{v_0, v_1, v_2, v_3\}$. We claim that $F' = \{v_0, v_1, v_2, v_3\}$ is a zero forcing set. From these four vertices, vertex v_5 can be forced from vertex v_0 as before. Also in the first iteration, vertex v_1 can force vertex v_4 since its other two adjacent vertices $\{v_0, v_2\}$ are already forced. After this step is completed all six vertices are forced, Thus F' is a zero forcing set. Therefore, $Z(G) = 4$ and $P(G) = 1$. The graph G turns out to be isomorphic to the complete bipartite $K_{3,3}$ which will be discussed in more detail later on (see Theorem 2.9).

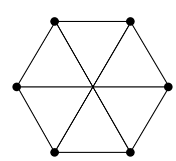

FIGURE 7. The circulant graph $C_6(1, 3) \simeq K_{3,3}$

After working through Example 1.43, it is more clear as to why the zero forcing number of a circulant graph G is not always straight forward and equal to the degree of each vertex. Now that the required background has been covered, we formally introduce the main question of this project.

QUESTION 1.44. *If* $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, *and if* $G = C_n(S)$ *is a circulant graph, then what is the zero forcing number of* G*?*

In the next chapter we will answer this question for some families of circulant graphs.

4. Literature Review

In this section, we discuss the origin of the zero forcing number, and review some of its known properties. The original idea of the zero forcing number was formally introduced in [**1**] to bound the minimum rank for a set of matrices to an associated graph, by bounding the maximum nullity. The minimum rank problem of a graph G is to determine the minimum rank over all of the $n \times n$ real symmetric of matrices with graph G.

4.1. Motivation: Minimum Rank and Zero Forcing. The set of $n \times n$ real symmetric matrices over $\mathbb{R}$ will be denoted by $S_n(\mathbb{R})$. For a matrix $A = [a_{ij}] \in S_n(\mathbb{R})$ there exists a graph of A, denoted $\mathbb{G}(A)$, with vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ and edges $\{\{v_i, v_j\} : a_{i+1,j+1} \neq 0, 0 \leq i < j \leq n-1\}$. We note that there are no restrictions on the diagonal elements of A.

Definition 1.45. The *set of* $n \times n$ *symmetric matrices* of a graph G (over $\mathbb{R}$) is

$$\mathbb{S}(G) = \{A \in S_n(\mathbb{R}) : \mathbb{G}(A) = G\}.$$

Example 1.46. In Figure 1.46 we see the graph $G = C_6(1,2)$ and two 6×6 symmetric matrices A_1 and A_2 in $\mathbb{S}(G)$. We see that the matrices A_1 and A_2 have a non-zero entry a_{ij} if and only if $\{v_{i-1}, v_{j-1}\} \in E(G)$. Otherwise, if no such edge exists, then the entry a_{ij} of A_1 and A_2 is zero.

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & 2 & 1 & 0 & 3 & 1 \\ 2 & 0 & 2 & 1 & 0 & 6 \\ 1 & 2 & 4 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 & 3 & 2 \\ 3 & 0 & 1 & 3 & 4 & 2 \\ 1 & 6 & 0 & 2 & 2 & 0 \end{bmatrix}$$

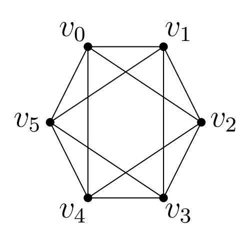

Figure 8. The circulant graph $G = C_6(1,2)$ and two elements of $\mathbb{S}(G)$.

Now that the relation between a set of matrices and its associated graph has been presented, we focus on why we would want such a relationship to exist.

Definition 1.47. The *minimum rank* of a graph G (over $\mathbb{R}$) is

$$mr(G) = min\{rank(A) : A \in \mathbb{S}(G)\}.$$

Example 1.48. Following from Example 1.46, we see matrix A_2 has six linearly independent rows. In contrast, we see that the first three rows of A_1 are linearly independent. Rows four, five, and six are copies of rows one, two and three, respectively. Thus $rank(A_2) = 6$, $rank(A_1) = 3$, and $mr(G) \leq 3$.

Definition 1.49. For $A \in \mathbb{R}^{n\times n}$, the *maximum nullity* of a graph G (over $\mathbb{R}$) is

$$M(G) = max\{nullity(A) : A \in \mathbb{S}(G)\}.$$

The Rank Nullity Theorem states that for any $n\times n$ matrix A, $rank(A)+nullity(A) = n$. Hence we have the following lemma which will be very useful.

Lemma 1.50. [**1**] *For any graph G, $mr(G) + M(G) = |V(G)|$.*

The minimum rank of G can often be difficult to compute directly. Instead, using Lemma 1.50, the maximum nullity is often used to find the solution for the minimum rank problem. The next few theorems demonstrate a relation between $Z(G)$ and $M(G)$ for a graph G. Theorem 1.51 illustrates that if $nullity(A) \geq 2$, then there is a vector in $Null(A)$ with a zero in any specified position.

Theorem 1.51. *Suppose there exists two linearly independent vectors $\vec{x}$ and $\vec{y} \in Null(A)$. Then for any i, $0 \leq i \leq n-1$, there exists a nonzero vector $\vec{z} = (z_0, z_1, \ldots, z_{n-1}) \in Null(A)$, such that $z_i = 0$.*

Proof. Without loss of generality, let $i = 0$. Let $\vec{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\vec{y} = (y_0, y_1, \ldots, y_{n-1})$ be linearly independent null vectors of A. If $x_0 = 0$ or $y_0 = 0$, then let $\vec{z} = \vec{x}$ or $\vec{z} = \vec{y}$ respectively. Otherwise, let $\vec{z} = y_0\vec{x} - x_0\vec{y}$. Then $z_0 = 0$ but $\vec{z} \neq 0$ since $\vec{x}$ and $\vec{y}$ are linearly independent and x_0 and y_0 are nonzero. □

The next theorem generalizes Theorem 1.51.

Theorem 1.52. [**1**] *If $nullity(A) > k$, then for any k specified positions there exists a non-zero vector in $Null(A)$ with zeros in those positions.*

Proof. Suppose A is an $n\times n$ matrix with $nullity(A) > k$. Without loss of generality, it is enough to show that there exists a non-zero vector $\vec{x} \in Null(A)$ such that $x_i = 0$ for $1 \leq i \leq k$. Let $\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{k+1}$ be linearly independent vectors in $Null(A)$. Consider the system of equations determined by the vector equation

$$c_1\vec{z}_1 + c_2\vec{z}_2 + \ldots + c_{k+1}\vec{z}_{k+1} = \vec{0}$$

restricted to the first k equations. Since there are more variables than equations, there is a non-trivial solution to these k equations with $c_1, c_2, \ldots, c_{k+1}$ not all zero. Since $\vec{z}_1, \ldots, \vec{z}_{k+1}$ are independent, then $\vec{x} = c_1\vec{z}_1 + c_2\vec{z}_2 + \ldots + c_{k+1}\vec{z}_{k+1} \in \mathbb{R}^n$ is a non-zero vector in $Null(A)$ with $x_i = 0$, $1 \leq i \leq k$. □

The proof of the next lemma illustrates why we use the term *zero-forcing*.

LEMMA 1.53. [**1**] *Let F be a zero forcing set of G and $A \in \mathbb{S}(G)$. If $\vec{x} \in Null(A)$ and $x_i = 0$ for all $x_i \in F$, then $\vec{x} = \vec{0}$.*

PROOF. We follow the proof of [**1**]. Let $A = [a_{ij}]$ be an $n \times n$ matrix with $A \in \mathbb{S}(G)$. Without loss of generality, assume that the first k vertices of $G = G(A)$ correspond to a zero forcing set of G. We assume that

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_{k+1} \\ \vdots \\ x_{n-1} \end{bmatrix} \in Null(A).$$

If $k = n-1$, then $\vec{x} = \vec{0}$. Suppose $k < n-1$. Then by the colour-change rule there exists some vertex $v_u \in F$, $0 \leq u \leq k$, which can force another vertex $v_w \notin F$. In particular, $k \leq w \leq n-1$. We claim that $x_w = 0$. To prove the claim, we consider $A\vec{x}$.

For convenience, if $v_a, v_b \in V(G)$, we write $v_a \sim v_b$ if v_a is adjacent to v_b, and $v_a \nsim v_b$ if $v_a \neq v_b$ and v_a is not adjacent to v_b. Then if $A \in \mathbb{G}(G)$ and $\vec{x} \in \mathbb{R}$

$$(A\vec{x})_u = a_{uu}x_u + \sum_{v_u \sim v_m} a_{um}x_m + \sum_{v_u \nsim v_m} a_{um}x_m = a_{uu}x_u + \sum_{v_u \sim v_m} a_{um}x_m. \tag{4.1}$$

The sum $\sum_{v_u \nsim v_m} a_{um}x_m = 0$ since $a_{uw} = 0$ if $\{v_u, v_m\} \notin E(G)$. Since $\vec{x} \in Null(A)$, by Equation 4.1 we have

$$a_{uu}x_u + \sum_{v_u \sim v_m} a_{um}x_m = 0.$$

However $x_u = 0$ since $u \in F$, so $a_{uu}x_u = 0$. Since v_w is a vertex forced by v_u, v_w is the only vertex adjacent to v_u which is not black by the colour-change rule. Hence,

$$a_{uu}x_u + \sum_{v_u \sim v_m} a_{um}x_m = 0 + a_{uw}x_w = 0. \text{ Thus } a_{uw}x_w = 0.$$

However $a_{uw} \neq 0$ because $\{v_u, v_w\} \in E(G)$, thus $x_w = 0$. In a similar way every other colour change corresponds to requiring another entry in $\vec{x}$ to be zero. Thus $\vec{x} = \vec{0}$. □

The next theorem links the zero forcing number with the maximum nullity.

THEOREM 1.54. [**1**] *Let G be a graph and let $F \subseteq V(G)$ be a zero forcing set of G. Then $M(G) \leq |F|$, and thus $M(G) \leq Z(G)$.*

PROOF. Suppose $M(G) > |F|$. So let $A \in \mathbb{G}(G)$ with $nullity(A) > |F|$. By Lemma 1.52, there exists a non-zero vector $\vec{x} \in Null(A)$ that vanishes on all vertices in F. But by Theorem 1.53, any vector that is zero on the vertices of F is zero everywhere. So our nonzero vector $\vec{x}$ is also the zero vector. This is a contradiction. Thus $M(G) \leq Z(G)$. □

Because of Lemma 1.50, the relationship $M(G) = n - mr(G) \leq Z(G)$ implies $n - Z(G) \leq mr(G)$. In other words the zero forcing number places a lower bound to the minimum rank of a graph G. This fact gives motivation to study $Z(G)$. The relationship between a set of matrices and their associated graph then takes a problem in linear algebra and gives it a pictorial representation as a graph.

4.2. Zero Forcing Number of General Graphs. The authors of [**1**] determined various properties of the zero forcing number for a few families of graphs. Some of these are useful for the study of the zero forcing number for circulant graphs. We state these results below.

LEMMA 1.55. *If $G = G_1 \cup \ldots \cup G_t$ is a disconnected graph with connected components $G_1, \ldots, G_t$, then $Z(G) = Z(G_1) + \ldots + Z(G_t)$ and $P(G) = max\{P(G_i), \ldots, P(G_t)\}$.*

THEOREM 1.56. [**1**] *For any graphs G, H, $Z(G \Box H) \leq min\{Z(G)|H|, |G|Z(H)\}$.*

COROLLARY 1.57. [**1**] *If $s \geq t$, then $Z(K_s \Box K_t) \leq st - s - t + 2$.*

THEOREM 1.58. [**1**] *For each of the following families of graphs, $Z(G) = M(G)$:*

(1) *Any graph G such that $|G| \leq 6$.*
(2) *K_n, C_n, P_n.*
(3) *Any tree T.*
(4) *All the graphs listed in Table 1 in* [**1**].

A few examples listed in Table 1 in [**1**] are:

- the hyercube, Q_n
- A graph G with a Hamiltonian path
- the supertriangle, T_n
- the Petersen graph
- the Mobius Ladder

Although the results are not all applicable to the family of circulant graphs, Theorem 1.58 (1) and (2) will be useful.

For a graph G and a subgraph $H \subseteq G$ it is not necessarily the case that $Z(G) \geq Z(H)$. For example, if $G = C_{12}(1,3,5,6)$ and $H = C_{12}(1,3,5)$, then $Z(G) = 9$ and $Z(H) = 10$, as shown in Table 2. Thus $Z(G) = 9 < 10 = Z(H)$.

4.3. Zero Forcing Number of Circulant Graphs. Since the introduction of the zero forcing number in [**1**], a number of authors have been interested in developing the properties of this invariant. Some of the references include [**2**], [**13**], [**12**], and [**3**]. We will now present some of their findings.

THEOREM 1.59. [**2**] *If G is a disconnected circulant graph, then there is a set of isomorphic subgraphs $G_1, \ldots, G_r$ such that $V(G) = V(G_1) \cup \ldots \cup V(G_r)$ and for $i = 1, \ldots, r$ then $Z(G) = rZ(G_i)$, and $P(G) = P(G_i)$.*

PROOF. Consider a disconnected circulant graph G on n vertices and set $S = \{s_1, s_2, \ldots, s_t\}$ such that $s_1 < s_2 < \ldots < s_t$, with connected components $\{G_1, \ldots, G_r\}$. If G is disconnected, then by Theorem 1.28, $gcd(n, s_1, s_2, \ldots, s_t) \neq 1$. The value of the greatest common denominator of the number of vertices and elements of S is equal to the number of connected components. This follows from how circulant graphs are constructed by the set S. If $gcd(n, s_1, s_2, \ldots, s_t) = r$, then r is the minimal generator, so $r = s_1$. For a vertex $v_i \in V(G)$, the minimal generator s_1 creates an edge $\{v_i, v_{i+s_1}\} \in E(G)$ and since s_1 is a divisor of n then $\frac{n}{s_1}$ is the length of the cycle generated by s_1 containing the vertices $\{v_i, v_{i+s_1}, v_{i+2s_1}, \ldots, v_{i+\frac{n}{s_1}-1s_1}\}$. Similarly for vertices $\{v_{i+1}, \ldots, v_{i+s_1-1}\}$ completely disjoint cycles of length $\frac{n}{s_1}$ are generated. If $gcd(n, s_1, s_2, \ldots, s_t) = r$, then all generators s_j for $j > 1$ will be a multiple of s_1, so the edges constructed by these generators will be contained within a single cycle generated by s_1.

By Theorem 1.28, if $gcd(n, s_1, s_2, \ldots, s_t) = 1$, then the graph is connected. If there are r connected components, then each will have $|G_i| = \frac{n}{r}$ and will have the set $S' = \{\frac{s_1}{r}, \frac{s_2}{r}, \ldots, \frac{s_t}{r}\}$. The components of G are thus on an equal number of vertices with a similar set of edge generators, thus the components G_i for $i = 1, \ldots, r$ are isomorphic.

EXAMPLE 1.60. Consider the graph $G = C_{16}(4, 8)$, since $gcd(16, 4, 8) = 4$, then for a vertex $v_i \in V(G)$ for $i = 0, 1, 2, 3$, $\{\{v_i, v_{i+4}\}, \{v_{i+4}, v_{i+8}\}, \{v_{i+8}, v_{i+12}\}, \{v_{i+12}, v_i\}\} \in E(G)$. Further, $\{v_i, v_{i+8}\} \in E(G)$. Similarly, $\{v_{i+4}, v_{i+12}\} \in E(G)$. The resulting edge set is a collection of four disjoint cubic graphs on four vertices. Thus $C_{16}(4, 8) = 4C_4(1, 2)$.

The value of $Z(G)$ is obtained by treating the components $G_1, \ldots, G_r$ individually when performing the zero forcing. The zero forcing process cannot force between disconnected subgraphs since there are no edges $\{v_i, v_j\} \in E(G)$ such that $v_i \in V(G_p)$ and $v_j \in V(G_q)$ for $1 \leq p < q \leq r$ by definition of component. Since each $G_i \subset G$ are isomorphic, then the zero forcing number of G_i is obtained from the colour-change rule and multiplied by the number of isomorphic copies of G_i. Thus $Z(G) = rZ(G_i)$. Since the zero forcing number is taken from a single subgraph, the propagation time of G is the same as the propagation time for a subgraph. Thus $P(G) = P(G_i)$. □

EXAMPLE 1.61. If $G = C_6(2)$, then G is isomorphic to two disconnected $C_3(1)$. See Figure 9. By Theorem 1.59 and Lemma 1.38, $Z(G) = 4$ and $P(G) = 1$.

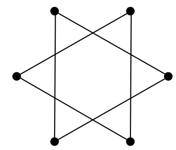

FIGURE 9. The circulant graph $C_6(2) \simeq 2C_3(1)$

EXAMPLE 1.62. If $G = C_{12}(2,6)$, then G is disconnected since $gcd(12,2,6) = 2$. The graph G is isomorphic to two disconnected $C_6(1,3)$ graphs. See Figure 10. By Theorem 1.59 and Example 1.43, $Z(G) = 8$ and $P(G) = 1$.

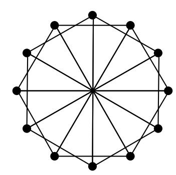

FIGURE 10. The circulant graph $C_{12}(2,6) \simeq 2C_6(1,3)$

An invariant of a graph which has not yet been discussed is the *girth* of a graph G.

DEFINITION 1.63. The *girth* of a graph G is the length of the shortest cycle in G.

In the paper [**3**], written by R. Davila and F. Kenter, they discuss and determine bounds for the zero forcing number of graphs with a large girth. Some of these results are presented below.

THEOREM 1.64. [**3**] *Let G be a graph with girth g, then $Z(G) \leq n - g + 2$.*

THEOREM 1.65. [**3**] *Let G be a graph with girth $g \geq 5$, and minimum degree $\delta \geq 2$, then $2\delta - 2 \leq Z(G)$.*

CHAPTER 2

Families of Circulant Graphs

In this chapter we determine the zero forcing number and propagation times for various families of circulant graphs. In the tables at the end of the project, the zero forcing numbers are shown for all circulant graphs, up to isomorphism, for $|G| \leq 16$. In this chapter, we determine $Z(G)$ for six families of circulant graphs. Examples of the families can be found in the tables.

1. The Circulant Graph $C_n(1, \ldots, d)$ with $1 \leq d \leq \lfloor \frac{n}{2} \rfloor$

The graphs in Figures 1,2,3, and 4 show the circulant graph $C_8(1, \ldots, d)$ for $d = 1, 2, 3, 4$.

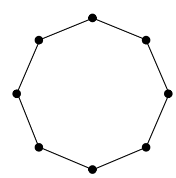

FIGURE 1. The circulant graph $C_8(1) \simeq C_8$

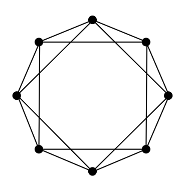

FIGURE 2. The circulant graph $C_8(1, 2)$

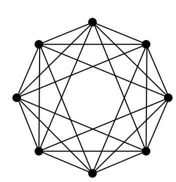

FIGURE 3. The circulant graph $C_8(1, 2, 3)$

For the family $C_n(1, \ldots, d)$, we have the following result.

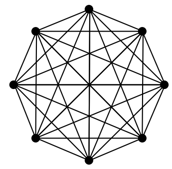

FIGURE 4. The circulant graph $C_8(1,2,3,4) \simeq K_8$

THEOREM 2.1. *Let $G = C_n(1,2,\dots,d)$.*

(1) *If $d < \lfloor \frac{n}{2} \rfloor$, then $Z(G) = 2d$ and $P(G) \le \lceil \frac{n-2d}{2} \rceil$.*
(2) *If $d = \lfloor \frac{n}{2} \rfloor$, then $Z(G) = n-1$ and $P(G) = 1$.*

PROOF. Let $G = C_n(1,2,\dots,d)$ for some $d < \lfloor \frac{n}{2} \rfloor$.

If $d = 1$, then $G = C_n(1) \simeq C_n$. By Lemma 1.41, $Z(C_n) = 2$ and $P(C_n) \le \lceil \frac{n-2}{2} \rceil$.

Suppose $2 \le d \le \lfloor \frac{n}{2} \rfloor - 1$. In this case, each vertex will have degree $2d$. By Lemma 1.35, $Z(G) \ge 2d$. To show that $Z(G) = 2d$, we must find a zero forcing set of size $2d$. We pick a vertex v_A and choose $2d-1$ of its adjacent vertices. The set of $2d$ vertices adjacent to vertex v_0 are $\{v_{n-d}, v_{n-(d-1)}, \dots, v_{n-1}, v_1, \dots, v_{d-1}, v_d\}$. Let $F = \{v_{n-(d-1)}, \dots, v_{n-1}, v_0, v_1, \dots, v_{d-1}, v_d\}$ and starting with all vertices in F black. We claim that F is a zero forcing set of G. Vertex v_{n-d} is the only vertex in the neighbour set of v_0 that is not in F. So v_{n-d} will be turned black in the first iteration of the colour-change rule. The vertex v_1 will also be adjacent to $2d-1$ vertices in F. Thus it will be able to force a vertex not yet in the set, specifically vertex v_{d+1}. This pattern continues in each iteration of the colour-change rule; in a given iteration t, $1 \le t \le P(G)$, vertices $v_{n-(t-1)}$ and v_t will force vertices $v_{n-(t+d-1)}$ and v_{t+d}, respectively. The iteration of the colour-change rule continues until all vertices are coloured black. Thus F is a zero forcing set of order $2d$ and therefore $Z(G) = 2d$. Since there are $n-2d$ remaining vertices and each iteration forces an additional two vertices, except possible the last one, then $P(C_n(1,\dots,d)) \le \lceil \frac{n-2d}{2} \rceil$.

If $d = \lfloor \frac{n}{2} \rfloor$, then $G = C_n(1,2,\dots,d)$ would be isomorphic to the complete graph K_n as noted in Example 1.40. By Lemma 1.42, $Z(K_n) = n-1$ and $P(G) = 1$. □

COROLLARY 2.2. *Suppose $G = C_n(s,2s,3s,\dots,ts)$ for $1 < ts < \frac{n}{2}$. If $gcd(n,s) \ne 1$, then $Z(G) = 2st$ and $P(G) \le \lceil \frac{\frac{n}{s}-2t}{2} \rceil$.*

PROOF. By Theorem 1.28, since $gcd(n,s,\dots,ts) \ne 1$, the graph G is disconnected. Further, by Theorem 1.59, there exist isomorphic subgraphs $G_1,\dots,G_s$ such that $V(G) = V(G_1) \cup \ldots \cup V(G_s)$ and $Z(G) = sZ(G_i)$, with $G_i = C_{\frac{n}{s}}(1,2,3,\dots,t)$. By Theorem 2.1 $Z(G_i) = 2t$ and $P(G_i) \le \lceil \frac{\frac{n}{s}-2t}{2} \rceil$ for $t < \lfloor \frac{n}{2s} \rfloor$. Thus $Z(G) = 2st$ and $P(G) \le \lceil \frac{\frac{n}{s}-2t}{2} \rceil$. □

COROLLARY 2.3. *If $G = C_n(s,2s,3s,\dots,ts)$ and $t = \lfloor \frac{n}{2s} \rfloor$, then $Z(G) = s(\frac{n}{s}-1)$ and $P(G) = 1$.*

PROOF. If $t = \lfloor \frac{n}{2s} \rfloor$, then by Lemma 1.28, G is isomorphic to s disjoint $K_{\frac{n}{s}}$ graphs. Then by Lemma 1.42, $Z(K_{\frac{n}{s}}) = \frac{n}{s} - 1$ and $P(G) = 1$. Thus $Z(G) = s(\frac{n}{s} - 1)$ and $P(G) = 1$. ☐

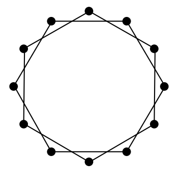

FIGURE 5. The circulant graph $C_{12}(2) \simeq 2C_6$

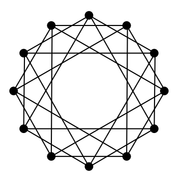

FIGURE 6. The circulant graph $C_{12}(2,4) \simeq 2C_6(1,2)$

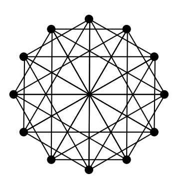

FIGURE 7. The circulant graph $C_{12}(2,4,6) \simeq 2C_6(1,2,3)$

COROLLARY 2.4. *If $gcd(s,n) = 1$ and $G = C_n(s, 2s, \ldots, ts)$, then $G \simeq C_n(1,2,3,\ldots,t)$, and hence $Z(G) = 2t$ and $P(G) \leq \lceil \frac{n-2t}{2} \rceil$.*

PROOF. We claim that the graphs $C_n(s, 2s, \ldots, ts)$ and $C_n(1,2,3,\ldots,d)$ are isomorphic graphs From Lemma 1.31, the graph isomorphism $f(x) : \mathbb{Z}_n \to \mathbb{Z}_n$, defined by $f(x) = sx$, where $gcd(n,s) = 1$, maps $C_n(1,2,3,\ldots,d)$ to $C_n(s,2s,\ldots,ts)$. Thus from Theorem 2.1, $Z(G) = 2t$ and $P(G) \leq \lceil \frac{n-2t}{2} \rceil$. ☐

COROLLARY 2.5. *If $G = C_{2n}(t)$ and $gcd(2n,t) = 1$, then $G \simeq C_{2n}$, and $Z(G) = 2$ and $P(G) \leq \lceil \frac{2n-2}{2} \rceil$.*

PROOF. If $gcd(2n,t) = 1$, the graph G is connected by Theorem 1.28. Since G is 2-regular and connected, G is a cycle. Thus $G \simeq C_{2n}$. Then by Lemma 1.41, $Z(G) = 2$ and $P(G) \leq \lceil \frac{2n-2}{2} \rceil$. ☐

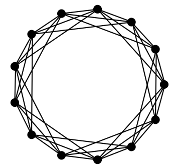

FIGURE 8. The circulant graph $C_{13}(1,2,3)$

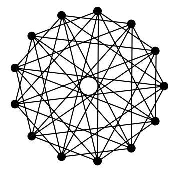

FIGURE 9. The circulant graph $C_{13}(2,4,6)$

If $gcd(2n,t) \neq 1$, then G is disconnected. If this is the case, then by Theorem 1.28, $G \simeq sC_{\frac{2n}{t}}(1)$. From here we apply Lemmas 1.41 and 1.59, resulting with $Z(G) = 2s$ and $P(G) \leq \lceil \frac{\frac{2n}{t}-2}{2} \rceil$.

EXAMPLE 2.6. Pictured in Figure 10 is the circulant bipartite graph $C_{10}(3)$. Since $gcd(10,3) = 1$, G is connected and it is equal to the graph C_{10}.

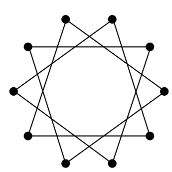

FIGURE 10. The circulant graph $C_{10}(3) \simeq C_{10}$

2. The Circulant Graph $C_{2n}(1,3,5,\ldots)$

By Theorem 1.32, the family of circulant graphs with an even number of vertices and all consecutive odd elements of S is equivalent to the family of complete bipartite graphs. See Figure 11 for an example of the complete bipartite circulant graph $K_{4,4}$, and Figure 12 for an image of the complete bipartite graph $K_{7,7}$.

LEMMA 2.7. *If $G = K_{a,b}$ for $1 < a \leq b$, then $Z(G) = a+b-2$ and $P(G) = 1$.*

PROOF. If $G = K_{a,b}$, consider a labeling of the vertices $\{v_0,\ldots,v_a,v_{a+1},\ldots,v_{a+b-1}\}$ such that v_i is adjacent to v_j if $0 \leq i \leq a-1$ and $a \leq j \leq a+b-1$. Since G is a complete bipartite graph each vertex of $\{v_0,\ldots,v_{a-1}\}$ is adjacent to every vertex of $\{v_a,\ldots,v_{a+b-1}\}$, and vice versa. Thus $\deg v_i = b$ for $0 \leq i \leq a-1$ and $\deg v_j = a$ for

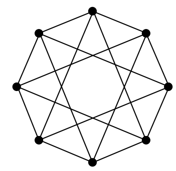

FIGURE 11. The circulant graph $C_8(1,3) \simeq K_{4,4}$

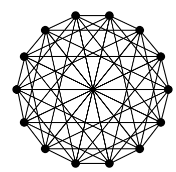

FIGURE 12. The circulant graph $C_{14}(1,3,5,7) \simeq K_{7,7}$

$a \le j \le a+b-1$. By the colour-change rule, for a vertex v_i to force a vertex v_j, all other vertices adjacent to v_i must be black. If there are more than two white vertices in either partition, then no vertex adjacent to them could force the two white vertices. Thus there can only be one white vertex in each partition of the vertices. So $Z(G) = a+b-2$ and $P(G) = 1$. □

COROLLARY 2.8. *If $G = K_{a,b}$ for $1 = a \le b$, then $Z(G) = a+b-2$ and $P(G) = 2$.*

PROOF. If $G = K_{a,b}$ such that $a = 1$, then we obtain a graph $G = K_{1,b}$, also known as a *star graph*. Let v_0 be the vertex of degree b, and label the remaining vertices $v_1, \ldots, v_b$. The same zero forcing set and reasoning is taken as in Lemma 2.7, thus $Z(G) = a+b-2$. Let vertex v_j be the vertex which is not in the initial zero forcing set for $1 \le j \le b$. The propagation time differs in this case since the vertex v_0 must be forced before the final unforced vertex v_j can be forced. Thus $P(G) = 2$. □

THEOREM 2.9. *If $G = C_{2n}(1,3,5,\ldots,t)$ with, $t = n$ if n is odd, and $t = n-1$ if n is even. Then $G \simeq K_{n,n}$ and $Z(G) = 2n-2$ and $P(G) = 1$.*

PROOF. In order to prove the lemma, it must be shown first that $G = C_{2n}(1,3,5,\ldots,t)$ is isomorphic to $K_{n,n}$. By Theorem 1.32, G is bipartite. In fact, the two parts correspond to vertices with even subscripts and those with odd subscripts.

If $t = n-1$, $|S| = \frac{n}{2}$, then G is regular of degree $2\frac{n}{2} = n$. If $t = n$, $|S| = \frac{n+1}{2}$. By Lemma 1.27, G is regular of degree $(2\frac{n+1}{2} - 1) = n$. Thus G is n regular, and so $G = K_{n,n}$.

By Lemma 2.7, $Z(G) = 2n-2$ and $P(G) = 1$. □

3. Cubic Circulant Graphs

We will now look at the family of cubic circulant graphs and determine the zero forcing number for this entire family. If G is a cubic circulant graph, then by Lemma 1.27, $G = C_{2n}(a, n)$ for some a, $1 \leq a < n$. Using the following theorem we are able to break the family of cubic circulants into two cases. From here, since all cubic circulants are isomorphic to one of the two cases, it will be much easier to determine the zero forcing number for this family of graphs.

Theorem 2.10. [**4**] *Let $G = C_{2n}(a, n)$ with $1 \leq a < n$, and let $t = gcd(a, 2n)$.*

(1) *If $\frac{2n}{t}$ is even, then $G \simeq tC_{\frac{2n}{t}}(1, \frac{n}{t})$.*
(2) *If $\frac{2n}{t}$ is odd, then $G \simeq \frac{t}{2}C_{\frac{4n}{t}}(2, \frac{2n}{t})$.*

Theorem 2.10 demonstrates that we can partition the family of cubic circulant graphs into two classes, either $\frac{2n}{t}$ is even or odd. From Lemma 1.59, we then know that if we can calculate the zero forcing number of the connected components isomorphic to $C_{\frac{2n}{t}}(1, \frac{n}{t})$ or $C_{\frac{4n}{t}}(2, \frac{2n}{t})$, then we can determine the zero forcing number of any cubic circulant graph. We characterize and determine the zero forcing number for these classes in Theorems 2.11, and 2.12 respectively.

Theorem 2.11. [**2**] *If $G = C_{2n}(1, n)$ and for $n \geq 3$, then $Z(G) = 4$ and $P(G) \leq \lceil \frac{n-2}{2} \rceil$.*

Proof. By Lemma 1.35, since G is 3-regular, then $Z(G) \geq 3$. To prove that $Z(G) \neq 3$, we must prove that no zero forcing set F exists such that $|F| = 3$, but one does exist such that $|F| = 4$.

Suppose F is a zero forcing set of G and $|F| = 3$. This means that some vertex $v_i \in V(G)$ in F will force one of its neighbours in the first iteration of the colour-change rule. Hence F must consist of v_i and all but one of its neighbours. But then the derived colouring will be v_i and its three neighbours since each of the neighbours of v_i will be adjacent to two vertices that are not black. Thus F is not a zero forcing set and so $|Z(G)| \geq 4$.

We now show that there exists a forcing set F such that $|F| = 4$. We claim that $F = \{v_0, v_1, v_n, v_{n+1}\}$ is a zero forcing set. Vertex v_0 can force vertex v_{2n-1}, and vertex v_1 can force vertex v_2. Vertices v_n and v_{n+1} can also force vertices v_{n-1} and v_{n+2} respectively. Repeating, vertices $\{v_{1+t}, v_{2n-t}, v_{n+1+t}, v_{n-t}\}$ will be forced in iteration t by vertices $\{v_t, v_{2n-(t-1)}, v_{n+t}, v_{n-(t-1)}\}$ respectively until all of the vertices are forced. Thus F is a zero forcing set, and $P(C_{2n}(1, n)) \leq \lceil \frac{n-2}{2} \rceil$. □

If $n = 2$, then $G = C_{2n}(1, n) = C_4(1, 2)$ is the complete graph K_4, which has a zero forcing number of three by Lemma 1.35.

Theorem 2.12. (1) *If n is even, then $C_{2n}(2, n) = 2C_n(1, \frac{n}{2})$.*
(2) *If n is odd, then $C_{2n}(2, n) = C_n \square P_2$.*

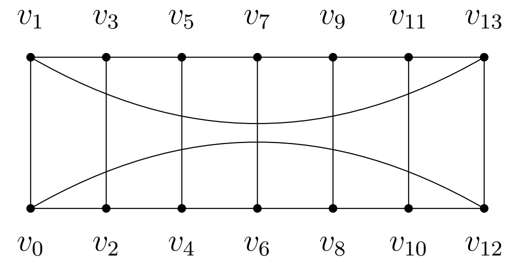

FIGURE 13. The Cartesian product of P_2 and C_7 results in the graph shown, $P_2 \Box C_7$.

PROOF. Case (1) of Theorem 2.12 follows from Theorem 2.11. It thus remains to prove case (2).

Let $G = C_{2n}(2, n)$ with n odd. Note that $C_{2n}(2)$ is a subgraph equal to two disjoint cycles of length n by Lemma 1.28. The vertices in the cycle containing vetex v_0 are $\{v_0, v_2, v_4, \ldots, v_{2n-4}, v_{2n-2}\}$. We observe that the cycle containing vertex v_0 contains only vertices with even subscripts. The other cycle will contain only vertices with odd valued subscripts. For any vertex $v_i \in V(G)$, its adjacent vertices will be $\{v_{i+n}, v_{i+2}, v_{i-2}\}$. Then the two cycles are connected such that the ordering of the cycles are both maintained. For example, in the cycle vertex v_i is adjacent to v_{i+2}, similarly vertex v_{i+n} is adjacent to vertex v_{i+n+2} which is diametrically opposite to vertex v_{i+2}. Thus $C_{2n}(2, n) = C_n \Box P_2$. □

Lemma 2.12, gives the result that $G = C_{2n}(2, n) = C_n \Box P_2$ for n odd. By Theorem 1.56, $Z(C_n \Box P_2) \leq min\{2Z(C_n), nZ(P_2)\} = min\{4, n\} = 4$ for $n \geq 5$. Thus for odd $n \geq 5$, $Z(C_{2n}(2, n)) = 4$ and if $n = 3$, then $Z(G) = 3$. See Figure 3 for an example of $P_2 \Box C_7$.This completes the computing the zero forcing number for the family of cubic circulant graphs for both classes of cubic whether $\frac{2n}{t}$ is even or odd. We summarize the final results in the following theorem combining Theorem 2.12 with Theorems 1.28 and 1.59.

THEOREM 2.13. *Let $G = C_{2n}(a, n)$ with $1 \leq a < n$, and let $t = gcd(a, 2n)$.*

(1) *If $\frac{2n}{t}$ is even, then $Z(G) = 4t$ for $\frac{n}{t} \geq 3$*
(2) *If $\frac{2n}{t}$ is odd, then $Z(G) = \frac{3t}{2}$ for $n = \frac{3t}{2}$ and $Z(G) = 2t$ for $n \geq 5$*

4. Future Research

As it was shown in the previous section, the zero forcing number for the cubic graphs can be determined for any graph in the family. The next family of graphs which were studied is the family of graphs on an vertices with the set $S = \{1, n, 2n, \ldots, bn\}$ for $b \leq \frac{a}{2}$. We make a conjecture about the zero forcing number of a graph $G = C_{2n}(1, n, 2n, \ldots, bn)$.

CONJECTURE 2.14. *If $G = C_{an}(1, n, 2n, \ldots, bn)$ with $b = \lfloor \frac{a}{2} \rfloor$, then $Z(G) = 2a$ and $P(G) = \lceil \frac{n-2}{2} \rceil$ for $n \geq 4$.*

See Figures 14 and 15 for examples. And see Tables 3 - 7 for the calculations done using SAGE.

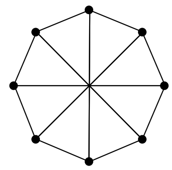

FIGURE 14. The circulant graph $C_8(1,4)$ where $a = 2$ and $n = 4$

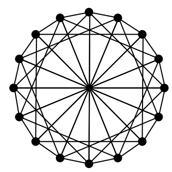

FIGURE 15. The circulant graph $C_{16}(1,4,8)$ where $a = 4$ and $n = 4$

Another family of circulant graphs $G = C_n(a, a+1, \ldots, b-1, b)$ for $1 < a \leq b \leq \frac{n}{2}$ was a point of focus. Certain families of circulants showed stabilizing patterns for a large number of vertices. For $n \geq 4$ we arrived at the following conjecture.

CONJECTURE 2.15. *If* $G = C_n(a, a+1, \ldots, b-1, b)$ *for* $1 < a \leq b \leq \frac{n}{2}$*, then* $Z(G) = n - 4$ *for* $n \geq 4$.

Theorem 2.9 gives the zero forcing number for $K_{n,n}$ and Theorem 2.5 gives the zero forcing number for a particular subgraph of $K_{n,n}$ which is 2-regular. Lemma 2.12 then gives the zero forcing number for the cubic bipartite graphs when a and n are both odd. It would be interesting to characterize the zero forcing number of all bipartite circulant graphs. This is a class that was explored in [**13**]. Using SAGE, the zero forcing number for the members of this family were calculated and are presented in Table 9. We arrived at the following conjecture.

CONJECTURE 2.16. *If* $G = C_{2n}(1, 3, 5, \ldots, t)$ *for* t *odd, then* $Z(G) = 2k - 2$ *where* $k = deg(v_i)$ *for* $v_i \in V(G)$.

In Theorem 2.1 we explored the family $G = C_n(1, 2, \ldots, d)$ for $d < \frac{n}{2}$. The set S for these graphs have a complement $S' = (d+1, \ldots, \lfloor \frac{n}{2} \rfloor)$ for $d \geq 1$. The zero forcing number of the family of circulant graphs $G = C_n(S')$ was tested using SAGE, the results shown in Table 8. We arrived at the following conjecture.

CONJECTURE 2.17. *If* $G = C_n(d+1, \ldots, \lfloor \frac{n}{2} \rfloor)$*, then* $Z(G) = n - (d+2)$ *for* $d \leq 5$ *and* $|S| > 2$.

Appendices

1. Zero Forcing Code

The following code was used to obtain the values of the zero forcing sets and the propagation times for the different circulant graphs.

```
def zero_forcing_power_prop(G):
    prop_number = G.order()
    for S in Combinations(G.vertices()):
        if len(S)>prop_number:
            break
        if len(S)<=prop_number:
            blue=set([])
            for s in S:
                blue.add(s)
            q = test_zero_forcing(G,blue)
            if q >= 0:
                prop_number = len(S)
                print len(S), blue, q

def test_zero_forcing(G,initial):
    blue = copy(initial)
    big_blue = copy(initial)
    if len(big_blue) == G.order():
        return 0
    prop_time = 0
    while True:
        propagated = False
        for b in blue:
            if len(set(G.neighbors(b)).difference(blue)) == 1:
                big_blue.add(set(G.neighbors(b)).difference(blue).pop())
                propagated = True
        if not propagated:
            return -1
        if propagated:
            prop_time+=1
```

```
        if len(big_blue) == G.order():
            return prop_time
        blue=copy(big_blue)
```

The code was obtained from: https://sage2.math.iastate.edu/home/pub/13/ [**15**].

2. Tables

TABLE 1. The circulant graphs on one to ten vertices.

G	K-Regularity	$Z(G)$	$P(G)$	$\simeq$
$C_2(1)$	1	1	1	P_2
$C_3(1)$	2	2	1	C_3
$C_4(1)$	2	2	1	C_4
$C_4(2)$	1	2	1	$2P_2$
$C_4(1,2)$	3	3	1	K_4
$C_5(1)$	2	2	2	C_5
$C_5(2)$	2	2	2	C_5
$C_5(1,2)$	4	4	1	K_5
$C_6(1)$	2	2	2	C_6
$C_6(2)$	2	4	1	$2K_3$
$C_6(3)$	1	3	1	$3P_2$
$C_6(1,2)$	4	4	2	
$C_6(1,3)$	3	4	1	$K_{3,3}$
$C_6(2,3)$	3	3	1	
$C_6(1,2,3)$	5	5	1	K_6
$C_7(1)$	2	2	3	C_7
$C_7(2)$	2	2	3	C_7
$C_7(3)$	2	2	3	C_7
$C_7(1,2)$	4	4	2	
$C_7(1,3)$	4	4	2	
$C_7(2,3)$	4	4	2	
$C_7(1,2,3)$	6	6	1	K_7

G	K-Regularity	$Z(G)$	$P(G)$	$\simeq$
$C_8(1)$	2	2	3	C_7
$C_8(2)$	2	4	1	$2C_4$
$C_8(3)$	2	2	3	C_8
$C_8(4)$	1	4	1	$4P_2$
$C_8(1,2)$	4	4	2	
$C_8(1,3)$	4	6	1	$K_{4,4}$
$C_8(1,4)$	3	4	2	
$C_8(2,3)$	4	4	3	
$C_8(2,4)$	3	6	1	$2K_4$
$C_8(3,4)$	3	4	1	
$C_8(1,2,3)$	6	6	1	
$C_8(1,2,4)$	5	5	2	
$C_8(1,3,4)$	5	6	1	
$C_8(2,3,4)$	5	5	2	
$C_8(1,2,3,4)$	7	7	1	K_8
$C_9(1)$	2	2	4	C_9
$C_9(2)$	2	2	4	C_9
$C_9(3)$	2	6	1	$3K_3$
$C_9(4)$	2	2	4	C_9
$C_9(1,2)$	4	4	3	
$C_9(1,3)$	4	5	2	
$C_9(1,4)$	4	4	3	
$C_9(2,3)$	4	5	4	
$C_9(2,4)$	4	4	3	
$C_9(3,4)$	4	5	2	
$C_9(1,2,3)$	6	6	2	
$C_9(1,2,4)$	6	7	1	
$C_9(1,3,4)$	6	6	3	
$C_9(2,3,4)$	6	6	2	
$C_9(1,2,3,4)$	8	8	1	K_9

G	K-Regularity	$Z(G)$	$P(G)$	$\simeq$
$C_{10}(1)$	2	2	4	C_{10}
$C_{10}(2)$	2	4	2	$2C_5$
$C_{10}(3)$	2	2	4	C_{10}
$C_{10}(4)$	2	4	2	$2C_5$
$C_{10}(5)$	1	5	1	$5P_2$
$C_{10}(1,2)$	4	4	3	
$C_{10}(1,3)$	4	6	1	
$C_{10}(1,4)$	4	6	2	
$C_{10}(1,5)$	3	4	3	
$C_{10}(2,3)$	4	6	2	
$C_{10}(2,4)$	4	8	1	$2K_5$
$C_{10}(2,5)$	3	4	3	
$C_{10}(3,4)$	4	4	3	
$C_{10}(3,5)$	3	4	3	
$C_{10}(4,5)$	3	4	2	
$C_{10}(1,2,3)$	6	6	2	
$C_{10}(1,2,4)$	6	6	3	
$C_{10}(1,2,5)$	5	6	2	
$C_{10}(1,3,4)$	6	6	3	
$C_{10}(1,3,5)$	5	8	1	$K_{5,5}$
$C_{10}(1,4,5)$	5	7	2	
$C_{10}(2,3,4)$	6	6	3	
$C_{10}(2,3,5)$	5	7	2	
$C_{10}(2,4,5)$	5	5	1	
$C_{10}(3,4,5)$	5	6	3	
$C_{10}(1,2,3,4)$	8	8	1	
$C_{10}(1,2,3,5)$	7	7	2	
$C_{10}(1,2,4,5)$	7	7	2	
$C_{10}(1,3,4,5)$	7	7	2	
$C_{10}(2,3,4,5)$	7	7	2	
$C_{10}(1,2,3,4,5)$	9	9	1	K_{10}

TABLE 2. These tables were obtained from [**14**] and modified to test if the zero forcing number correlated with a graph being well-covered. Notation and other calculations for well-covered circulant graphs is explained in [**14**].

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_3(1)$	2	2	1	V	1
$C_4(1)$	2	2	1	B	
$C_4(2)$	1	2	1	V	1
$C_4(1,2)$	3	3	1	V	1
$C_5(1)$	2	2	2	V	1
$C_5(1,2)$	4	4	1	V	1
$C_6(2)$	2	4	1	V	1
$C_6(3)$	1	3	1	V	1
$C_6(1,2)$	4	4	2	B	
$C_6(1,3)$	3	4	1	B	
$C_6(2,3)$	3	3	1	V	1
$C_6(1,2,3)$	5	5	1	V	1
$C_7(1)$	2	2	3	B	
$C_7(1,2,3)$	6	6	1	V	1
$C_8(2)$	2	4	1	N	
$C_8(4)$	1	4	1	V	1
$C_8(1,2)$	4	4	2	V	1
$C_8(1,3)$	4	6	1	B	
$C_8(1,4)$	3	4	2	B	
$C_8(2,4)$	3	6	1	V	1
$C_8(1,2,3)$	6	6	1	B	
$C_8(1,2,4)$	5	5	2	V	1
$C_8(1,3,4)$	5	6	1	B	1
$C_8(1,2,3,4)$	7	7	1	V	1

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_9(3)$	2	6	1	V	1
$C_9(1,3)$	4	5	2	B	
$C_9(1,2,3)$	6	6	2	V	1
$C_9(1,2,4)$	6	7	1	B	
$C_9(1,2,3,4)$	8	8	1	V	1
$C_{10}(2)$	2	4	2	V	1
$C_{10}(5)$	1	5	1	V	1
$C_{10}(1,4)$	4	6	2	N	
$C_{10}(2,4)$	4	8	1	V	1
$C_{10}(2,5)$	3	4	3	B	
$C_{10}(1,2,3)$	6	6	2	V	1
$C_{10}(1,2,4)$	6	6	3	V	1
$C_{10}(1,2,5)$	5	6	2	B	
$C_{10}(1,3,5)$	5	8	1	B	
$C_{10}(1,4,5)$	5	7	2	V	1
$C_{10}(2,4,5)$	5	5	1	V	1
$C_{10}(1,2,3,4)$	8	8	1	B	
$C_{10}(1,2,3,5)$	7	7	2	B	1
$C_{10}(1,2,4,5)$	7	7	2	V	1
$C_{10}(1,2,3,4,5)$	9	9	1	V	1
$C_{11}(1,2)$	4	4	4	B	
$C_{11}(1,3)$	4	6	2	B	
$C_{11}(1,2,3)$	6	6	3	V	1
$C_{11}(1,2,4)$	6	7	2	B	
$C_{11}(1,2,3,4)$	8	8	2	V	1

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_{11}(1,2,3,4,5)$	10	10	1	V	1
$C_{12}(3)*$	2	6	2	N	
$C_{12}(4)*$	2	8	1	V	1
$C_{12}(6)*$	1	6	1	V	1
$C_{12}(1,4)$	4	6	2	N	
$C_{12}(2,4)*$	4	8	2	N	
$C_{12}(2,6)*$	3	8	1	N	
$C_{12}(3,4)$	4	6	3	B	
$C_{12}(3,6)*$	3	9	1	V	1
$C_{12}(4,6)*$	3	6	1	V	1
$C_{12}(1,2,6)$	5	7	2	B	1
$C_{12}(1,3,5)$	6	10	1	B	
$C_{12}(1,3,6)$	5	7	2	V	1
$C_{12}(1,4,6)$	5	7	3	B	
$C_{12}(2,3,4)$	6	8	2	N	
$C_{12}(2,3,6)$	5	7	2	B	1
$C_{12}(2,4,6)*$	5	10	1	V	1
$C_{12}(3,4,6)$	5	7	2	B	1
$C_{12}(1,2,3,4)$	8	8	2	V	1
$C_{12}(1,2,4,5)$	8	10	1	B	
$C_{12}(1,2,4,6)$	7	7	3	V	1
$C_{12}(1,3,4,5)$	8	10	1	B	1
$C_{12}(1,3,4,6)$	7	8	2	B	
$C_{12}(1,3,5,6)$	7	9	1	B	1
$C_{12}(1,4,5,6)$	7	8	2	V	1

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_{12}(2,3,4,6)$	7	9	1	V	1
$C_{12}(1,2,3,4,5)$	10	10	1	B	
$C_{12}(1,2,3,4,6)$	9	9	2	V	1
$C_{12}(1,2,3,5,6)$	9	10	1	B	
$C_{12}(1,2,4,5,6)$	9	10	1	B	1
$C_{12}(1,3,4,5,6)$	9	9	1	B	1
$C_{12}(1,2,3,4,5,6)$	11	11	1	V	1
$C_{13}(1,3)$	4	6	2	B	
$C_{13}(1,5)$	4	6	4	V	1
$C_{13}(1,2,4)$	6	8	2	B	
$C_{13}(1,2,5)$	6	8	3	B	
$C_{13}(1,3,4)$	6	8	3	B	1
$C_{13}(1,2,3,4)$	8	8	3	V	1
$C_{13}(1,2,3,5)$	8	9	2	V	1
$C_{13}(1,2,3,6)$	8	9	2	B	
$C_{13}(1,2,3,4,5)$	10	10	2	V	1
$C_{13}(1,2,3,4,5,6)$	12	12	1	V	1
$C_{14}(2)*$	2	4	3	N	
$C_{14}(7)*$	1	7	1	V	1
$C_{14}(1,6)$	4	8	3	N	
$C_{14}(2,4)*$	4	8	2	V	1
$C_{14}(1,2,5)$	6	8	3	B	
$C_{14}(1,4,6)$	6	8	3	B	
$C_{14}(1,4,7)$	5	8	3	B	
$C_{14}(1,6,7)$	5	9	3	B	1

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_{14}(2,4,6)*$	6	12	1	V	1
$C_{14}(2,4,7)$	5	7	2	B	
$C_{14}(1,2,3,4)$	8	8	3	V	1
$C_{14}(1,2,3,7)$	7	9	2	B	
$C_{14}(1,2,4,6)$	8	8	3	V	1
$C_{14}(1,2,4,7)$	7	9	3	B	
$C_{14}(1,2,5,6)$	8	10	2	N	
$C_{14}(1,2,5,7)$	7	9	3	B	1
$C_{14}(1,3,5,7)$	7	12	1	B	
$C_{14}(1,4,6,7)$	7	9	3	B	1
$C_{14}(2,4,6,7)$	7	7	2	V	1
$C_{14}(1,2,3,4,5)$	10	10	2	V	1
$C_{14}(1,2,3,4,6)$	10	10	2	V	1
$C_{14}(1,2,3,4,7)$	9	10	2	V	1
$C_{14}(1,2,3,5,7)$	9	9	3	B	
$C_{14}(1,2,3,6,7)$	9	10	2	B	
$C_{14}(1,2,4,6,7)$	9	9	3	V	1
$C_{14}(1,2,5,6,7)$	9	11	2	V	1
$C_{14}(1,2,3,4,5,6)$	12	12	1	B	
$C_{14}(1,2,3,4,5,7)$	11	11	2	B	1
$C_{14}(1,2,3,4,6,7)$	11	11	2	V	1
$C_{14}(1,2,3,4,5,6,7)$	13	13	1	V	1
$C_{15}(3)*$	2	6	2	V	1
$C_{15}(5)*$	2	10	1	V	1
$C_{15}(1,5)$	4	6	2	N	

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_{15}(3,5)$	4	6	2	N	
$C_{15}(3,6)*$	4	12	1	V	1
$C_{15}(1,2,3)$	6	6	5	B	
$C_{15}(1,3,5)$	6	9	3	B	
$C_{15}(1,3,6)$	6	9	2	V	1
$C_{15}(1,4,6)$	6	11	2	N	
$C_{15}(3,5,6)$	6	9	2	V	1
$C_{15}(1,2,3,6)$	8	10	3	N	
$C_{15}(1,2,3,7)$	8	10	3	B	1
$C_{15}(1,2,5,6)$	8	10	3	B	
$C_{15}(1,3,4,5)$	8	10	3	B	1
$C_{15}(1,3,4,6)$	8	10	3	B	1
$C_{15}(1,3,5,6)$	8	10	3	B	
$C_{15}(1,4,5,6)$	8	12	2	V	1
$C_{15}(1,2,3,4,5)$	10	10	3	V	1
$C_{15}(1,2,3,5,6)$	10	11	2	B	
$C_{15}(1,2,3,5,7)$	10	11	2	V	1
$C_{15}(1,2,4,5,7)$	10	13	1	B	
$C_{15}(1,3,4,5,6)$	10	11	2	V	1
$C_{15}(1,2,3,4,5,6)$	12	12	2	V	1
$C_{15}(1,2,3,4,5,7)$	12	12	2	B	1
$C_{15}(1,2,3,4,6,7)$	12	13	1	B	
$C_{15}(1,2,3,4,5,6,7)$	14	14	1	V	1
$C_{16}(4)*$	2	8	1	N	
$C_{16}(8)*$	1	8	1	V	1

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_{16}(2,4)*$	4	8	2	V	1
$C_{16}(2,6)*$	4	12	1	N	
$C_{16}(2,8)*$	3	8	2	N	
$C_{16}(4,8)*$	3	12	1	V	1
$C_{16}(1,2,4)$	6	8	2	B	
$C_{16}(1,2,6)$	6	9	3	B	1
$C_{16}(1,4,6)$	6	8	4	V	1
$C_{16}(1,4,7)$	6	10	3	N	
$C_{16}(1,4,8)$	5	8	2	S	1
$C_{16}(1,6,8)$	5	8	2	B	
$C_{16}(2,4,6)*$	6	12	1	N	
$C_{16}(2,4,8)*$	5	10	2	V	1
$C_{16}(2,6,8)*$	5	12	1	N	1
$C_{16}(1,2,3,8)$	7	10	2	B	1
$C_{16}(1,2,4,7)$	8	10	3	B	
$C_{16}(1,2,5,8)$	7	10	3	B	
$C_{16}(1,2,6,7)$	8	12	1	N	
$C_{16}(1,2,6,8)$	7	9	4	N	
$C_{16}(1,2,7,8)$	7	8	3	B	
$C_{16}(1,3,5,7)$	8	14	1	B	
$C_{16}(1,4,6,8)$	7	9	4	B	
$C_{16}(1,4,7,8)$	7	12	1	B	1
$C_{16}(2,4,6,8)*$	7	14	1	V	1
$C_{16}(1,2,3,4,5)$	10	10	3	V	1
$C_{16}(1,2,3,4,6)$	10	11	3	V	1

G	K-Regularity	$Z(G)$	$P(G)$	V/B/N	-/1
$C_{16}(1,2,3,6,8)$	9	11	3	B	1
$C_{16}(1,2,3,7,8)$	9	11	3	B	
$C_{16}(1,2,4,5,8)$	9	11	3	B	
$C_{16}(1,2,4,6,7)$	10	12	2	N	
$C_{16}(1,2,4,6,8)$	9	9	4	V	1
$C_{16}(1,2,4,7,8)$	9	11	4	B	1
$C_{16}(1,2,6,7,8)$	9	12	2	V	1
$C_{16}(1,3,4,5,7)$	10	12	1	N	
$C_{16}(1,3,5,7,8)$	9	12	1	B	1
$C_{16}(1,2,3,4,5,6)$	12	12	2	V	1
$C_{16}(1,2,3,4,5,7)$	12	12	2	B	1
$C_{16}(1,2,3,4,5,8)$	11	12	2	V	1
$C_{16}(1,2,3,4,6,8)$	11	11	3	V	1
$C_{16}(1,2,3,4,7,8)$	11	12	2	B	
$C_{16}(1,2,3,5,6,7)$	12	14	1	B	
$C_{16}(1,2,3,5,6,8)$	11	12	2	V	1
$C_{16}(1,2,3,5,7,8)$	11	12	2	B	
$C_{16}(1,2,4,6,7,8)$	11	13	2	V	1
$C_{16}(1,3,4,5,7,8)$	11	14	1	B	1
$C_{16}(1,2,3,4,5,6,7)$	14	14	1	B	
$C_{16}(1,2,3,4,5,6,8)$	13	13	2	V	1
$C_{16}(1,2,3,4,5,7,8)$	13	13	2	B	1
$C_{16}(1,2,3,5,6,7,8)$	13	14	1	B	1
$C_{16}(1,2,3,4,5,6,7,8)$	15	15	1	V	1

TABLE 3. The Family $C_{2n}(1,n)$ for $n = 1,\ldots,10$, and 20.

G	K-Regularity	$Z(G)$	$P(G)$
$C_2(1)$	1	1	1
$C_4(1,2)$	3	3	1
$C_6(1,3)$	3	4	1
$C_8(1,4)$	3	4	2
$C_{10}(1,5)$	3	4	3
$C_{12}(1,6)$	3	4	3
$C_{14}(1,7)$	3	4	3
$C_{16}(1,8)$	3	4	3
$C_{18}(1,9)$	3	4	4
$C_{20}(1,10)$	3	4	4
$C_{40}(1,20)$	3	4	9

TABLE 4. The Family $C_{3n}(1, n)$ for $n = 1, \ldots, 10$, and 20.

G	K-Regularity	$Z(G)$	$P(G)$
$C_3(1)$	2	2	1
$C_6(1,2)$	4	4	1
$C_9(1,3)$	4	5	2
$C_{12}(1,4)$	4	6	1
$C_{15}(1,5)$	4	6	2
$C_{18}(1,6)$	4	6	2
$C_{21}(1,7)$	4	6	3
$C_{24}(1,8)$	4	6	3
$C_{27}(1,9)$	4	6	4
$C_{30}(1,10)$	4	6	4
$C_{60}(1,20)$	4	6	8

TABLE 5. The Family $C_{4n}(1, n, 2n)$ for $n = 1, \ldots, 10$, and 20.

G	K-Regularity	$Z(G)$	$P(G)$
$C_4(1,2)$	3	3	1
$C_8(1,2,4)$	5	5	2
$C_{12}(1,3,6)$	5	7	2
$C_{16}(1,4,8)$	5	8	1
$C_{20}(1,5,10)$	5	8	2
$C_{24}(1,6,12)$	5	8	2
$C_{28}(1,7,14)$	5	8	3
$C_{32}(1,8,16)$	5	8	3
$C_{36}(1,9,18)$	5	8	4
$C_{40}(1,10,20)$	5	8	4
$C_{80}(1,20,40)$	5	8	8

TABLE 6. The Family $C_{5n}(1, n, 2n)$ for $n = 1, \ldots, 10$, and 20.

G	K-Regularity	$Z(G)$	$P(G)$
$C_5(1,2)$	4	4	1
$C_{10}(1,2,4)$	6	6	2
$C_{15}(1,3,6)$	6	9	2
$C_{20}(1,4,8)$	6	10	1
$C_{25}(1,5,10)$	6	10	2
$C_{30}(1,6,12)$	6	10	2
$C_{35}(1,7,14)$	6	10	3
$C_{40}(1,8,16)$	6	10	3
$C_{45}(1,9,18)$	6	10	4
$C_{50}(1,10,20)$	6	10	4
$C_{100}(1,20,40)$	6	10	8

TABLE 7. The Family $C_{6n}(1, n, 2n, 3n)$ for $n = 1, \ldots, 10$, and 20.

G	K-Regularity	$Z(G)$	$P(G)$
$C_6(1, 2, 3)$	5	5	1
$C_{12}(1, 2, 4, 6)$	7	7	3
$C_{18}(1, 3, 6, 9)$	7	11	2
$C_{24}(1, 4, 8, 12)$	7	12	1
$C_{30}(1, 5, 10, 15)$	6	12	2
$C_{36}(1, 6, 12, 18)$	6	12	2
$C_{42}(1, 7, 14, 21)$	6	12	3
$C_{48}(1, 8, 16, 24)$	6	12	3
$C_{54}(1, 9, 18, 27)$	6	12	4
$C_{60}(1, 10, 20, 30)$	6	12	4
$C_{120}(1, 20, 40, 60)$	6	12	8

TABLE 8. The Family of circulant graphs of the form $C_n(d, ..., n/2)$ for $d > 1$.

G	K-Regularity	$Z(G)$	$P(G)$
$C_6(2,3)$	3	3	1
$C_7(2,3)$	4	4	2
$C_8(3,4)$	3	4	2
$C_8(2,3,4)$	5	5	2
$C_9(3,4)$	4	5	2
$C_9(2,3,4)$	6	6	2
$C_{10}(4,5)$	3	4	2
$C_{10}(3,4,5)$	5	6	2
$C_{10}(2,3,4,5)$	7	7	2
$C_{11}(4,5)$	4	6	2
$C_{11}(3,4,5)$	6	7	2
$C_{11}(2,3,4,5)$	8	8	2
$C_{12}(5,6)$	3	4	2
$C_{12}(4,5,6)$	5	7	3
$C_{12}(3,4,5,6)$	7	8	2
$C_{12}(2,3,4,5,6)$	9	9	2
$C_{13}(5,6)$	4	6	2
$C_{13}(4,5,6)$	6	8	3
$C_{13}(3,4,5,6)$	8	9	2
$C_{13}(2,3,4,5,6)$	10	10	2
$C_{14}(6,7)$	3	4	3
$C_{14}(5,6,7)$	5	8	3
$C_{14}(4,5,6,7)$	7	9	3
$C_{14}(3,4,5,6,7)$	9	10	2
$C_{14}(2,3,4,5,6,7)$	11	11	2
$C_{15}(6,7)$	4	6	3
$C_{15}(5,6,7)$	6	9	3
$C_{15}(4,5,6,7)$	8	10	3
$C_{15}(3,4,5,6,7)$	10	11	2
$C_{15}(2,3,4,5,6,7)$	12	12	2
$C_{16}(7,8)$	3	4	3
$C_{16}(6,7,8)$	5	8	3
$C_{16}(5,6,7,8)$	7	10	3
$C_{16}(4,5,6,7,8)$	9	11	3
$C_{16}(3,4,5,6,7,8)$	11	12	2
$C_{16}(2,3,4,5,6,7,8)$	13	13	2

TABLE 9. The family of circulant graphs of the form $C_{2n}(1,3,5,\ldots)$ for $n \leq 8$.

G	K-Regularity	$Z(G)$	$P(G)$
$C_2(1)$	1	1	1
$C_4(1)$	2	2	1
$C_6(1)$	2	2	2
$C_6(1,3)$	3	4	1
$C_8(1)$	2	2	3
$C_8(1,3)$	4	6	1
$C_{10}(1)$	2	2	4
$C_{10}(1,3)$	4	6	1
$C_{10}(1,3,5)$	5	8	1
$C_{12}(1)$	2	2	5
$C_{12}(1,3)$	4	6	2
$C_{12}(1,3,5)$	6	10	1
$C_{14}(1)$	2	2	6
$C_{14}(1,3)$	4	6	2
$C_{14}(1,3,5)$	6	10	1
$C_{14}(1,3,5,7)$	7	12	1
$C_{16}(1)$	2	2	7
$C_{16}(1,3)$	4	6	3
$C_{16}(1,3,5)$	6	10	2
$C_{16}(1,3,5,7)$	8	14	1

TABLE 10. The family of circulant graphs of the form $C_{2n+1}(2, 4, 6, \ldots)$ for $n \leq 8$.

G	K-Regularity	$Z(G)$	$P(G)$
$C_3(2)$	2	2	1
$C_5(2)$	2	2	2
$C_7(2)$	2	2	3
$C_9(2)$	2	2	4
$C_9(2,4)$	4	4	3
$C_{11}(2)$	2	2	5
$C_{11}(2,4)$	4	4	4
$C_{13}(2)$	2	2	6
$C_{13}(2,4)$	4	4	5
$C_{13}(2,4,6)$	6	6	4
$C_{15}(2)$	2	2	7
$C_{15}(2,4)$	4	4	6
$C_{15}(2,4,6)$	6	6	5
$C_{17}(2)$	2	2	8
$C_{17}(2,4)$	4	4	7
$C_{17}(2,4,6)$	6	6	6
$C_{17}(2,4,6,8)$	8	8	5
$C_{19}(2)$	2	2	9
$C_{19}(2,4)$	4	4	8
$C_{19}(2,4,6)$	6	6	7
$C_{19}(2,4,6,8)$	8	8	6
$C_{21}(2)$	2	2	10
$C_{21}(2,4)$	4	4	9
$C_{21}(2,4,6)$	6	6	8
$C_{21}(2,4,6,8)$	8	8	7
$C_{21}(2,4,6,8,10)$	10	10	6
$C_{23}(2)$	2	2	11
$C_{23}(2,4)$	4	4	10
$C_{23}(2,4,6)$	6	6	9
$C_{23}(2,4,6,8)$	8	8	8
$C_{23}(2,4,6,8,10)$	10	10	7
$C_{25}(2)$	2	2	12
$C_{25}(2,4)$	4	4	11
$C_{25}(2,4,6)$	6	6	10
$C_{25}(2,4,6,8)$	8	8	9
$C_{25}(2,4,6,8,10)$	10	10	8
$C_{25}(2,4,6,8,10,12)$	12	12	7

TABLE 11. The family of circulant graphs $C_n(a, a+1, \ldots, b-1, b)$ for $1 < a \leq b < \frac{n}{2}$.

G	K-Regularity	$Z(G)$	$P(G)$
$C_6(2)$	2	4	1
$C_7(2)$	2	2	3
$C_8(2,3)$	4	4	2
$C_9(2,3)$	4	5	2
$C_{10}(2,3,4)$	6	6	2
$C_{11}(2,3,4)$	6	7	2
$C_{12}(2,3,4,5)$	8	8	2
$C_{13}(2,3,4,5)$	8	9	2
$C_{14}(2,3,4,5,6)$	10	10	2
$C_{15}(2,3,4,5,6)$	10	11	2
$C_{16}(2,3,4,5,6,7)$	12	12	2

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