An Introduction to f-Ideals and their Complements

by
Samuel Budd

A project submitted to the Department of
Mathematics & Statistics in conformity with the requirements
for the degree of Master of Science

McMaster University
Hamilton, Ontario, Canada

To Emily,
for her love and support.

Abstract

Given a squarefree monomial ideal I, one can associate to I two simplicial complexes, namely, the facet complex, $\delta_{\mathcal{F}}(I)$, and the non-face complex, $\delta_{\mathcal{N}}(I)$. When $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ have the same f-vector, I is said to be an f-ideal. In this project, we summarize known results about f-ideals, and present our new results. In particular, we perform a count of f-ideals generated in degree d inside various polynomial rings. We also show that for a squarefree monomial ideal I, then I is an f-ideal if and only if its complement $\hat{I}$ is an f-ideal, where $\hat{I}$ is the generalized Newton complementary dual of I. This result gives a new characterization of f-ideals.

Acknowledgements

First and foremost, I would like to thank my supervisor Dr. Adam Van Tuyl for a multitude of reasons: his patience and support during times of struggle, his guidance and expertise when I encountered mental blocks, and how he always made himself available, to only name a few. He is an outstanding professor and mentor, and I will forever be grateful that I had the privilege to work with him.

I would also like to thank the Department of Mathematics and Statistics at McMaster University for the financial support. Furthermore, interacting with the various members of Faculty and Staff was a pleasure throughout my time at McMaster.

My thanks also go out to my friends. For those at McMaster, thank you for the new friendships, the support, your knowledge, and your time. It was a pleasure to get to know you all, and you made this year better than I could have imagined. For my friends back in Thunder Bay and elsewhere, your continued support and your various communications provided me with comfort in times of stress.

Lastly, I express my most profound gratitude to my family, in particular, my parents. You have always supported me no matter what, and I would not be where I am today if it weren't for you.

Contents

CHAPTER 1

Introduction

This project will work within the realm of combinatorics and commutative algebra. Specifically, we will examine the relationship between squarefree monomial ideals and simplicial complexes. In particular, given a squarefree monomial ideal I, one can construct two different simplicial complexes, namely, the facet complex, $\delta_{\mathcal{F}}(I)$, and the non-face complex, $\delta_{\mathcal{N}}(I)$. One can also compute the f-vector of these simplicial complexes, which counts the number of faces of a given dimension. When the f-vectors for both $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ are the same, then I is said to be an *f-ideal*. An example of an f-ideal is provided below:

FIGURE 1. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_2, x_1x_4, x_2x_3\rangle$, both of which have the same f-vector.

The goal of this project is to summarize many of the known results about f-ideals, and to provide some new results on f-ideals. For detailed definitions of terminology used throughout this chapter, please see Chapters 2, 3, 4, and 5.

1. History of f-Ideals

We begin by providing a brief timeline of the advances in this area of math. To begin, we make note of the influential work done by Richard P. Stanley and Gerald Reisner in developing the field of combinatorial commutative algebra in the early 1970's [**6**]. In particular, Stanley-Reisner theory examines the relationship between simplicial complexes and their associated rings and ideals, and has laid the ground work for many of the concepts we discuss throughout this paper, especially in the construction of f-ideals. Some of these results are summarized in the next chapter.

An alternative correspondence between simplicial complexes and squarefree monomial ideals was first developed by Sara Faridi in 2001 [**5**]. Her approach generalized the edge ideal construction of Villarreal from 1990 [**12**]. In her construction, Faridi introduced the concept of a facet ideal, as well as a facet complex.

The concept of an f-ideal was introduced by G.Q. Abbasi, S. Ahmad, I. Anwar, and W.A. Baig in 2012 [**1**]. In their paper, they classify all f-ideals generated in degree 2. Not long after this classification, in 2014, I. Anwar, H. Mahmood, M. A. Binyamin, and M. K. Zafar [**3**] characterized unmixed f-ideals generated in degree d. Some of the work done by Abbasi et al. and Anwar et al. is summarized in Chapters 2 and 3.

Moving forward, J. Guo, T. Wu, and Q. Liu [**9**] proposed a different approach to characterizing f-ideals of degree 2. In particular, they used the concept of perfect sets to classify f-ideals. They also examined other interesting properties of f-ideals, such as counting how many f-ideals exist in a given polynomial ring, and whether or not f-ideals can be unmixed. This work was done around 2013, although it was not published until 2016. Building upon results discovered by Guo et al., in 2015, Guo and Wu [**10**] expanded upon their work on f-ideals generated in degree 2 to provide some algorithms for constructing certain types of f-ideals. They also found an example of an f-ideal generated in mixed degree, a concept we further investigate in this paper. The constructions given by Guo et al. in [**9**] are described in Chapter 4.

2. Results

We now summarize some of our new contributions in this project. As part of our research, we carried out a computer search for f-ideals. One of the main results of our computations is given below:

THEOREM 1.1 (See Thoerem 5.7). *Let $I \subseteq R = k[x_1, x_2, x_3, x_4]$ be a squarefree monomial ideal. If I is an f-ideal, then I must have all of its generators in degree d. As a consequence, there are no f-ideals generated in mixed degree in the polynomial ring $R = k[x_1, \ldots, x_n]$ for $n \leqslant 4$.*

One of our key interests was the relationship between f-ideals and the complementary ideals they admit. Initially defined by B. Costa and A. Simis [**4**] as the Newton complementary dual, we use the construction given by K. Ansaldi, K. Lin, and Y. Shen [**2**] to examine the generalized Newton complementary dual of an ideal. In particular, the generalized Newton complementary dual takes the complement of each generator for a given ideal I, and yields a complementary generating set. For the purposes of this project, we were interested in a specific case of the generalized Newton complementary dual of I. More specifically, given a squarefree monomial ideal $I = \langle g_1, \ldots, g_p \rangle \subseteq R = k[x_1, \ldots, x_n]$ with $\{g_1, \ldots, g_p\}$ a minimal generating set for I, one can construct a complementary generator using the following map,

$$g_i \xmapsto{\varphi} \frac{x_1 \cdots x_n}{g_i} = \hat{g}_i.$$

Using this map, we obtain a minimal generating set for the generalized Newton complementary dual of I, or simply, the *complement* of I, denoted by $\hat{I}$. In other words, we have that $\hat{I} = \langle \hat{g}_1, \ldots, \hat{g}_p \rangle$. Our second main result is the following new classification of f-ideals:

THEOREM 1.2 (See Thoerem 5.19). *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Then I is an f-ideal if and only if $\hat{I}$ is an f-ideal.*

In order to prove this, we use a result given by Guo et al. in [**9**], which characterizes f-ideals based on the number of generators of a certain degree. The main consequence that arises from this theorem is the idea that f-ideals come in pairs. In fact, using the theorem above, we are able to expand upon many of the known results simply by looking at ideals generated in degree $n-d$ rather than in degree d. This becomes useful in counting f-ideals, as well as constructing different f-ideals generated in both mixed and unmixed degree.

3. Structure of the Paper

Here, we provide a brief overview of this paper. In Chapter 2, we introduce several concepts from both combinatorics and commutative algebra. We refer to this chapter for the remainder of the paper.

Moving forward, Chapter 3 formally introduces the concept of an f-ideal, and provides some basic properties of such ideals. The chapter concludes with some characterizations on f-ideals, specifically, an initial characterization for f-ideals generated in degree 2 given by Abbasi et al. [**1**], followed by a more general characterization for unmixed f-ideals generated in degree d, given by Anwar et al. [**3**].

Chapter 4 then introduces an alternative way to look at f-ideals, using the concept of perfect sets. A complete and explicit characterizations for f-ideals generated in degree 2 is given. This is followed by a more general theorem, which we will use to prove our main result, Theorem 5.19. This material is based upon the work done by Guo et al. in [**9**] and [**10**].

Chapter 5 contains our new results. It provides an example of an f-ideal generated in mixed degree, and also outlines some algorithms for constructing f-ideals of certain types. We also introduce the complement of an ideal, and prove Theorem 5.19.

We use Chapter 6 to illustrate some of the computational results we have obtained, as well as some implications from our main result in Chapter 5. We then briefly summarize our results in Chapter 7 to conclude the paper. An appendix containing our Macaulay2 computer code is also included.

CHAPTER 2

Background Information

In this chapter, we will introduce the relevant background for the results contained in this paper. We first examine some basic definitions and results regarding combinatorics, as well as terminology to connect the algebra and combinatorics found in this paper.

1. Combinatorics

In this section, we introduce the necessary background in combinatorics for this project. We focus on simplicial complexes and some of the terminology required for understanding these mathematical objects.

DEFINITION 2.1. Given a set $S = \{x_1, x_2, \ldots, x_n\}$ of n elements, a *simplicial complex* Δ over S is a collection of subsets of S, such that,

(i) for any $x_i \in S$ with $i = 1, 2, \ldots, n$, then $\{x_i\} \in \Delta$, and
(ii) for any subset $K \subseteq \Delta$, all subsets of K are also in Δ.

In other words, a simplicial complex is a subset of the *power set* of S (i.e., $\Delta \subseteq \mathcal{P}(S)$) with some additional conditions.

EXAMPLE 2.2. Let $S = \{x_1, x_2, x_3, x_4\}$. Then an example of a simplicial complex Δ over S is

$$\Delta = \{\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_3, x_4\}, \{x_1, x_3, x_4\}\}.$$

We observe that all elements of S are contained in Δ, as well as all subsets of larger sets. For example, $\{x_1, x_3, x_4\} \in \Delta$, and therefore $\{x_1, x_3\}$, $\{x_1, x_4\}$, and $\{x_3, x_4\}$ are also all in Δ.

DEFINITION 2.3. Given a simplicial complex Δ, we call an element of Δ a *face* and denote it by F. Moreover, for a face F composed of m elements from S, that is $|F| = m$, then the *dimension* of F is defined as $\dim(F) = |F| - 1$. Sometimes F is called a face of *degree* m. We also note that the dimension of $\varnothing$ is -1.

We briefly mention the *degree* of a face in the above definition; however, we will provide a formal definition later on. It is now convenient to define the dimension of a simplicial complex Δ:

DEFINITION 2.4. The *dimension* of a simplicial complex Δ, denoted $\dim(\Delta)$, is equal to the largest dimension of any of the faces of Δ, i.e.,

$$\dim(\Delta) = \max \{\dim(F) \mid F \in \Delta\}.$$

EXAMPLE 2.5. From the previous example, we see that $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, and $\{x_4\}$ all have dimension 0, $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_1, x_4\}$ and $\{x_3, x_4\}$ have dimension 1, and $\{x_1, x_3, x_4\}$ has dimension 2. Furthermore, we see that the simplicial complex Δ has $\dim(\Delta) = 2$.

DEFINITION 2.6. We say that the face F of Δ is a *maximal face* if $F \nsubseteq G$ for all $F \neq G \in \Delta$. In other words, a face F is maximal if all the elements composing F are not contained in another face of Δ. These maximal faces are known as the *facets* of Δ.

EXAMPLE 2.7. Consider the simplicial complex in Example 2.2. We see that $\{x_1, x_2\}$ and $\{x_1, x_3, x_4\}$ are the two unique facets of Δ. All other faces of Δ are contained in one of these two facets.

We note that a simplicial complex can be completely described using its facets. That is, for facets $F_1, \ldots, F_p \in \Delta$, we write $\Delta = \langle F_1, \ldots, F_p \rangle$.

2. Commutative Algebra

We also require some background in commutative algebra. We use the following section to outline some basic concepts required for the rest of the paper. In what follows, k denotes a field.

DEFINITION 2.8. Given the polynomial ring $R = k[x_1, \ldots, x_n]$, a *monomial* $x^a \in R$ is a product of the indeterminants, i.e., $x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ with $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$. If $a_1, \ldots, a_n$ are all 0 or 1, then we say x^a is a *squarefree monomial.*

DEFINITION 2.9. Given a monomial $g = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in k[x_1, \ldots, x_n]$ with $a_i \geqslant 0$ for $i = 1, 2, \ldots, n$, we say that g has *degree* d, denoted by $\deg(g) = d$, if

$$d = a_1 + a_2 + \ldots + a_n.$$

In other words, d is the sum of the powers of the indeterminants composing g (or simply the number of indeterminants when referring to squarefree monomials).

DEFINITION 2.10. Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal, i.e., an ideal generated by monomials. The set of squarefree monomials in I is denoted by $sm(I)$, while the set of squarefree monomials in R is denoted by $sm(R)$. Furthermore, the set of squarefree monomials of degree d in R (I, respectively) is denoted by $sm(R)_d$ ($sm(I)_d$, respectively).

A natural next step is to examine the construction of objects using squarefree monomials.

DEFINITION 2.11. Let $I = \langle g_1, \ldots, g_p \rangle \subseteq R = k[x_1, \ldots, x_n]$ be an ideal. If $g_1, \ldots, g_p$ are all squarefree monomials, i.e., the ideal is generated solely by squarefree monomials, then I is referred to as a *squarefree monomial ideal.*

LEMMA 2.12. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal and let $G(I) = \{g_1, g_2, \ldots, g_r\}$ be the set of monomials in I that are minimal with respect to divisibility. Then $G(I)$ is the unique set of minimal monomial generators of I.*

Proof. See Proposition 1.1.6 of [**11**] for details of this proof. □

With a general definition for *degree* given above, we will also be interested in a specific case, that is, when the monomials generating a certain ideal all have degree 2.

Definition 2.13. We define a *quadratic* squarefree monomial ideal as a squarefree monomial ideal generated by degree two squarefree monomials. In other words, taking the product of two distinct indeterminants x_i and x_j from the polynomial ring $R = k[x_1, \ldots, x_n]$ (i.e., $x^a = x_i x_j$ for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$) yields a quadratic squarefree monomial.

We now explain how a squarefree monomial ideal can be constructed in two different ways from a simplicial complex Δ.

Definition 2.14. Let Δ be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, and denote the number of facets of Δ by p.

(i) We define the *facet ideal* of Δ, denoted by $I_{\mathcal{F}}$, as the ideal generated by the monomials $x_{i1}x_{i2}\cdots x_{im_i}$, given that $\{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$ is a facet of Δ, for $i = 1, 2, \ldots, p$.

(ii) We define the *non-face ideal* of Δ (also known as the *Stanley-Reisner ideal* of Δ), denoted by $I_{\mathcal{N}}$, as the ideal generated by the monomials $x_{i1}x_{i2}\cdots x_{im_i}$, given that $\{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$ is *not* a face of Δ.

For the above definition, recall that m_i is the number of squarefree monomials composing a given facet F.

Example 2.15. Returning to Example 2.2, we can construct the facet ideal and the non-face ideal as follows. First, we construct the facet ideal of Δ. From Example 2.7, we found the facets of Δ to be $\{x_1, x_2\}$ and $\{x_1, x_3, x_4\}$. Therefore we find that

$$I_{\mathcal{F}} = \langle x_1x_2, x_1x_3x_4\rangle.$$

We now wish to find the non-face ideal of Δ, in which case we need to find all the non-faces of Δ. To do so, we first calculate the power set of S, the set over which Δ is created, which yields

$$\begin{aligned}\mathcal{P}(S) = \{&\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\},\\ &\{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}.\end{aligned}$$

Now, taking the first facet in Δ, namely $F_1 = \{x_1, x_2\}$, we remove from the power set of S the set corresponding to this facet, as well as all subsets of F_1. This removes the elements $\{x_1, x_2\}$, $\{x_1\}$, and $\{x_2\}$, and yields the set

$$\begin{aligned}\mathcal{P}_1(S) = \{&\varnothing, \{x_3\}, \{x_4\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\},\\ &\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}.\end{aligned}$$

Next, we take the second facet, $F_2 = \{x_1, x_3, x_4\}$, and once again, remove the set corresponding to F_2 from the power set of S, as well as all subsets of F_2. This removes the elements $\{x_1, x_3, x_4\}$, $\{x_1, x_3\}$, $\{x_1, x_4\}$, $\{x_3, x_4\}$, $\{x_1\}$, $\{x_3\}$, and $\{x_4\}$, yielding the set

$$\mathcal{P}_{1,2}(S) = \{\{x_2, x_3\}, \{x_2, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}.$$

Keeping only the minimal subsets of $\mathcal{P}_{1,2}(S)$, we discard $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_2, x_3, x_4\}$, and $\{x_1, x_2, x_3, x_4\}$ yielding the final set of $\mathcal{P}_{1,2}(S)' = \{\{x_2, x_3\}, \{x_2, x_4\}\}$. The squarefree monomials corresponding to the elements of this set are then taken as a minimal set of generators for the non-face ideal of Δ, as seen below:

$$I_{\mathcal{N}} = \langle x_2x_3, x_2x_4\rangle.$$

With an understanding of how ideals can be generated, we also wish to provide a definition for the degree of an ideal I, namely;

Definition 2.16. Let $I \subseteq R$ be a squarefree monomial ideal minimally generated by squarefree monomials $g_1, g_2, \ldots, g_p$. Then the *degree* of I, denoted $\deg(I)$, is defined as

$$\deg(I) = \max\{\deg(g_i) \mid i = 1, 2, \ldots, p\}.$$

So far, we have the ability to construct two different squarefree monomial ideals based on a given simplicial complex Δ. Naturally, we are also interested in being able to proceed in the reverse direction, that is, we wish to be able to construct a simplicial complex given a squarefree monomial ideal I. In particular, we will see that this can also be done in two ways, yielding two different simplicial complexes given one squarefree monomial ideal I. This is formally defined below:

Definition 2.17. Let $I \subseteq R$ be a squarefree monomial ideal generated by a minimal set of squarefree monomials $G(I) = \{g_1, g_2, \ldots, g_p\} \subseteq I$.

(i) We define the *facet complex* of I, denoted by $\delta_{\mathcal{F}}(I)$, as the simplicial complex obtained by constructing facets $\{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$ for $i = 1, 2, \ldots, p$, given that $g_i = x_{i1}x_{i2}\cdots x_{im_i}$ is a generator of I.

(ii) We define the *non-face complex* (or the *Stanley-Reisner complex*) of I, denoted by $\delta_{\mathcal{N}}(I)$, as the simplicial complex with vertex set $V = \{x_i \mid x_i \notin I\}$ obtained by constructing faces $\{x_{i1}, x_{i2}, \ldots, x_{is}\}$ only if, for $g_j \in G(I)$, $g_j \nmid x_{i1}x_{i2}\cdots x_{is}$ for all j.

Below, we construct an example of both the facet complex and the non-face complex of a given squarefree monomial ideal $I \subseteq R = k[x_1, x_2, x_3, x_4]$.

Example 2.18. Consider the ideal $I = \langle x_1x_2, x_1x_3x_4\rangle \subseteq R = k[x_1, x_2, x_3, x_4]$. To construct the facet complex, we simply let each generator of I be a facet of $\delta_{\mathcal{F}}(I)$. Therefore we obtain an edge from x_1 to x_2, as well as a triangle between x_1, x_3, and x_4; this is illustrated on the left of Figure 1.

To obtain the non-face complex, we must compute all squarefree monomials in R and find those that are not contained in I. Recall from above that the set of all squarefree monomials in R is denoted by $sm(R)$. Thus

$$\begin{aligned} sm(R) = \{&1, x_1, x_2, x_3, x_4, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4,\\ &x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4, x_1x_2x_3x_4\}. \end{aligned}$$

Considering I is a minimal set of generators, then x_1x_2 and $x_1x_3x_4$ are obviously contained in I, but so are all multiples of x_1x_2 and $x_1x_3x_4$. Thus we must find all squarefree monomials that are not x_1x_2, $x_1x_3x_4$, or any multiples of them.

Working with one generator of I at a time, we remove x_1x_2 and all its multiples, namely, $x_1x_2x_3$, $x_1x_2x_4$, and $x_1x_2x_3x_4$. In other words, $\delta_{\mathcal{N}}(I)$ can therefore not contain $\{x_1, x_2\}$, $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, and $\{x_1, x_2, x_3, x_4\}$. Removing the corresponding monomials from the set $sm(R)$, we obtain

$$sm(R)\backslash sm(\langle x_1x_2\rangle) = \{1, x_1, x_2, x_3, x_4, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_3x_4, x_2x_3x_4\}.$$

Taking the next generator of I, we remove $x_1x_3x_4$ and all of its multiples, which in this case is only $x_1x_2x_3x_4$. This implies that $\delta_{\mathcal{N}}(I)$ can therefore not contain the faces $\{x_1, x_3, x_4\}$ and $\{x_1, x_2, x_3, x_4\}$. Removing the corresponding monomials from $sm(R)\backslash sm(\langle x_1x_2\rangle)$ (the last of which we already removed due to the previous step), we obtain the set

$$sm(R)\backslash sm(\langle x_1x_2, x_1x_3x_4\rangle) = \{1, x_1, x_2, x_3, x_4, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2x_3x_4\}.$$

The set $sm(R)\backslash sm(\langle x_1x_2, x_1x_3x_4\rangle) = sm(R)\backslash sm(I)$ therefore contains all squarefree monomials of R that are not contained in I, and hence, correspond to the faces of the non-face complex, $\delta_{\mathcal{N}}(I)$. We can also reduce this set to contain only maximal faces of $\delta_{\mathcal{N}}(I)$, thus yielding a minimal generating set of $\delta_{\mathcal{N}}(I)$. Consequently, we find that the facets of $\delta_{\mathcal{N}}(I)$ are $\{x_2, x_3, x_4\}$, $\{x_1, x_3\}$ and $\{x_1, x_4\}$. The non-face complex of I is pictured on the right in Figure 1 below.

FIGURE 1. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_2, x_1x_3x_4\rangle$.

We provide one final example illustrating these constructions.

Example 2.19. We begin by starting with a squarefree monomial ideal

$$I = \langle x_1x_5, x_2x_3, x_2x_5, x_1x_3x_4 \rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5].$$

We can therefore construct the facet and the non-face complexes in the following way. For the facet complex, we simply take each generator of I to be a facet of $\delta_{\mathcal{F}}(I)$, thus yielding the simplicial complex on the left in Figure 2. As for the non-face complex, we construct faces of $\delta_{\mathcal{N}}(I)$ only if they correspond to monomials that are not divisible by any of the generators of I. We find both simplicial complexes below in Figure 2.

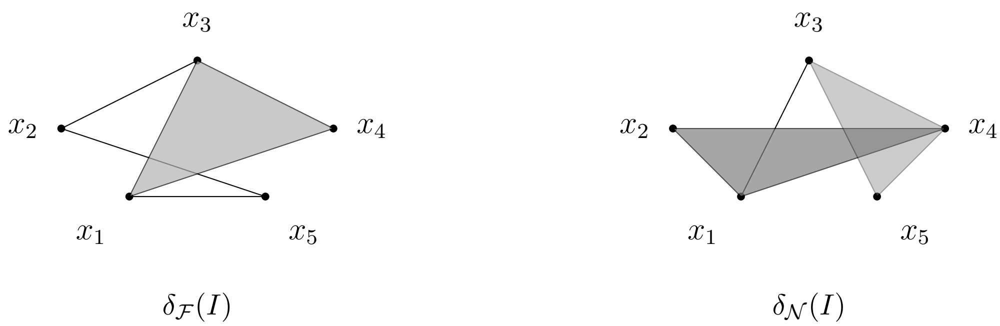


Figure 2. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_5, x_2x_3, x_2x_5, x_1x_3x_4 \rangle$.

On the other hand, if we begin with a simplicial complex Δ, we can instead obtain the facet and the non-face ideal of Δ. In order to compare results, we let Δ be

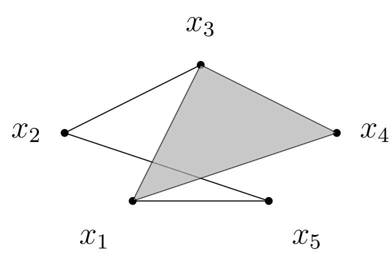


Figure 3. Simplicial complex $\Delta = \langle \{x_1, x_5\}, \{x_2, x_3\}, \{x_2, x_5\}, \{x_1, x_3, x_4\} \rangle$.

In order to obtain the facet ideal, we let each generator of $I_{\mathcal{F}}$ correspond to a facet of Δ, thus $I_{\mathcal{F}} = \langle x_1x_5, x_2x_3, x_2x_5, x_1x_3x_4 \rangle$. As for the non-face ideal, we let the collection of minimal non-faces of Δ generate $I_{\mathcal{N}}$, thus yielding $I_{\mathcal{N}} = \langle x_1x_2, x_2x_4, x_4x_5, x_3x_5 \rangle$.

It is important to note that the generators of the non-face ideal are not equal to the faces of the non-face complex when starting from the opposite direction. It is clear from Figure 2 that the facets of $\delta_{\mathcal{N}}(I)$ are in fact $\{x_1, x_2, x_4\}$, $\{x_3, x_4, x_5\}$, and $\{x_1, x_3\}$, while the generators of $I_{\mathcal{N}}$ are x_1x_2, x_2x_4, x_4x_5, and x_3x_5.

Before we move into the next chapter, we introduce the remaining background in commutative algebra required for this paper.

Definition 2.20. The *support* of a monomial $x^a = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$, denoted by $\operatorname{Supp}(x^a)$, is defined as the set of all indeterminants with powers strictly greater than zero, namely

$$\operatorname{Supp}(x^a) = \{x_i \mid a_i > 0\}.$$

Furthermore, the support of an ideal I is defined as the union of the individual supports of each monomial generator of I. More specifically, let g_i for $i = 1, 2, \dots, p$ be all the monomials generating I. Then

$$\operatorname{Supp}(I) = \bigcup_{i=1}^{p} \operatorname{Supp}(g_i).$$

Definition 2.21. Let k be a field and let $R = k[x_1, x_2, \dots, x_n]$ be a polynomial ring in n variables. Additionally, let $x^a = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ with $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$. Then we define the *graded component* of R, denoted R_d, as

$$R_d = \left\{ \sum_{a=(a_1,a_2,\dots,a_n)\in\mathbb{N}^n} r_a x^a \mid r_a \in k \text{ and } a_1 + a_2 + \dots + a_n = d \right\}.$$

Definition 2.22. Let R be a ring and let $\{R_i\}$ be a family of subgroups in R for i in some index set I. Then we say that the ring R is *graded* if

(i) $R = \bigoplus_{i\in I} R_i$, and
(ii) for subgroups R_j and R_k, then $R_j \cdot R_k \subseteq R_{j+k}$ for all $j, k \in I$.

Definition 2.23. Let $g_1, g_2, \dots, g_p$ be squarefree monomials in R_d, the graded component of R, for some $d > 0$. We call a squarefree monomial ideal $I = \langle g_1, g_2, \dots, g_p \rangle \subseteq R$ a *pure squarefree monomial ideal of degree* d when $\deg(g_1) = \deg(g_2) = \dots = \deg(g_p) = d$.

We will explore some results regarding the prime decomposition of ideals. This material will help with our understanding of the classification of f-ideals, the topic of the next chapter. To begin, we define a prime ideal.

Definition 2.24. Let R be a ring and let I be an ideal, with $I \subsetneq R$. We say that the ideal I is a *prime ideal* (sometimes denoted by $I = \mathfrak{p}$) if, for the element $xy \in I$, then we have that either $x \in I$, $y \in I$, or $x, y \in I$. In other words, if I is a prime ideal of R, then for elements $x \in R\backslash I$ and $y \in R\backslash I$, then $xy \in R\backslash I$.

Definition 2.25. The set of all prime ideals $\mathfrak{p}$ of the quotient ring R/I with the property that there is an injective homomorphism $\varphi_j : R/\mathfrak{p} \to R/I$, for $j = 1, \dots, s$, denoted by

$$\operatorname{Ass}(R/I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_s\},$$

is known as the set of *associated primes* of R. Sometimes we say $\mathfrak{p}$ is an *associated prime* of I.

THEOREM 2.26. *Let I be a squarefree monomial ideal of $R = k[x_1, x_2, \ldots, x_n]$ with associated primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$, where each associated prime $\mathfrak{p}_i$ is generated by a subset of $\{x_1, x_2, \ldots, x_n\}$. Then I can be written as the intersection of its prime ideals, that is,*

$$I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_s,$$

where the above notation is known as the primary decomposition of I.

PROOF. See Theorem 6.1.4 of [**13**] for a proof of this theorem. □

Below, we provide a simple method for computing the primary decomposition of a squarefree monomial ideal.

ALGORITHM 2.27. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. If g is a minimal generator of I, with $g = g_1 g_2$ and $\gcd(g_1, g_2) = 1$, then we can construct the primary decomposition of I by repeatedly applying the formula below:*

$$I = (I + \langle g_1 \rangle) \cap (I + \langle g_2 \rangle).$$

This process yields the primary decomposition of I.

EXAMPLE 2.28. Let $I = \langle x_1x_2, x_1x_3x_4 \rangle$. We can find a primary decomposition of I as follows.

$$\begin{aligned} I &= \langle x_1x_2, x_1x_3x_4 \rangle \\ &= \langle x_1, x_1x_3x_4 \rangle \cap \langle x_2, x_1x_3x_4 \rangle = \langle x_1 \rangle \cap \langle x_2, x_1x_3x_4 \rangle \\ &= \langle x_1 \rangle \cap \langle x_1, x_2 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle = \langle x_1 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle. \end{aligned}$$

Note that throughout the process we have collapsed the expressions where possible by either removing duplicates or removing *redundant* terms, i.e., if $I_j \subseteq I_k$, then we say that I_k is *redundant*.

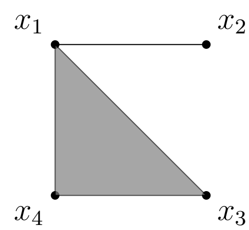
THEOREM 2.29. *Let Δ be a simplicial complex with vertices $x_1, x_2, \ldots, x_n$. Then the primary decomposition of the non-face ideal of Δ is*

$$I_{\mathcal{N}} = \bigcap_F \mathfrak{p}_F$$

where the intersection is taken over all facets F of Δ, and $\mathfrak{p}_F = \langle x_i \mid x_i \notin F \rangle$, the prime ideal generated by all x_i such that $x_i \notin F$.

PROOF. See Proposition 5.3.10 of [**13**] for a proof of this theorem. □

EXAMPLE 2.30. Consider the simplicial complex Δ from Example 2.2. We can use the above theorem to obtain the non-face ideal of Δ using its facets, without needing to compute the power set over which Δ is generated. First, recall the simplicial complex from Example 2.2, pictured below:



Δ

From here, we break Δ into its facets, and for each facet F_i, we note the elements of $\{x_1, x_2, x_3, x_4\}$ not found in F_i;

$$\begin{array}{rccc} \Delta = \langle & \{x_1, x_3, x_4\}, & \{x_1, x_2\} & \rangle \\ & \downarrow & \downarrow & \\ I_{\mathcal{N}} = & \langle x_2 \rangle & \cap \quad \langle x_3, x_4 \rangle. & \end{array}$$

By Theorem 2.29, we therefore have that $I_{\mathcal{N}} = \langle x_2 \rangle \cap \langle x_3, x_4 \rangle = \langle x_2x_3, x_2x_4 \rangle$. As expected, this yields the same non-face ideal as was obtained in Example 2.15.

Another application of Theorem 2.29 allows one to find the facets of the non-face complex without having to compute all squarefree monomials in R, as was done in Example 2.18.

Example 2.31. Considering the ideal from Example 2.28, we found it to have the following primary decomposition:

$$I = \langle x_1x_2, x_1x_3x_4 \rangle = \langle x_1 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle.$$

Taking the complement of each of the 3 sets in the above intersection, we obtain 3 other sets, each of which contain the elements x_i if and only if x_i is not a generator in the respective set above:

$$\begin{array}{rcccc} I = & \langle x_1 \rangle \;\cap & \langle x_2, x_3 \rangle \cap & \langle x_2, x_4 \rangle & \\ & \downarrow & \downarrow & \downarrow & \\ \delta_{\mathcal{N}}(I) = \langle & \{x_2, x_3, x_4\}, & \{x_1, x_4\}, & \{x_1, x_3\} & \rangle. \end{array}$$

Keeping only the maximal subsets of $\delta_{\mathcal{N}}(I)$ (which in this case are all the subsets), we obtain $\delta_{\mathcal{N}}(I) = \langle \{x_2, x_3, x_4\}, \{x_1, x_4\}, \{x_1, x_3\} \rangle$, the elements of which correspond precisely to the facets of the non-face complex of I.

Definition 2.32. Let R be a ring and let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_n$ be a chain of prime ideals in R. The *Krull dimension*, denoted $\dim R$, is the upper bound on the length of chains of prime ideals in R.

Definition 2.33. Let I be an ideal and let $\mathfrak{p}$ be a prime ideal in a ring R.

(i) The *height* of a prime ideal $\mathfrak{p}$, denoted $\text{ht}(\mathfrak{p}) = n - 1$, is the upper bound on the lengths of chains of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_n$ where the final prime ideal is $\mathfrak{p}$ itself.

(ii) The *height* of an ideal I, denoted $\text{ht}(I)$, is defined as

$$\text{ht}(I) = \min \{ \text{ ht}(\mathfrak{p}) \mid I \subset \mathfrak{p} \text{ } \},$$

where $\mathfrak{p}$ ranges over all the prime ideals containing I.

We note that if $\mathfrak{p}$ is a prime monomial ideal, and hence generated by a subset of $\{x_1, x_2, \ldots, x_n\}$, i.e., $\mathfrak{p} = \langle x_{i1}, x_{i2}, \ldots, x_{in_i} \rangle$, then the height of $\mathfrak{p}$ is equal to the number of variables generating $\mathfrak{p}$, that is,

$$\text{ht}(\mathfrak{p}) = n_i.$$

Restricting ourselves to the case of squarefree monomial ideals, we have the following results regarding height.

Definition 2.34. Let $\Delta = \langle F_1, F_2, \ldots, F_p \rangle$ be a simplicial complex. We define the height of the non-face ideal of Δ, denoted by $\text{ht}(I_{\mathcal{N}})$, as

$$\text{ht}(I_{\mathcal{N}}) = \min\{n - |F_i| \mid F_i \in \langle F_1, F_2, \ldots, F_p \rangle\},$$

where $|F_i| = m_i$, the degree of F_i.

Equivalently, we can say that the height of the non-face ideal $I_{\mathcal{N}}$ is the difference between the number of indeterminants and the degree of the largest facet in Δ.

Definition 2.35. Let I be a squarefree monomial ideal with minimal prime decomposition $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n$. We say that I is *unmixed* when the height of each associated prime ideal of I is the same, i.e., $\text{ht}(\mathfrak{p}_1) = \text{ht}(\mathfrak{p}_2) = \ldots = \text{ht}(\mathfrak{p}_n)$.

Lemma 2.36. *Let I be a squarefree monomial ideal. Then I is unmixed if and only if $\delta_{\mathcal{N}}(I)$ is pure.*

Proof. Let I be a squarefree monomial ideal. We know I is unmixed if and only if the height of each associated prime of I is the same. But by Theorem 2.29, the height of each associated prime is the same if and only if each facet of the non-face complex has the same dimension, and this occurs if and only if $\delta_{\mathcal{N}}(I)$ is pure, therefore we obtain the result we are looking for. □

CHAPTER 3

f-Ideals

In this chapter, we introduce f-ideals. In Section 1, we provide a formal definition for f-ideals, and follow this with several useful properties of f-ideals in Section 2. Moving into the the classification of f-ideals, Section 3 summarizes the work done by Abbasi et al. [**1**] in classifying f-ideals generated in degree 2, and further outlines a more general characterization of unmixed f-ideals generated in degree d, given by Anwar et al. [**3**].

1. Introduction to f-Ideals

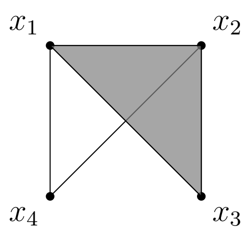
We use this section to formally define an f-ideal. We also provide a simple example of an f-ideal generated in degree 2.

Definition 3.1. Let Δ be a d-dimensional simplicial complex over n vertices. We define the *f-vector* of the simplicial complex Δ, as

$$f(\Delta) = (f_0, f_1, \dots, f_d),$$

a $(d+1)$-tuple where each f_i is the number of faces of dimension i in Δ.

Example 3.2. Let Δ be the simplicial complex below:



Then the f-vector of the simplicial complex Δ is $f(\Delta) = (4, 5, 1)$. This is because there are precisely 4 faces of dimension 0, (i.e., $f_0 = 4$), 5 faces of dimension 1 (i.e., $f_1 = 5$), and 1 face of dimension 2 (i.e., $f_2 = 1$).

Lemma 3.3. *Let $I = \langle g_1, g_2, \dots, g_p\rangle \subseteq R = k[x_1, \dots, x_n]$ be a squarefree monomial ideal generated in degree d, where $g_1, g_2, \dots, g_p$ form a minimal generating set for I. Then $f_{d-1}(\delta_{\mathcal{F}}(I)) = p$.*

Proof. Let $I = \langle g_1, g_2, \dots, g_p\rangle$ be a squarefree monomial ideal generated in degree d where $g_1, g_2, \dots, g_p$ form a minimal generating set for I. By definition, it is clear that each facet of $\delta_{\mathcal{F}}(I)$ will have dimension $d-1$, and since there are exactly p generators composing the minimal generating set, then there will be precisely p facets of dimension $d-1$. Thus $f_{d-1}(\delta_{\mathcal{F}}(I)) = p$. $\square$

We now describe the concept of an f-ideal. The definition of an f-ideal was originally defined by Abbasi et al. [**1**].

DEFINITION 3.4. Let $I \subseteq R$ be a squarefree monomial ideal, and let $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ be the facet and non-face complex, respectively, of I. Then I is said to be an *f-ideal* if the f-vectors of both $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ are the same.

Below, we provide an example to illustrate the concept of an f-ideal.

EXAMPLE 3.5. Let $I = \langle x_1x_2, x_1x_4, x_2x_3\rangle \subseteq R = k[x_1, x_2, x_3, x_4]$ be an ideal and consider its facet and non-face complex below.

FIGURE 1. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_2, x_1x_4, x_2x_3\rangle$.

From above, the f-vectors of the facet and non-face complex of I are $f(\delta_{\mathcal{F}}(I)) = (4, 3)$ and $f(\delta_{\mathcal{N}}(I)) = (4, 3)$, respectively. Since $f(\delta_{\mathcal{F}}(I)) = f(\delta_{\mathcal{N}}(I))$, I is an f-ideal.

2. Constructing f-Ideals

Before characterizing f-ideals, we state several lemmas that will be useful in proving some of the main results of this chapter.

LEMMA 3.6. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Then*

$$\dim(\delta_{\mathcal{F}}(I)) = \dim(\delta_{\mathcal{N}}(I))$$

if and only if $\mathrm{ht}(I) + \deg(I) = n$.

PROOF. Observe that $\dim(\delta_{\mathcal{F}}(I)) = \max\{\dim(F) | F \in \delta_{\mathcal{F}}(I)\}$, and $\dim(F) = \deg(F) - 1$. But $\deg(I) = \max\{\deg(F_i) | F_i$ is a facet of $\delta_{\mathcal{F}}(I)\}$, thus $\dim(\delta_{\mathcal{F}}(I)) = \deg(I) - 1$.

By Theorem 2.29, we know that the largest dimension of a facet in $\delta_{\mathcal{N}}(I)$ must be generated by a monomial of degree $n - \mathrm{ht}(I)$ therefore $\dim(\delta_{\mathcal{N}}(I)) = n - \mathrm{ht}(I) - 1$. From this, it is clear that $\dim(\delta_{\mathcal{F}}(I)) = \dim(\delta_{\mathcal{N}}(I))$ if and only if $n - \mathrm{ht}(I) - 1 = \deg(I) - 1$ and thus $n = \deg(I) + \mathrm{ht}(I)$. $\square$

Lemma 3.7. *Let $I = \langle g_1, g_2, \ldots, g_p \rangle \subseteq R = k[x_1, \ldots, x_n]$ be a pure squarefree monomial ideal generated in degree d. Then*

$$\binom{n}{d} = f_{d-1}(\delta_{\mathcal{F}}(I)) + f_{d-1}(\delta_{\mathcal{N}}(I)).$$

Proof. Let I be a squarefree monomial ideal generated by monomials $\{g_1, g_2, \ldots, g_p\}$, with $\deg(g_i) = d$ for $i = 1, \ldots, p$. Furthermore, note that there are precisely $\binom{n}{d}$ ways to construct faces of dimension $d-1$ for a simplicial complex over n vertices. Since I is a pure squarefree monomial ideal in degree d, then the faces of dimension $d-1$ not in $\delta_{\mathcal{F}}(I)$ must lie in $\delta_{\mathcal{N}}(I)$, and thus $\binom{n}{d} - f_{d-1}(\delta_{\mathcal{F}}(I)) = f_{d-1}(\delta_{\mathcal{N}}(I))$. Therefore $\binom{n}{d} = f_{d-1}(\delta_{\mathcal{F}}(I)) + f_{d-1}(\delta_{\mathcal{N}}(I))$. $\square$

Lemma 3.8. *Let $I = \langle g_1, g_2, \ldots, g_p \rangle \subseteq R = k[x_1, \ldots, x_n]$ be a pure squarefree monomial ideal generated in degree d. If $F \in \delta_{\mathcal{F}}(I)$ is a face with dimension less than $d-1$, the face F must also be in $\delta_{\mathcal{N}}(I)$. Consequently,*

$$f_i(\delta_{\mathcal{F}}(I)) \leqslant f_i(\delta_{\mathcal{N}}(I)) \quad \textit{for all} \quad i < d-1.$$

Proof. Let $I = \langle g_1, g_2, \ldots, g_p \rangle$ be a pure squarefree monomial ideal generated in degree d with $\{g_1, g_2, \ldots, g_p\}$ a minimal set of generators. After relabelling, let $F = \{x_1, \ldots, x_r\}$ be a face of $\delta_{\mathcal{F}}(I)$ with dimension less than $d-1$. Furthermore, let $m = x_1 x_2 \cdots x_r$ be the squarefree monomial corresponding to the face F. Then if F is a face of $\delta_{\mathcal{F}}(I)$, by definition, m divides some monomial g_i in I, for some $i \in 1, 2, \ldots, p$. Suppose now that $F \notin \delta_{\mathcal{N}}(I)$. Then by definition, if $F \notin \delta_{\mathcal{N}}(I)$, then $m = x_1 x_2 \cdots x_r \in I$. But from above, F is a face of dimension strictly less than $d-1$, and hence m is a monomial of degree strictly less than d. Since I is a squarefree monomial generated in degree d, then $m = x_1 x_2 \cdots x_r \in I$ yields a contradiction since $x_1 x_2 \cdots x_r$ is a monomial in I of degree smaller than d, thus completing the proof. $\square$

Lemma 3.9. *Let $I = \langle g_1, g_2, \ldots, g_p \rangle \subseteq R = k[x_1, \ldots, x_n]$ be an f-ideal generated in degree d. Then*

$$f_{d-2}(\delta_{\mathcal{F}}(I)) = \binom{n}{d-1}.$$

Proof. Let $I = \langle g_1, g_2, \ldots, g_p \rangle \subseteq R$ be an f-ideal generated in degree d. Since I is generated in degree d, there are no squarefree monomials in I of degree $d-1$. Thus, Lemma 3.8 implies that all faces of dimension $d-2$ will be contained in $\delta_{\mathcal{N}}(I)$. Since there are $\binom{n}{d-1}$ ways to choose a face of dimension $d-2$, then we have that $f_{d-2}(\delta_{\mathcal{N}}(I)) = \binom{n}{d-1}$. Since I is an f-ideal, the f-vectors of $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ are equivalent, and hence $f_{d-2}(\delta_{\mathcal{F}}(I)) = f_{d-2}(\delta_{\mathcal{N}}(I)) = \binom{n}{d-1}$, therefore completing the proof. $\square$

3. Classification of f-Ideals

We use this section to state and prove a classification of unmixed f-ideals. We begin by examining the case in which the ideal I is generated in degree two, originally proven by Abbasi et al. [**1**]. We follow this discussion with a more general result obtained by Anwar et al. [**3**].

THEOREM 3.10 (Abbasi, Ahmad, Anwar & Baig [**1**], Theorem 3.5). *Let* $I = \langle g_1, \ldots, g_p \rangle$ *be a squarefree monomial ideal generated in degree 2 inside* $k[x_1, \ldots, x_n]$. *Then* I *is an* f*-ideal if and only if*

(i) I *is unmixed with* $ht(I) = n - 2$,
(ii) $\binom{n}{2} \equiv 0 \pmod 2$, *and*
(iii) $p = \frac{1}{2}\binom{n}{2}$.

EXAMPLE 3.11. Let k be a field and let $R = k[x_1, x_2, x_3, x_4, x_5]$. Then the (quadratic) squarefree monomial ideal $I = \langle x_1x_2, x_2x_5, x_3x_4, x_3x_5, x_4x_5 \rangle$ is an f-ideal.

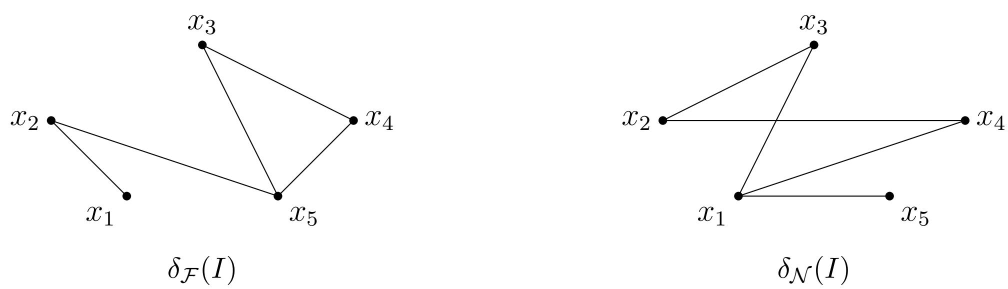


FIGURE 2. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_2, x_2x_5, x_3x_4, x_3x_5, x_4x_5 \rangle$.

From the above diagram, we can clearly see that $f(\delta_{\mathcal{F}}(I)) = (5,5)$ and $f(\delta_{\mathcal{N}}(I)) = (5,5)$, as both simplicial complexes have 5 faces of dimension 0 (i.e., the vertices), and 5 faces of dimension 1 (i.e., the edges). Furthermore, we see that all conditions of Theorem 3.10 are met. Computing the prime decomposition of I, we find that

$$\begin{aligned} I &= \langle x_1x_2, x_2x_5, x_3x_4, x_3x_5, x_4x_5 \rangle \\ &= \langle x_1, x_3, x_5 \rangle \cap \langle x_1, x_4, x_5 \rangle \cap \langle x_2, x_3, x_4 \rangle \cap \langle x_2, x_3, x_5 \rangle \cap \langle x_2, x_4, x_5 \rangle. \end{aligned}$$

It is clear from above that the height of each prime ideal is 3, and thus the same, therefore the ideal I is unmixed. This agrees with the fact that $\mathrm{ht}(I) = 5 - 2 = 3$. Furthermore, $\binom{5}{2} = 10 \equiv 0 \pmod 2$, and lastly $p = \frac{1}{2}\binom{5}{2} = 5$, thus satisfying all conditions of Theorem 3.10.

In [**3**], Anwar et. al investigate how to classify f-ideals generated in degree greater than 2. Anwar et al. first provide necessary but not sufficient conditions for squarefree monomial ideals generated in degree greater than or equal to 2. In fact, it is simply an extension of Theorem 3.10, where the conditions are amended to a slightly more general setting. We see their initial theorem below:

Theorem 3.12 (Anwar, Mahmood, Binyamin & Zafar [**3**], Theorem 3.3). *Let* $I = \langle g_1, g_2, \ldots, g_p\rangle \subseteq R = k[x_1, x_2, \ldots, x_n]$ *be a squarefree monomial ideal generated in degree* d. *If* I *is an* f*-ideal, then*

(i) I *is unmixed with* $ht(I) = n - d$,
(ii) $\binom{n}{d} \equiv 0 \pmod 2$, *and*
(iii) $p = \frac{1}{2}\binom{n}{d}$.

In the next example, we find a squarefree monomial ideal with degree higher than 2 that satisfies the necessary conditions of Theorem 3.12 but fails to be an f-ideal.

Example 3.13. Let $R = k[x_1, x_2, x_3, x_4, x_5]$ be a polynomial ring in five variables and take $I = \langle x_1x_2x_3, x_1x_2x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5\rangle \subseteq R$. We can check that I satisfies (i), (ii), and (iii) of Theorem 3.12. However we find that I is not an f-ideal. First, observe that

$$\begin{aligned} I &= \langle x_1x_2x_3, x_1x_2x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5\rangle \\ &= \langle x_1, x_5\rangle \cap \langle x_2, x_3\rangle \cap \langle x_2, x_4\rangle \cap \langle x_2, x_5\rangle \cap \langle x_3, x_4\rangle. \end{aligned}$$

Therefore the height of each prime ideal is equal to 2 which agrees with I being unmixed with $\mathrm{ht}(I) = n - d = 5 - 3 = 2$. Furthermore, we have that $\binom{n}{d} = \binom{5}{3} = 10 \equiv 0 \pmod 2$, and $p = \frac{1}{2}\binom{5}{3} = 5$, and thus, we find that all conditions are met. On the other hand, we can construct the facet and non-face complex of I (as seen in Figure 3) and compute their respective f-vectors.

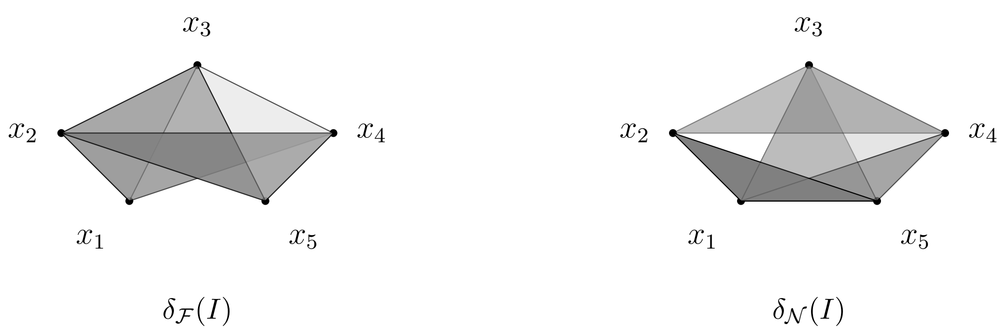


Figure 3. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_2x_3, x_1x_2x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5\rangle$.

From the simplicial complexes in Figure 3, we find that $f(\delta_{\mathcal{F}}(I)) = (5, 9, 5)$ and $f(\delta_{\mathcal{N}}(I)) = (5, 10, 5)$. We can conclude from this that I is not an f-ideal, even though the conditions of Theorem 3.12 satisfied.

Now, we introduce the main result on f-ideals for any degree d.

Theorem 3.14 (Anwar, Mahmood, Binyamin & Zafar [**3**], Theorem 3.8). *Let* $I = \langle g_1, g_2, \ldots, g_p\rangle \subseteq R = k[x_1, x_2, \ldots, x_n]$ *be a pure squarefree monomial ideal generated in* *degree d. Then I is an unmixed f-ideal if and only if each of the following conditions are satisfied.*

(i) $\mathrm{ht}(I) = n - d$,
(ii) $\binom{n}{d} \equiv 0 \pmod 2$,
(iii) $|\mathrm{Ass}(R/I)| = p = \frac{1}{2}\binom{n}{d}$, *and*
(iv) $f_{d-2}(\delta_{\mathcal{F}}(I)) = \binom{n}{d-1}$.

PROOF. ($\Rightarrow$) We will begin by proving the forward direction and assuming that I is a squarefree unmixed f-ideal generated in degree d. Since we have that I is an unmixed f-ideal with $\deg(I) = d$, the simplicial complexes $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ have the same dimension. By Lemma 3.6, we have that $\dim(\delta_{\mathcal{F}}(I)) = \dim(\delta_{\mathcal{N}}(I))$ implies that $\mathrm{ht}(I) + \deg(I) = n$. Since $\deg(I) = d$, we can rearrange to obtain $\mathrm{ht}(I) = n - d$, thus proving part (i). Now, since I is a squarefree monomial ideal generated in degree d, then by Lemma 3.7, we have $\binom{n}{d} = f_{d-1}(\delta_{\mathcal{F}}(I)) + f_{d-1}(\delta_{\mathcal{N}}(I))$. But by Lemma 3.3 we have that $f_{d-1}(\delta_{\mathcal{F}}(I)) = p$, and thus $\binom{n}{d} = p + f_{d-1}(\delta_{\mathcal{N}}(I))$. Since I is an f-ideal, we must have that $f_{d-1}(\delta_{\mathcal{F}}(I)) = f_{d-1}(\delta_{\mathcal{N}}(I))$, and thus $\binom{n}{d} = 2p$. Rearranging we obtain $p = \frac{1}{2}\binom{n}{d}$. Additionally, since I is an f-ideal generated in degree d and $f_{d-1}(\delta_{\mathcal{F}}(I)) = p$, then both $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ have p facets. But by Theorem 2.29, we know that the number of facets of $\delta_{\mathcal{N}}(I)$ is equal to the number of associated primes of R/I, that is, $p = |\mathrm{Ass}(R/I)|$, thus proving part (iii). From the previous argument, we can also see that $\binom{n}{d} = 2p \equiv 0 \pmod 2$, therefore satisfying (ii). Lastly, since I is an f-ideal generated in degree d, the statement $f_{d-2}(\delta_{\mathcal{F}}(I)) = \binom{n}{d-1}$ follows directly from Lemma 3.9.

($\Leftarrow$) We will now prove the converse. We begin by assuming that I is a pure squarefree monomial ideal generated in degree d that satisfies conditions (i) through (iv). First, we note that $f_{d-2}(\delta_{\mathcal{F}}(I)) = \binom{n}{d-1}$. In other words, we have that there are exactly $\binom{n}{d-1}$ faces of dimension $d-2$ in $\delta_{\mathcal{F}}(I)$. But since we are choosing all faces of degree $d-1$, the simplicial complex $\delta_{\mathcal{F}}(I)$ must contain all faces of dimension $d-2$. But if $\delta_{\mathcal{F}}(I)$ contains all faces of dimension $d-2$, then it must also contain all subfaces of those faces, and hence $\delta_{\mathcal{F}}(I)$ must contain all faces of dimension less than $d-2$. This can be continued until we reach subfaces of dimension $d-d=0$, where all these subfaces are still contained in $\delta_{\mathcal{F}}(I)$. But for any i, there are $\binom{n}{i+1}$ many ways to choose a subset of degree $i+1$ elements from n, and hence the maximal number of faces of dimension i must be $\binom{n}{i+1}$. On the other hand, we know that there are exactly $\binom{n}{i+1}$ ways to choose $i+1$ indeterminants from $x_1, x_2, \ldots, x_n$, which therefore implies that $f_i(\delta_{\mathcal{N}}(I)) = \binom{n}{i+1}$. Using these two facts and applying Lemma 3.8, we have that $\binom{n}{i+1} = f_i(\delta_{\mathcal{F}}(I)) \leqslant f_i(\delta_{\mathcal{N}}(I)) = \binom{n}{i+1}$ which implies that $f_i(\delta_{\mathcal{F}}(I)) = f_i(\delta_{\mathcal{N}}(I))$ for $i \leqslant d-2$. From above we have that each f_i of $f(\Delta)$ is equivalent for both $f_i(\delta_{\mathcal{F}}(I))$ and $f_i(\delta_{\mathcal{N}}(I))$ for $i \leqslant d-2$.

We know from Lemma 3.3 however, that $f_{d-1}(\delta_{\mathcal{F}}(I)) = p$. But by (iii), we know $p = \frac{1}{2}\binom{n}{d}$, and by Lemma 3.7, we have $\binom{n}{d} = f_{d-1}(\delta_{\mathcal{F}}(I)) + f_{d-1}(\delta_{\mathcal{N}}(I))$, thus rearranging, we find $2f_{d-1}(\delta_{\mathcal{F}}(I)) = f_{d-1}(\delta_{\mathcal{F}}(I)) + f_{d-1}(\delta_{\mathcal{N}}(I))$ and therefore $f_{d-1}(\delta_{\mathcal{F}}(I)) = f_{d-1}(\delta_{\mathcal{N}}(I))$. From this we conclude that I is an f-ideal.

We now show that I is unmixed. By our hypothesis, we know that $|\mathrm{Ass}(R/I)| = p$. This implies that we have prime ideals $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_p\}$ such that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_p$. Now suppose I is not unmixed. Since $\mathrm{ht}(I) = \min\{\mathrm{ht}(\mathfrak{p}_i) | I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_p\}$, then there exists a $\mathfrak{p}_i \in \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_p\}$ for some $i \in \{1, 2 \ldots, p\}$ such that $\mathrm{ht}(\mathfrak{p}_i) > \mathrm{ht}(I)$. From our hypothesis, we know $\mathrm{ht}(I) = n - d$, thus $n - d < \mathrm{ht}(\mathfrak{p}_i)$ and hence $n - (d - 1) = n - d + 1 \leqslant \mathrm{ht}(\mathfrak{p}_i)$.

By Theorem 2.29, there is a one-to-one correspondence between the associated primes of I and the facets of the non-face complex of I, and hence $\delta_{\mathcal{N}}(I)$ has precisely p facets (where each facet is built from the complement of each $\mathfrak{p}_j$ for $j = 1, 2, \ldots, p$). We also see that the facet obtained from the associated prime $\mathfrak{p}_i$ has dimension less than or equal to $d - 2$. But then $f_{d-1}(\delta_{\mathcal{N}}(I)) \leqslant p - 1$ since one of the facets has already been created from $\mathfrak{p}_i$ and the remaining $p - 1$ may have dimension $d - 1$. By Lemma 3.3, we have $f_{d-1}(\delta_{\mathcal{F}}(I)) = p$ and from above we know $f_{d-1}(\delta_{\mathcal{N}}(I)) \leqslant p - 1$. Using the fact that $f_{d-1}(\delta_{\mathcal{F}}(I)) = f_{d-1}(\delta_{\mathcal{N}}(I))$ we find $p = f_{d-1}(\delta_{\mathcal{F}}(I)) = f_{d-1}(\delta_{\mathcal{N}}(I)) \leqslant p - 1$. But this implies $p \leqslant p - 1$, a contradiction. From this, we can therefore conclude that I is unmixed, and hence I is an unmixed f-ideal. $\square$

CHAPTER 4

A Parallel Approach to f-Ideals

As mentioned in the introduction, Guo, Wu, and Liu provide an alternative method for studying f-ideals. In this chapter, we highlight the main ideas of [**9**].

As discussed in Chapter 2, one can construct a facet and a non-face complex from a given squarefree monomial ideal I. Essentially the same as the definitions provided earlier, Guo et al. [**9**] make a clear reference to the bijection between $sm(R)$, the set of squarefree monomials in R, and $2^{[n]}$, with $[n] = \{1, 2, \ldots, n\}$. In particular, the natural bijection $\sigma : sm(R) \longrightarrow 2^{[n]}$ has the form

$$x_{i_1} x_{i_2} \cdots x_{i_k} \overset{\sigma}{\longmapsto} \{i_1, i_2, \ldots, i_k\},$$

where squarefree monomials are simply transformed into faces of a simplicial complex. In particular, the facet and the non-face complex are defined below using this alternative terminology:

Definition 4.1. Let $I = \langle g_1, \ldots, g_p \rangle \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal with $g_1, \ldots, g_p$ a minimal generating set of I. Then the *facet complex* of I is defined as

$$\delta_{\mathcal{F}}(I) = \langle \sigma(g_i) \mid i = 1, 2, \ldots, p \rangle,$$

while the *non-face complex* of I is defined as

$$\delta_{\mathcal{N}}(I) = \{\sigma(g) \mid g \in sm(R) \backslash sm(I)\}.$$

1. Perfect Sets

The above definitions are by no means drastically different from those in Chapter 2. However, we now introduce material that begins to differ from the constructions we have already presented.

Definition 4.2. Let S be a set of squarefree monomials in $R = k[x_1, \ldots, x_n]$. Then the set

$$\sqcup(S) = \{g x_i \mid g \in S, x_i \nmid g, 1 \leqslant i \leqslant n\}$$

is called the *upper generated set of* S, and the set

$$\sqcap(S) = \{g/x_i \mid g \in S,\ x_i | g,\ \text{and } g/x_i \neq 1\}$$

is called the *lower cover set* of S. Similarly, when applying $\sqcup$ and $\sqcap$ multiple times, we define $\sqcup^k(S) = \sqcup(\sqcup^{k-1}(S))$ and $\sqcap^k(S) = \sqcap(\sqcap^{k-1}(S))$. Moreover, $\sqcup^\infty(S) = \bigcup_{i=1}^{\infty} \sqcup^i(S)$ and $\sqcap^\infty(S) = \bigcup_{i=1}^{\infty} \sqcap^i(S)$.

DEFINITION 4.3. Let $R = k[x_1, \ldots, x_n]$ and let $S \subseteq sm(R)_d$. If $\sqcup(S) = sm(R)_{d+1}$, then S is said to be *upper perfect*. In contrast, if $\sqcap(S) = sm(R)_{d-1}$, then S is said to be *lower perfect*. If a set S is both upper and lower perfect, then S is said to be $(n, d)^{th}$ *perfect*.

EXAMPLE 4.4. Let $R = k[x_1, x_2, x_3, x_4, x_5]$ and consider the following subsets of $sm(R)_2$:

$$S_1 = \{x_1x_2, x_2x_3, x_2x_4, x_4x_5\} \quad S_2 = \{x_1x_2, x_1x_3, x_1x_5, x_2x_3, x_2x_5, x_3x_5\}.$$

Furthermore, note that $sm(R)_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and

$$\begin{aligned} sm(R)_3 = \{&x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, \\ &x_1x_4x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5\}. \end{aligned}$$

Clearly, $\sqcap(S_1) = \{x_1, x_2, x_3, x_4, x_4\} = sm(R)_1$ but $x_1x_3x_5 \notin \sqcup(S_1)$, thus $\sqcup(S_1) \neq sm(R)_3$, and hence S_1 is lower perfect, but not upper perfect. On the other hand, $\sqcup(S_2)$ is clearly equal to $sm(R)_3$ but $x_4 \notin \sqcap(S_2)$, therefore $\sqcap(S_2) \neq sm(R)_1$. Hence S_2 is upper perfect, but not lower perfect.

EXAMPLE 4.5. Consider the set $S = \{x_1x_3, x_1x_5, x_2x_4, x_3x_5\} \subseteq sm(R)_2$ in the polynomial ring $R = k[x_1, x_2, x_3, x_4, x_5]$. By computing the upper generated set and the lower cover set, we find that $\sqcup(S) = sm(R)_3$ and $\sqcap(S) = sm(R)_1$, and therefore S is $(5, 2)^{th}$ perfect, or simply, *perfect*.

A helpful theorem which allows one to determine whether an ideal is an f-ideal is given below.

THEOREM 4.6 (J. Guo, T. Wu, Q. Liu, [**9**], Theorem 2.3). *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in degree d with minimal generating set $G(I) = \{g_1, \ldots, g_p\}$. Then I is an f-ideal if and only if $G(I)$ is $(n, d)^{th}$ perfect and $|G(I)| = \frac{1}{2}\binom{n}{d}$.*

In addition, we also have an algorithm for finding all perfect subsets of degree d in the polynomial ring $k[x_1, \ldots, x_n]$.

ALGORITHM 4.7. *Let $R = k[x_1, \ldots, x_n]$ and let $sm(R)_d$ be the set of squarefree monomials in R generated in degree d. Then the following procedure yields all f-ideals generated in degree d in R.*

(1) List elements of $sm(R)_d$, $sm(R)_{d-1}$, and $sm(R)_{d+1}$
(2) For each $S \subseteq sm(R)_d$, if $|S| = \frac{1}{2}\binom{n}{d}$, then go to step 3.
(3) Compute $\sqcup(S)$ and $\sqcap(S)$ and check if $\sqcup(S) = sm(R)_{d+1}$ and $\sqcap(S) = sm(R)_{d-1}$.

Ultimately, this provides a brute force method for computing all f-ideals generated in degree d within a certain polynomial ring. Nevertheless, this process becomes arduous and inefficient for large n and d.

PROPOSITION 4.8. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be an f-ideal generated in degree d, with minimal generating set $G(I) = \{g_1, \ldots, g_p\}$. Then I is unmixed if and only if $sm(R)_d \backslash G(I)$ is lower perfect.*

In relation to a complete classification of f-ideals generated in degree 2, Abbasi et al. provide a characterization in [**1**]. However, Guo et al. [**9**] expand upon this by completely and explicitly classifying all f-ideals generated in degree 2. Their classification is breifly described in the following section.

2. Structure of all Quadratic f-Ideals

As mentioned above, we use this section to summarize the results obtained by Guo et al. [**9**] on the characterization of all f-ideals generated in degree two, that is, all quadratic f-ideals. In particular, the following results on quadratic f-ideals are found in [**9**].

PROPOSITION 4.9. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in degree 2. If I is an f-ideal, then I is unmixed.*

PROOF. See Proposition 5.2 of [**9**] for details on this proof. □

DEFINITION 4.10. Let S be a subset of $[n] = \{1, 2, \ldots, n\}$, with $1 < |S| < n - 1$, and set

$$W_S = \{x_i x_j \mid i, j \in S \text{ or } i, j \in \bar{S}\}, \text{ where } \bar{S} = [n] \backslash S.$$

For an f-ideal I, generated in degree 2 with generating set $G(I)$, if there exists an $S \subseteq [n]$ such that $W_S \subseteq G(I)$, then I is called a *quadratic f-ideal of r type*, where $r = \min(|S|, |\bar{S}|)$. The set of all f-ideals of r type is denoted by W_r.

THEOREM 4.11 (J. Guo, T. Wu, Q. Liu, [**9**], Theorem 4.7). *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring and let $V(n, 2)$ be the set of all f-ideals generated in degree 2. Then*

$$V(n,2) = \begin{cases} \bigcup\limits_{0 \leqslant i \leqslant \sqrt{k}} W_{2k-i} & \text{if } n = 4k; \\ \bigcup\limits_{0 \leqslant i \leqslant \frac{\sqrt{1+4k}-1}{2}} W_{2k-i} & \text{if } n = 4k+1, \ k \neq 1; \\ W_2 \cup C_5 & \text{if } n = 5; \\ \varnothing & \text{if } n = 4k+2 \text{ or } n = 4k+3. \end{cases}$$

where W_r represents the set of all f-ideals of r type, and C_5 represents the cycle graph on five vertices.

In addition to the classification of f-ideals generated in degree 2, Guo et al. [**9**] also counted the number of quadratic f-ideals in any polynomial ring $k[x_1, \ldots, x_n]$. This is given as the following theorem:

THEOREM 4.12 (J. Guo, T. Wu, Q. Liu, [**9**], Proposition 4.10). *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring and let $V(n,2)$ be the set of all f-ideals generated in degree 2 in R. Then the number of f-ideals generated in degree 2 for a given n, that is, $|V(n,2)|$, is given by*

$$|V(n,2)| = \begin{cases} \frac{1}{2}\binom{4k}{2k}\binom{4k^2}{k} + \sum\limits_{1\leqslant i\leqslant\sqrt{k}} \binom{4k}{2k-i}\binom{4k^2-i^2}{k-i^2} & \text{if } n = 4k; \\ \sum\limits_{0\leqslant i\leqslant \frac{\sqrt{1+4k}-1}{2}} \binom{4k+1}{2k-i}\binom{4k^2+2k-i-i^2}{k-i-i^2} & \text{if } n = 4k+1,\ k \neq 1; \\ 72 & \text{if } n = 5; \\ 0 & \text{if } n = 4k+2 \text{ or } n = 4k+3. \end{cases}$$

3. Unmixed f-Ideals Generated in Degree d

Guo et al. also provide a characterization which allows one to determine if an unmixed squarefree monomial ideal I generated in degree d is an f-ideal. We use this section to outline this result, as well as mention some facts that we will need for this characterization.

DEFINITION 4.13. Let Δ be a simplicial complex over $S = \{x_1, \ldots, x_n\}$. If every minimal non-face of Δ has dimension $d-1$, then Δ is said to be a *d-flag complex*.

DEFINITION 4.14. Let Δ be a simplicial complex over $S = \{x_1, \ldots, x_n\}$. Then the set

$$\Delta^\vee = \{\{x_1, \ldots, x_n\} \backslash F \mid F \notin \Delta\}$$

is called the *Alexander dual* of Δ.

EXAMPLE 4.15. Consider the simplicial complex $\Delta = \langle \{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\} \rangle$. As seen in Example 3.5, we have that the minimal non-faces of Δ are $\{x_1, x_3\}$, $\{x_2, x_4\}$, and $\{x_3, x_4\}$.

Letting $\mathcal{N}_\Delta$ denote the set of minimal non-faces of Δ, we compute the facets of the Alexander dual below:

$$\begin{array}{rl} \Delta = & \langle\ \{x_1, x_2\},\ \{x_1, x_4\},\ \{x_2, x_3\}\ \rangle, \\ & \qquad\qquad \downarrow \\ \mathcal{N}_\Delta = & \{\ \{x_1, x_3\},\ \{x_2, x_4\},\ \{x_3, x_4\}\ \}, \\ & \quad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ \Delta^\vee = & \langle\ \{x_2, x_4\},\ \{x_1, x_3\},\ \{x_1, x_2\}\ \rangle. \end{array}$$

From the above computations, we obtain the following simplicial complexes:

FIGURE 1. Simplicial complex $\Delta = \langle\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}\rangle$ and its Alexander dual $\Delta^\vee$.

We also observe that every minimal non-face of the simplicial complex Δ has dimension 1, thus we have that Δ is a 2-*flag* complex.

We can now state the main classification on unmixed f-ideals generated in degree d, given by Guo et al. [**10**]:

THEOREM 4.16 (J. Guo, T. Wu, [**10**], Proposition 2.1). *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in degree d with minimal generating set $G(I) = \{g_1, \ldots, g_p\}$. Then I is an unmixed f-ideal if and only if I satisfies the following conditions:*

(i) $|G(I)| = p = \frac{1}{2}\binom{n}{d}$,
(ii) $\dim \delta_{\mathcal{F}}(I)^\vee = n - d - 1$, *and*
(iii) $\langle\sigma(h) \mid h \in sm(R)_d \backslash G(I)\rangle$ *is a d-flag complex.*

EXAMPLE 4.17. Consider the squarefree monomial ideal $I = \langle x_1x_2, x_2x_3, x_1x_4\rangle \subseteq R = k[x_1, x_2, x_3, x_4]$ from Example 3.5. We can use Theorem 4.16 to verify that I is an f-ideal.

We observe that $|G(I)| = 3$, and compute $\frac{1}{2}\binom{4}{2} = 3$, thus condition (i) holds. Since $\delta_{\mathcal{F}}(I) = \langle\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}\rangle$, then we know from Example 4.15 that $\delta_{\mathcal{F}}(I)^\vee = \langle\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_4\}\rangle$, and hence $\dim \delta_{\mathcal{F}}(I)^\vee = 1$. Computing $n-d-1 = 4-2-1 = 1$, we see that condition (ii) also holds. Lastly, we also know from Example 4.15 that

$$\langle\sigma(h) \mid h \in sm(R)_d \backslash G(I)\rangle = \langle\{x_1, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\rangle.$$

Hence, $\langle\sigma(h) \mid h \in sm(R)_d \backslash G(I)\rangle$ is a 2-flag complex. Since $d = 2$, condition (iii) holds, and I is an f-ideal.

4. Generalizations for f-Ideals

So far in this chapter, we have focused on the case that I is generated in degree d. We now examine some of the results seen in [**9**], which allow one to determine if a squarefree monomial ideal generated in mixed degree is an f-ideal. We begin by providing a necessary and sufficient condition for a squarefree monomial ideal generated in mixed degree to be an f-ideal.

PROPOSITION 4.18. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal with minimal generating set $G(I) = \bigcup_{i=1}^{k} G_{d_i}(I)$, where $G_{d_i}(I)$ represents the set of minimal generators of degree d_i. Then I is an f-ideal if and only if the following conditions hold:*

(i) For each positive $l \in \{d_1, \ldots, d_k\}$,

$$|G_l(I)| = \frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{d_i > l}(\sqcap^{d_i - l}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i < l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right|\right).$$

(ii) For each positive $l \notin \{d_1, \ldots, d_k\}$,

$$\bigcup_{d_i > l} \sqcap^{d_i - l}(G_{d_i}(I)) = sm(R)_l \setminus \bigcup_{d_i < l} \sqcup^{l-d_i}(G_{d_i}(I)).$$

PROOF. See Proposition 6.1 of [**8**] for details on this proof. □

In particular, the above proposition can be simplified, as we show in Theorem 4.20 below. In order to prove this theorem, we state a lemma that we will use.

LEMMA 4.19. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Then*

(i) $\sigma(\sqcap^\infty(G(I))) = \delta_{\mathcal{F}}(I) \cap \delta_{\mathcal{N}}(I)$,
(ii) $\sigma(\sqcup^\infty(G(I))) \cap \delta_{\mathcal{F}}(I) = \varnothing$ and $\sigma(\sqcup^\infty(G(I))) \cap \delta_{\mathcal{N}}(I) = \varnothing$.

PROOF. (i) Let $G(I) = \bigcup_{j=1}^{k} G_{d_j}(I)$ and let $F \in \sigma(\sqcap^\infty(G(I)))$. By definition, $\sqcap^\infty(G(I))$ is the set of all squarefree monomials $\bigcup_{i=1}^{\infty} \sqcap^i(\bigcup_{j=1}^{k} G_{d_j}(I))$. This implies that $\sigma(\sqcap^\infty(G(I)))$ will include all faces of $\delta_{\mathcal{F}}(I)$, except for its facets. But since $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ share all faces of dimension $d_1 - 2, \ldots, d_k - 2$, then $\sigma(\sqcap^\infty(G(I))) \subseteq \delta_{\mathcal{F}}(I) \cap \delta_{\mathcal{N}}(I)$

Now, let $F \in \delta_{\mathcal{F}}(I) \cap \delta_{\mathcal{N}}(I)$ be a face corresponding to the monomial m. Then there must be a minimal generator $g \in G(I)$ such that m strictly divides g. Because of this, we must have that $m \in \sqcap^\infty(G(I))$. Since m corresponds to the face F, then we have that $F \in \sigma(\sqcap^\infty(G(I)))$. The result follows, and hence $\delta_{\mathcal{F}}(I) \cap \delta_{\mathcal{N}}(I) \subseteq \sigma(\sqcap^\infty(G(I)))$. Therefore $\sigma(\sqcap^\infty(G(I))) = \delta_{\mathcal{F}}(I) \cap \delta_{\mathcal{N}}(I)$.

(ii) Let $G(I) = \bigcup_{j=1}^{k} G_{d_j}(I)$ be a minimal generating set of I. Then $\sigma(\sqcup^\infty(G(I)))$ is the set of faces generated by monomials in $\bigcup_{i=1}^{\infty} \sqcup^i(\bigcup_{j=1}^{k} G_{d_j}(I))$. But these monomials have degrees no less than $d_1 + 1, d_2 + 1, \ldots, d_k + 1$, respectively. Since $G(I)$ is a minimal generating set, we have that any face in $\sigma(\sqcup^\infty(G(I)))$ will not be contained in $\delta_{\mathcal{F}}(I)$. Hence, $\sigma(\sqcup^\infty(G(I))) \cap \delta_{\mathcal{F}}(I) = \varnothing$.

Now, suppose $F \in \sigma(\sqcup^\infty(G(I))) \cap \delta_{\mathcal{N}}(I)$. Since $F \in \sigma(\sqcup^\infty(G(I)))$, there exists a monomial $g \in G(I)$ and a squarefree monomial m such that $\gcd(g, m) = 1$ and $\sigma(gm) = F$. But $gm \in sm(I)$, thus, by definition, $\sigma(gm) \notin \delta_{\mathcal{N}}(I)$. But this implies that $F \notin \delta_{\mathcal{N}}(I)$, and hence contradicts $F \in \sigma(\sqcup^\infty(G(I))) \cap \delta_{\mathcal{N}}(I)$. We therefore conclude that $\sigma(\sqcup^\infty(G(I))) \cap \delta_{\mathcal{N}}(I) = \varnothing$. □

THEOREM 4.20 (J. Guo, T. Wu, Q. Liu, [**9**], Theorem 7.2). *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a squarefree monomial ideal with minimal generating set $G(I) = \bigcup_{i=1}^{k} G_{d_i}(I)$. Then I is an f-ideal if and only if, for each $l \in \{1, 2, \dots, n\}$,*

$$|G_l(I)| = \frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right|\right).$$

PROOF. We begin by observing that for each $l \in \{1, \dots, n\}$, the set of squarefree monomials in R can be represented as the disjoint union of four sets, that is,

$$sm(R)_l = (G_l(I)) \cup \left(\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right) \cup \left(\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right) \cup \left(\sigma^{-1}(F_l)\right),$$

where $F_l = (\delta_{\mathcal{N}}(I)\backslash\delta_{\mathcal{F}}(I)) \cap \sigma(sm(R)_l)$. We also note that since $\sigma(\sqcup^{\infty}(G(I))) \cap \delta_{\mathcal{F}}(I) = \varnothing$ by Lemma 4.19, then $f_{l-1}(\delta_{\mathcal{F}}(I)) = |G_l(I)| + \left|\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right|$, as none of the faces of $\delta_{\mathcal{F}}(I)$ will be contained in $\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))$.

But Lemma 4.19 also implies that since $\sigma(\sqcup^{\infty}(G(I))) \cap \delta_{\mathcal{N}}(I) = \varnothing$, then none of the faces in the non-face complex of I will be contained in $\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))$, and further, they cannot correspond to a facet of $\delta_{\mathcal{F}}(I)$, therefore $f_{l-1}(\delta_{\mathcal{N}}(I)) = |\sigma^{-1}(F_l)| + \left|\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right|$.

However, we know that I is an f-ideal if and only if $f_{l-1}(\delta_{\mathcal{F}}(I)) = f_{l-1}(\delta_{\mathcal{N}}(I))$ for each $l \in \{1, \dots, n\}$. Therefore I must be an f-ideal if and only if

$$|G_l(I)| + \left|\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right| = |\sigma^{-1}(F_l)| + \left|\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right|.$$

This implies that $|G_l(I)| = |\sigma^{-1}(F_l)|$.

Now, examining the cardinality of $sm(R)_l$, it is clear that

$$|sm(R)_l| = \left|(G_l(I)) \cup \left(\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right) \cup \left(\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right) \cup \left(\sigma^{-1}(F_l)\right)\right|.$$

But since the above unions are disjoint, we can turn them into summations, and using the fact that $|G_l(I)| = |\sigma^{-1}(F_l)|$, we obtain

$$\begin{aligned}|sm(R)_l| &= |(G_l(I))| + \left|\left(\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right)\right| + \left|\left(\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right)\right| + \left|\left(\sigma^{-1}(F_l)\right)\right| \\ &= 2\,|(G_l(I))| + \left|\left(\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right)\right| + \left|\left(\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right)\right|.\end{aligned}$$

Finally, there are precisely $\binom{n}{l}$ ways to construct a squarefree monomial of degree l in a polynomial ring of n variables, and thus $|sm(R)_l| = \binom{n}{l}$. Applying this to the above formula and rearranging, we find

$$\binom{n}{l} = 2\left|(G_l(I))\right| + \left|\left(\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right)\right| + \left|\left(\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right)\right|,$$

or equivalently,

$$\left|(G_l(I))\right| = \frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{d_i>l}(\sqcap^{d_i-l}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i<l}(\sqcup^{l-d_i}(G_{d_i}(I)))\right|\right),$$

the result we are looking for. □

We will use this result to prove our main theorem in the next chapter.

CHAPTER 5

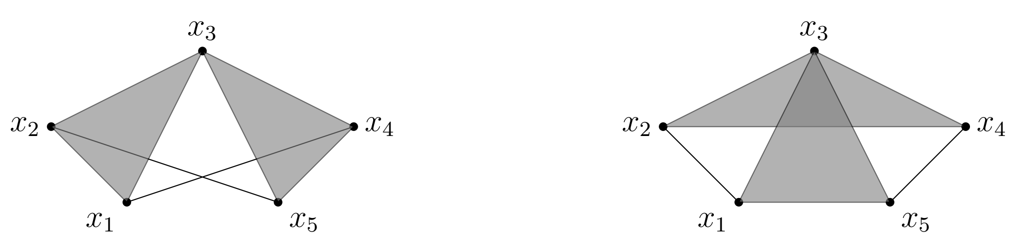
Finding New f-Ideals

In this chapter, we discuss our new results about f-ideals. We begin by providing an example of an f-ideal generated in mixed degree, and state various results that we found regarding properties of f-ideals. In particular, we show that f-ideals in the polynomial ring $R = k[x_1, \ldots, x_n]$ cannot contain a generator of degree 1 or n, and we further classify f-ideals in $R = k[x_1, \ldots, x_n]$ for $n \leqslant 4$. We follow these results by describing some algorithms for constructing f-ideals, given by Guo and Wu [**10**]. We then introduce the complement of an ideal, and investigate the implications this has on f-ideals. In particular, we show that if I is an f-ideal, its complement must also be an f-ideal, and vice versa.

1. Non-Pure f-Ideals

Up until now, we have been looking at ideals generated by squarefree monomials all having the same degree. In this section, we find f-ideals that are generated by squarefree monomials of different degrees, that is, non-pure f-ideals, and examine some of their properties. We begin with an example:

EXAMPLE 5.1. Let $I = \langle x_1x_4, x_2x_5, x_1x_2x_3, x_3x_4x_5\rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5]$ be a squarefree monomial ideal. Then the following facet and non-face complexes are associated with I.



$\delta_{\mathcal{F}}(I)$ $\delta_{\mathcal{N}}(I)$

FIGURE 1. Facet and non-face complex of the squarefree monomial ideal $I = \langle x_1x_4, x_2x_5, x_1x_2x_3, x_3x_4x_5\rangle$.

It is easy to see that $f(\delta_{\mathcal{F}}(I)) = f(\delta_{\mathcal{N}}(I)) = (5, 8, 2)$, and therefore I is an f-ideal.

We can also show that the ideal from the previous example is an f-ideal using Theorem 4.20. We do this next.

EXAMPLE 5.2. Consider the ideal

$$I = \langle x_1x_4, x_2x_5, x_1x_2x_3, x_3x_4x_5\rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5]$$

from the previous example. We can verify that I is in fact an f-ideal using Theorem 4.20. We first observe that $G(I) = G_2(I) \cup G_3(I) = \{x_1x_4, x_2x_5\} \cup \{x_1x_2x_3, x_3x_4x_5\}$. We also note that $|G_2(I)| = 2$ and $|G_3(I)| = 2$. Working with each l in $\{1,\dots, 5\}$, we compute the cardinality of $G_1(I)$ through $G_5(I)$ to show that I is an f-ideal.

Letting $l = 1$, we have

$$|G_1(I)| = \frac{1}{2}\left(\binom{5}{1} - \left|\bigcup_{d_i>1}(\sqcap^{d_i-1}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i<1}(\sqcup^{1-d_i}(G_{d_i}(I)))\right|\right).$$

Since there is no $d_i < 1$ we have $\bigcup_{d_i<1}(\sqcup^{1-d_i}(G_{d_i}(I))) = \varnothing$, thus $\left|\bigcup_{d_i<1}(\sqcup^{1-d_i}(G_{d_i}(I)))\right| = 0$. Furthermore,

$$\left|\bigcup_{d_i>1}(\sqcap^{d_i-1}(G_{d_i}(I)))\right| = \left|\sqcap^{2-1}(G_2(I)) \cup \sqcap^{3-1}(G_3(I))\right| = |\{x_1, x_2, x_3, x_4, x_5\}| = 5.$$

Therefore,

$$|G_1(I)| = \frac{1}{2}\left(\binom{5}{1} - \left|\bigcup_{d_i>1}(\sqcap^{d_i-1}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i<1}(\sqcup^{1-d_i}(G_{d_i}(I)))\right|\right) = \frac{1}{2}(5-5-0) = 0.$$

For $l = 2$, we have $|G_2(I)| = \frac{1}{2}\left(\binom{5}{2} - \left|\bigcup_{d_i>2}(\sqcap^{d_i-2}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i<2}(\sqcup^{2-d_i}(G_{d_i}(I)))\right|\right)$. Once again, there is no $d_i < 2$ and hence $\left|\bigcup_{d_i<2}(\sqcup^{2-d_i}(G_{d_i}(I)))\right| = 0$. Moreover,

$$\left|\bigcup_{d_i>2}(\sqcap^{d_i-2}(G_{d_i}(I)))\right| = \left|\sqcap^{3-2}(G_3(I))\right| = |\{x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_4x_5\}| = 6,$$

thus $|G_2(I)| = \frac{1}{2}\left(\binom{5}{2} - 6 - 0\right) = \frac{1}{2}(10-6) = 2$, which does indeed agree with our previous observation.

When $l = 3$, we find that $\left|\bigcup_{d_i>3}(\sqcap^{d_i-3}(G_{d_i}(I)))\right| = 0$ since there is no $d_i > 3$. Furthermore,

$$\begin{aligned}\left|\bigcup_{d_i<3}(\sqcup^{3-d_i}(G_{d_i}(I)))\right| &= |\sqcup(G_2(I))| \\ &= |\{x_1x_2x_4, x_1x_3x_4, x_1x_4x_5, x_1x_2x_5, x_2x_3x_5, x_2x_4x_5\}| = 6,\end{aligned}$$

hence,

$$\begin{aligned}|G_3(I)| &= \frac{1}{2}\left(\binom{5}{3} - \left|\bigcup_{d_i>3}(\sqcap^{d_i-3}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i<3}(\sqcup^{3-d_i}(G_{d_i}(I)))\right|\right) \\ &= \frac{1}{2}\left(\binom{5}{3} - 0 - 6\right) = \frac{1}{2}(10-6) = 2.\end{aligned}$$

For $l = 4$, we have $\left|\bigcup_{d_i>4}(\sqcap^{d_i-4}(G_{d_i}(I)))\right| = 0$, and

$$\begin{aligned}\left|\bigcup_{d_i<4}(\sqcup^{4-d_i}(G_{d_i}(I)))\right| &= \left|\sqcup(G_3(I)) \cup \sqcup^2(G_2(I))\right| \\ &= |\{x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_3x_4x_5, x_2x_3x_4x_5, x_1x_2x_4x_5\}| = 5.\end{aligned}$$

Therefore, $|G_4(I)| = \frac{1}{2}\left(\binom{5}{4} - 0 - 5\right) = \frac{1}{2}(5-5) = 0$.

Lastly, when $l = 5$, we find that $\left|\bigcup_{d_i>5}(\sqcap^{d_i-5}(G_{d_i}(I)))\right| = 0$, and

$$\left|\bigcup_{d_i<5}(\sqcup^{5-d_i}(G_{d_i}(I)))\right| = \left|\sqcup^2(G_3(I)) \cup \sqcup^3(G_2(I))\right| = |\{x_1x_2x_3x_4x_5\}| = 1.$$

Thus, we find $|G_5(I)| = \frac{1}{2}\left(\binom{5}{5} - 0 - 1\right) = \frac{1}{2}(1-1) = 0$. We have therefore shown that $|G_1(I)| = 0$, $|G_2(I)| = 2$, $|G_3(I)| = 2$, $|G_4(I)| = 0$, and $|G_5(I)| = 0$ through the above computations, and since these numbers agree with the respective number of generators of each degree for I, we have once again shown that I is an f-ideal.

Lemma 5.3. *Let $I = \langle g_1, g_2, \ldots, g_p\rangle \subseteq R = k[x_1, x_2, \ldots, x_n]$ be a squarefree monomial ideal. If I contains at least one monomial of degree 1, then I cannot be an f-ideal. In other words, if I is an f-ideal,* $\deg(g_i) > 1$ *for all i.*

Proof. Let I be an f-ideal and suppose that I contains a monomial of degree 1, say g_k. Then each face of the non-face complex will not be divisible by g_k by definition. But since $g_k \mid g_k$, then $\delta_{\mathcal{N}}(I)$ cannot contain g_k or any faces containing g_k. But this implies that $\delta_{\mathcal{N}}(I)$ will contain one less vertex than $\delta_{\mathcal{F}}(I)$, and hence the f-vectors will not be the same for both the facet and the non-face complex. This is a contradiction as I is an f-ideal. Hence there are no monomials of degree 1 in I. $\square$

Lemma 5.4. *Let $I = \langle g_1, g_2, \ldots, g_p\rangle \subseteq R = k[x_1, x_2, \ldots, x_n]$ be a squarefree monomial ideal. If $g_i = x_1x_2\cdots x_n \in I$ for some $1 \leqslant i \leqslant p$, i.e., $I = \langle x_1\cdots x_n\rangle$, then I cannot be an f-ideal. In other words, I cannot be generated by a squarefree monomial of degree n.*

Proof. Since by definition $\{g_1, \ldots, g_p\}$ is a minimal generating set for I, then if there exists a monomial generator $g_i = x_1x_2\cdots x_n$ for some $i \in \{1, \ldots, p\}$, then we must have that $I = \langle x_1x_2\cdots x_n\rangle$. But then $f_{n-1} = 1$ in $f(\delta_{\mathcal{F}}(I))$. Since $x_1x_2\cdots x_n \in I$ and $I \subseteq R = k[x_1, x_2, \ldots, x_n]$, then $\delta_{\mathcal{N}}(I)$ will only contain faces of dimension strictly less than $n-1$. But this implies that $f_{n-1} = 0$ in $f(\delta_{\mathcal{N}}(I))$, and hence $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ have different f-vectors. From this we conclude that I is not an f-ideal. $\square$

Theorem 5.5. *Let $R = k[x_1, x_2, x_3, x_4]$. Then R contains precisely 12 f-ideals, all of which are generated in degree 2. As a consequence, the polynomial ring R contains no f-ideals generated in mixed degree.*

Proof. By a brute force count, out of the 32,767 squarefree monomial ideals found in $R = k[x_1, x_2, x_3, x_4]$, we find that only 12 of them are actually f-ideals (See Appendix

A for a list of all twelve of these f-ideals). Because we found that there were 12 f-ideals generated in degree 2 in Chapter 6, then we can conclude that there are no f-ideals generated by mixed degree in the polynomial ring $R = k[x_1, x_2, x_3, x_4]$. $\square$

LEMMA 5.6. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ for $n \leqslant 3$ be a squarefree monomial ideal. Then I cannot be an f-ideal.*

PROOF. We must consider f-ideals in polynomial rings $R = k[x_1, \ldots, x_i]$ for $i = 1, 2, 3$. For ideals $I \subseteq R = k[x_1]$ then we can only have squarefree monomials of degree 1 or 0. If $I = \langle x_1 \rangle$, then by Lemma 5.3, I cannot be an f-ideal as it contains a squarefree monomial of degree 1.

For the case of $R = k[x_1, x_2]$, then the only possible squarefree monomials (up to permutation of vertices) are $I = \langle x_1 \rangle$, $I = \langle x_1, x_2 \rangle$, and $I = \langle x_1 x_2 \rangle$. But by Lemma 5.3, the first and second case cannot be f-ideals, and by Lemma 5.4, the third case cannot be an f-ideal.

For squarefree monomial ideals in $R = k[x_1, x_2, x_3]$, then Lemma 5.3 implies that we can eliminate all squarefree monomials containing at least one monomial of degree 1, and Lemma 5.4 allows us to eliminate the case when $I = \langle x_1 x_2 x_3 \rangle$. We are therefore left with two cases up to permutation of vertices, namely, when $I = \langle x_1 x_2, x_1 x_3 \rangle$ and $I = \langle x_1 x_2, x_1 x_3, x_2 x_3 \rangle$. If $I = \langle x_1 x_2, x_1 x_3 \rangle$, then $f(\delta_{\mathcal{F}}(I)) = (3, 2)$ while $f(\delta_{\mathcal{N}}(I)) = (3, 1)$, and thus I is not an f-ideal. Lastly, if $I = \langle x_1 x_2, x_1 x_3, x_2 x_3 \rangle$, then $f(\delta_{\mathcal{F}}(I)) = (3, 3)$ while $f(\delta_{\mathcal{N}}(I)) = (3, 0)$. Hence, I cannot be an f-ideal. $\square$

THEOREM 5.7. *Let $I \subseteq R = k[x_1, x_2, x_3, x_4]$ be a squarefree monomial ideal. If I is an f-ideal, then I must have all of its generators in the same degree. As a consequence, there are no f-ideals generated in mixed degree in the polynomial ring $R = k[x_1, \ldots, x_n]$ for $n \leqslant 4$.*

PROOF. Let $R = k[x_1, x_2, x_3, x_4]$. By a count of all f-ideals in $R = k[x_1, x_2, x_3, x_4]$ using Macaulay2 [**7**], we know there are only 12 such ideals. Since we know that there are 12 f-ideals generated in degree 2 in $R = k[x_1, x_2, x_3, x_4]$ by Table 2 in Chapter 6, then we can conclude that there are no f-ideals in $R = k[x_1, x_2, x_3, x_4]$ generated in mixed degree. By Lemma 5.6, it follows directly that there are no f-ideals generated in mixed degree in the polynomial ring $R = k[x_1, \ldots, x_n]$ for $n \leqslant 4$. $\square$

2. Creating New f-Ideals

In this section, we discuss the ability to construct an f-ideal. To begin, we outline two methods given by Guo and Wu [**10**] for constructing f-ideals generated in degree 2.

ALGORITHM 5.8 (J. Guo, T. Wu, [**10**]). *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring and let $[n] = \{1, \ldots, n\}$. Furthermore, let S be a subset of $[n]$, and $\bar{S} = [n] \backslash S$, the complement of S in $[n]$, with $||S| - |\bar{S}|| \leqslant 1$.*

Step 1: Construct the set X as follows:

$$X \coloneqq \{x_i x_j \mid i, j \in S \text{ or } i, j \in \bar{S}\}.$$

Step 2: Set $|S| = m_1$ and $|\bar{S}| = m_2$ and compute

$$m \coloneqq \frac{1}{2}\binom{n}{2} - \left(\frac{m_1(m_1-1) + m_2(m_2-1)}{2}\right).$$

Step 3: Construct a set $\tilde{X}$ such that:

$$\tilde{X} \subseteq \{x_i x_j \mid i \in S \text{ and } j \in \bar{S}\} \text{ with } |\tilde{X}| = m.$$

Step 4: Construct the ideal I as follows:

$$I = \langle X \cup \tilde{X} \rangle.$$

The ideal generated in Step 4 is an f-ideal.

In the above proposition, note that the choice of $x_i x_j$'s in $\tilde{X}$ was arbitrary, as long as we choose exactly m such squarefree monomials with $i \in S$ and $j \in \bar{S}$. Another way to go through this process is given below:

Algorithm 5.9. *To construct an f-ideal $I \subseteq R = k[x_1, \ldots, x_n]$ generated in degree 2 in n-variables, consider the following steps.*
Step 1: Construct the sets X and $\bar{X}$ such that

$$X \subseteq \{x_1, \ldots, x_n\} \text{ and } \bar{X} = \{x_1, \ldots, x_n\} \backslash X \text{ with } ||X| - |\bar{X}|| \leqslant 1.$$

Step 2: Set $|X| = m_1$ and $|\bar{X}| = m_2$, and compute m as follows:

$$m = \frac{1}{2}\binom{n}{2} - \left(\frac{m_1(m_1-1) + m_2(m_2-1)}{2}\right),$$

Step 3: Construct complete graphs K_{m_1} and K_{m_2} and connect K_{m_1} and K_{m_2} using m edges. Label the resulting graph (simplicial complex) as G.
Step 4: Compute the facet ideal of G. The resulting ideal will be an f-ideal.

The above algorithms provide a straight-forward method of constructing f-ideals generated in degree 2, inside large polynomial rings. To illustrate this, we look at an example in $R = k[x_1, \ldots, x_8]$.

Example 5.10. Suppose we wish to find a quadratic f-ideal in $R = k[x_1, \ldots, x_8]$. We first construct two sets, namely X and $\bar{X}$. We choose

$$X = \{x_1, x_2, x_7, x_8\} \text{ and } \bar{X} = \{x_3, x_4, x_5, x_6\}.$$

From here, we construct two K_4 complete graphs, labelling the vertices with $x_1, \ldots, x_8$.

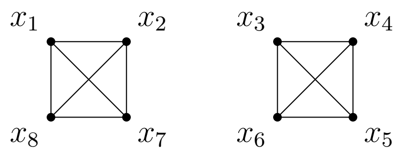


FIGURE 2. Complete graphs K_4.

Since $m_1 = m_2 = 4$, then $m = \frac{1}{2}\binom{8}{2} - \left(\frac{4(4-1)+4(4-1)}{2}\right) = 2$. We therefore need to connect the two K_4 graphs using $m = 2$ edges. We choose to add edges $\{x_2, x_3\}$ and $\{x_6, x_7\}$, and hence we obtain

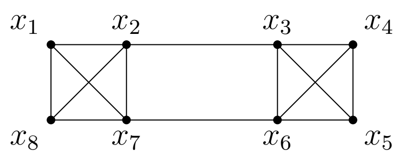


FIGURE 3. $G = K_4 \cup K_4 \cup \{x_2, x_3\} \cup \{x_6, x_7\}$.

Computing the facet (edge) ideal $I_{\mathcal{F}}$ of the graph G, we obtain

$$I_{\mathcal{F}} = \langle x_1x_2, x_1x_7, x_1x_8, x_2x_3, x_2x_7, x_2x_8, x_3x_4,$$
$$x_3x_5, x_3x_6, x_4x_5, x_4x_6, x_5x_6, x_6x_7, x_7x_8\rangle.$$

The associated facet and non-face complexes of $I_{\mathcal{F}}$ are given below.

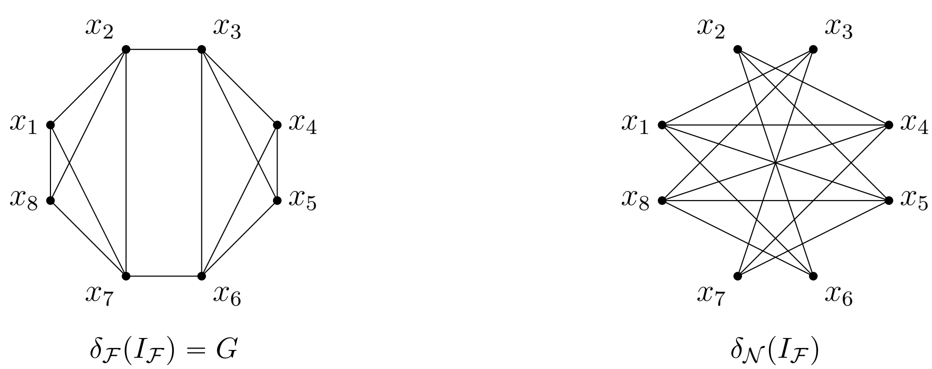


FIGURE 4. Facet and non-face complex of the squarefree monomial ideal I.

We observe that $f(\delta_{\mathcal{F}}(I_{\mathcal{F}})) = f(\delta_{\mathcal{N}}(I_{\mathcal{F}})) = (8, 14)$, and hence $I_{\mathcal{F}}$ is an f-ideal.

3. Ideals and Their Complements

In this section we examine the complement of an ideal, and some of the implications this has on f-ideals. We begin by defining the generalized Newton complementary dual of an ideal, originally given by Ansaldi et al. [**2**], based upon ideas of Costa and Simis [**4**]. We follow this with some examples.

Definition 5.11. Let $I \subseteq R = k[x_1, \dots, x_n]$ with $G(I) = \{g_1, \dots, g_p\}$ a minimal generating set of I and $g_i = x_i^{\alpha_i} = x_{i,1}^{\alpha_{i,1}} \cdots x_{i,j}^{\alpha_{i,j}}$ for all $i = 1, \dots, p$. Furthermore, set $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ with $\beta_k \geqslant \alpha_{i,k}$ for all $k = 1, \dots, n$ and all $i = 1, \dots, p$. Then the ideal $\hat{I}^{[\beta]}$ with generating set

$$\left\{\hat{g}_i = \frac{x^\beta}{g_i} \mid g_i \in G\right\} = \left\{\frac{x^\beta}{g_1}, \frac{x^\beta}{g_2}, \dots, \frac{x^\beta}{g_p}\right\}$$

is called the *generalized Newton complementary dual of* I determined by β.

Remark 5.12. Let $I = \langle g_1, g_2, \dots, g_p\rangle \subseteq R = k[x_1, \dots, x_n]$ be a squarefree monomial ideal and let $\hat{I}^{[1]}$ be the generalized Newton complementary dual of I determined by $\beta = (1, 1, \dots, 1)$. Then we obtain a complementary squarefree monomial ideal generated by squarefree monomials $\hat{g}_1, \hat{g}_2, \dots, \hat{g}_p$. For simplicity, we will denote $\hat{I}^{[1]}$ by $\hat{I}$ and call it the *complement* of I.

Example 5.13. Consider the ideals $I_1 = \langle xz^4, x^3y^7z^2, y^3z\rangle$ and $I_2 = \langle xy, xz, yz\rangle$ in $k[x, y, z]$. Then if we set $\beta_1 = (4, 7, 5)$ and $\beta_2 = (1, 1, 1)$ we obtain

$$\hat{I_1}^{[\beta_1]} = \left\langle \frac{x^4y^7z^5}{xz^4}, \frac{x^4y^7z^5}{x^3y^7z^2}, \frac{x^4y^7z^5}{y^3z} \right\rangle = \langle x^3y^7z, xz^3, x^4y^4z^4\rangle,$$

and

$$\hat{I_2}^{[\beta_2]} = \hat{I_2} = \left\langle \frac{xyz}{xy}, \frac{xyz}{xz}, \frac{xyz}{yz} \right\rangle = \langle z, y, x\rangle.$$

As a major motivation for the section that follows, we are interested in the relationship between I and its complement, $\hat{I}$. In particular, we examine various cases in which we compute the complement of an f-ideal I, and test whether $\hat{I}$ is also an f-ideal. These examples are presented below:

Example 5.14. Consider the f-ideal from Example 3.11:

$$I = \langle x_1x_2, x_2x_5, x_3x_4, x_3x_5, x_4x_5\rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5].$$

We can construct another ideal from I by forming the complement, $\hat{I}$.

$$\begin{array}{rcccccl} I = \langle & x_1x_2, & x_2x_5, & x_3x_4, & x_3x_5, & x_4x_5 & \rangle \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \hat{I} = \langle & x_3x_4x_5, & x_1x_3x_4, & x_1x_2x_5, & x_1x_2x_4, & x_1x_2x_3 & \rangle. \end{array}$$

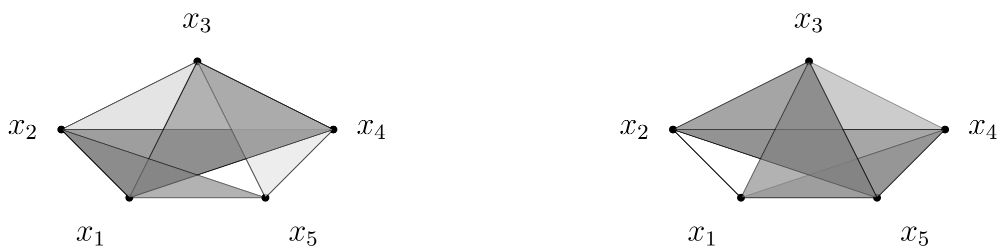
Finding the facet and the non-face complex of $\hat{I}$ yields

$$\delta_{\mathcal{F}}(\hat{I}) = \langle \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_3, x_4, x_5\}\rangle,$$

and,

$$\delta_{\mathcal{N}}(\hat{I}) = \langle \{x_2, x_4, x_5\}, \{x_1, x_4, x_5\}, \{x_2, x_3, x_5\}, \{x_1, x_3, x_5\}, \{x_2, x_3, x_4\}, \{x_1, x_2\}\rangle.$$

These simplicial complexes are illustrated below:



$\delta_{\mathcal{F}}(\hat{I})$ $\delta_{\mathcal{N}}(\hat{I})$

FIGURE 5. Facet and non-face complex of the squarefree monomial ideal $\hat{I} = \langle x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_3x_4x_5 \rangle$.

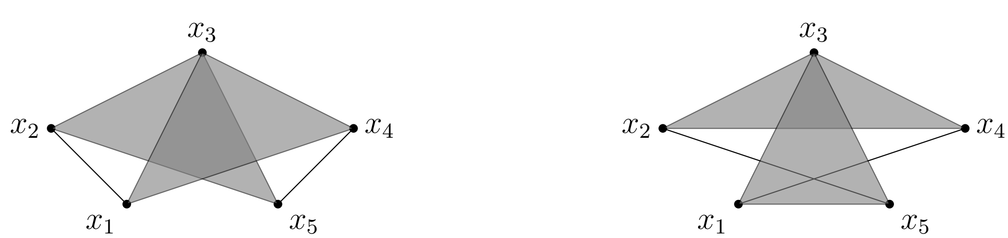
Computing the f-vectors of the facet and non-face complex, we see that $f(\delta_{\mathcal{F}}(\hat{I})) = f(\delta_{\mathcal{N}}(\hat{I})) = (5, 10, 5)$, and hence $\hat{I}$ is indeed an f-ideal.

Being able to construct another f-ideal using a given f-ideal was an unexpected result, and is something we examine further throughout the remainder of this project. As a result, one may also ask whether this works for f-ideals generated in mixed degree. For an ideal in $k[x_1, \ldots, x_5]$, we provide an example that illustrates that it does.

EXAMPLE 5.15. Consider the f-ideal generated in mixed degree from Example 5.1. We compute its complement:

$$\begin{array}{rccccc} I = \langle & x_1x_4, & x_2x_5, & x_1x_2x_3, & x_3x_4x_5 & \rangle \\ & \downarrow & \downarrow & \downarrow & \downarrow & \\ \hat{I} = \langle & x_2x_3x_5, & x_1x_3x_4, & x_4x_5, & x_1x_2 & \rangle. \end{array}$$

Constructing the facet and the non-face complex of $\hat{I}$, we obtain:



$\delta_{\mathcal{F}}(\hat{I})$ $\delta_{\mathcal{N}}(\hat{I})$

FIGURE 6. Facet and non-face complex of the squarefree monomial ideal $\hat{I} = \langle x_1x_2, x_4x_5, x_1x_3x_4, x_2x_3x_5 \rangle$.

The f-vectors of both $\delta_{\mathcal{F}}(\hat{I})$ and $\delta_{\mathcal{N}}(\hat{I})$ are $(5, 8, 2)$, and therefore $\hat{I}$ is an f-ideal.

We now consider one last example for a quadratic f-ideal in $R = k[x_1, \ldots, x_8]$.

Example 5.16. Consider the following quadratic f-ideal from Example 5.10:

$$I = \langle x_1x_2, x_1x_7, x_1x_8, x_2x_3, x_2x_7, x_2x_8, x_3x_4, x_3x_5, x_3x_6, x_4x_5, x_4x_6, x_5x_6, x_6x_7, x_7x_8\rangle.$$

Constructing the complement ideal,

$$\hat{I} = \langle x_3x_4x_5x_6x_7x_8, x_1x_4x_5x_6x_7x_8, x_1x_2x_5x_6x_7x_8, x_1x_2x_4x_6x_7x_8, x_1x_2x_3x_6x_7x_8, x_1x_2x_4x_5x_7x_8, x_1x_2x_3x_5x_7x_8, x_1x_2x_3x_4x_7x_8, x_2x_3x_4x_5x_6x_8, x_1x_3x_4x_5x_6x_8, x_1x_2x_3x_4x_5x_8, x_2x_3x_4x_5x_6x_7, x_1x_3x_4x_5x_6x_7, x_1x_2x_3x_4x_5x_6\rangle,$$

we find that $f(\delta_{\mathcal{F}}(\hat{I})) = f(\delta_{\mathcal{N}}(\hat{I})) = (8, 28, 56, 70, 56, 14)$, and hence $\hat{I}$ is an f-ideal.

4. Complementary f-Ideals

In this section, we introduce the main result of this paper. In order to prove this result, we begin with several lemmas, one of which is crucial in the proof of the main theorem.

Lemma 5.17. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring and let S be a subset of $sm(R)_d$. Furthermore, set $\hat{S} = \{m/s \mid s \in S\}$ with $m = x_1x_2 \ldots x_n$. Then the following conditions hold:*

(i) S is a lower perfect subset of $sm(R)_d$ if and only if $\hat{S}$ is an upper perfect subset of $sm(R)_{n-d}$
(ii) S is an upper perfect subset of $sm(R)_d$ if and only if $\hat{S}$ is a lower perfect subset of $sm(R)_{n-d}$

Proof. See Lemma 3.1 of [**10**] for details of this proof. □

Lemma 5.18. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal with minimal generating set $G(I) = \bigcup_{i=1}^{k} G_{d_i}(I)$. Then for each $l \in \{1, 2, \ldots, n\}$,*

(i) $\left|\bigcup_{d_i<l} \left(\sqcap^{l-d_i} G_{n-d_i}(I)\right)\right| = \left|\bigcup_{d_i<l} \left(\sqcup^{l-d_i} G_{d_i}(\hat{I})\right)\right|$, and
(ii) $\left|\bigcup_{d_i>l} \left(\sqcup^{d_i-l} G_{n-d_i}(I)\right)\right| = \left|\bigcup_{d_i>l} \left(\sqcap^{d_i-l} G_{d_i}(\hat{I})\right)\right|$.

Proof. (i) To begin, we define a map φ as follows:

$$\varphi : \bigcup_{d_i<l} \left(\sqcup^{l-d_i} G_{d_i}(I)\right) \longrightarrow \bigcup_{l<d_i} \left(\sqcap^{l-d_i} G_{n-d_i}(\hat{I})\right)$$
$$m \longmapsto \frac{x_1 \cdots x_n}{m}.$$

We first ensure this map is well-defined. Let m be a squarefree monomial with $m \in \bigcup_{d_i<l} \left(\sqcup^{l-d_i} G_{d_i}(I)\right)$. Then there exists a $d_j < l$ such that $m \in \sqcup^{l-d_j}(G_{d_j}(I))$. Then by definition, there exists an $m' \in G_{d_j}(I)$ and a squarefree monomial g, with $\deg(g) = l - d_j$

and $\gcd(g, m') = 1$ such that $m = m'g$. Because we have $m' \in G_{d_j}(I)$, then $\frac{x_1 \cdots x_n}{m'} \in G_{n-d_j}(\hat{I})$, and hence $g \mid \frac{x_1 \cdots x_n}{m'}$ implies $\frac{x_1 \cdots x_n}{m'g} \in \sqcap^{l-d_j}(G_{n-d_j}(\hat{I}))$ since $\deg(g) = l - d_j$. We therefore have that

$$\varphi(m) = \frac{x_1 \cdots x_n}{m} = \frac{x_1 \cdots x_n}{m'g} \in \bigcup_{d_i<l} (\sqcap^{l-d_j}(G_{n-d_j}(\hat{I})))$$

and hence, φ is well-defined.

For injectivity, it is clear that

$$\varphi(\alpha) = \varphi(\beta) \quad \Rightarrow \quad \frac{x_1 \cdots x_n}{\alpha} = \frac{x_1 \cdots x_n}{\beta} \quad \Rightarrow \quad \alpha = \beta,$$

and hence φ is injective.

Lastly, we check that φ is surjective. To begin, let $u \in \bigcup_{d_i<l}(\sqcap^{l-d_i}(G_{n-d_i}(\hat{I})))$. Then there exists a $d_k < l$ such that $u \in \sqcap^{l-d_k}(G_{n-d_k}(\hat{I}))$. But then there must be a squarefree monomial g, with $\deg(g) = l - d_k$, such that $ug \in G_{n-d_k}(\hat{I})$ and $\gcd(g, u) = 1$. This implies that $\frac{x_1 \cdots x_n}{ug} \in G_{d_k}(I)$. Multiplying by g, we find

$$g \cdot \frac{x_1 \cdots x_n}{ug} = \frac{x_1 \cdots x_n}{u} \in \sqcup^{l-d_k} G_{d_k}(I) \subseteq \bigcup_{d_i<l} (\sqcup^{l-d_i} G_{d_i}(I)).$$

But from this we see that

$$\varphi\left(\frac{x_1 \cdots x_n}{u}\right) = \frac{x_1 \cdots x_n}{\frac{x_1 \cdots x_n}{u}} = u.$$

We therefore have that φ is a well-defined bijective map. Since we have a bijection between the two finite sets $\bigcup_{d_i<l}\left(\sqcup^{l-d_i} G_{d_i}(I)\right)$ and $\bigcup_{d_i<l}\left(\sqcap^{l-d_i} G_{n-d_i}(\hat{I})\right)$, then

$$\left|\bigcup_{d_i<l}\left(\sqcup^{l-d_i} G_{d_i}(I)\right)\right| = \left|\bigcup_{d_i<l}\left(\sqcap^{l-d_i} G_{n-d_i}(\hat{I})\right)\right|.$$

Lastly, since $\hat{\hat{I}} = I$, then

$$\left|\bigcup_{d_i<l}\left(\sqcup^{l-d_i} G_{d_i}(\hat{I})\right)\right| = \left|\bigcup_{d_i<l}\left(\sqcap^{l-d_i} G_{n-d_i}(I)\right)\right|.$$

(ii) We now define a map ψ as follows:

$$\psi : \bigcup_{d_i>l}\left(\sqcap^{d_i-l} G_{d_i}(I)\right) \longrightarrow \bigcup_{d_i>l}\left(\sqcup^{d_i-l} G_{n-d_i}(\hat{I})\right)$$
$$m \longmapsto \frac{x_1 \cdots x_n}{m}.$$

Once again, we ensure this map is well-defined. Let m be a squarefree monomial with $m \in \bigcup_{d_i>l}\left(\sqcap^{d_i-l} G_{d_i}(I)\right)$. Then there exists a $d_j > l$ such that $m \in \sqcap^{d_j-l}(G_{d_j}(I))$. But by definition, there exists an $m' \in G_{d_j}(I)$ and a squarefree monomial g, with $\deg(g) = d_j - l$

and $g|m'$ such that $m = m'/g$. But $m' \in G_{d_j}(I)$ implies that $\frac{x_1\cdots x_n}{m'} \in G_{n-d_j}(\hat{I})$, and since $\deg(g) = d_j - l$, then $g \cdot \frac{x_1\cdots x_n}{m'} \in \sqcup^{d_j-l}G_{n-d_j}(\hat{I})$. We therefore have that

$$\psi(m) = \frac{x_1\cdots x_n}{m} = \frac{x_1\cdots x_n}{m'/g} = g\cdot\frac{x_1\cdots x_n}{m'} \in \bigcup_{d_i>l}(\sqcup^{d_i-l}(G_{n-d_i}(\hat{I})))$$

and hence ψ is well-defined.

For injectivity, we again see that

$$\psi(\alpha) = \psi(\beta) \quad\Rightarrow\quad \frac{x_1\cdots x_n}{\alpha} = \frac{x_1\cdots x_n}{\beta} \quad\Rightarrow\quad \alpha = \beta,$$

and hence ψ is injective.

Lastly, we check that ψ is surjective. To begin, let $u \in \bigcup_{d_i>l}\left(\sqcup^{d_i-l}G_{n-d_i}(\hat{I})\right)$. Then there exists a $d_k > l$ such that $u \in \sqcup^{d_k-l}(G_{n-d_k}(\hat{I}))$. But then there must be $v \in G_{n-d_k}(\hat{I})$ and a squarefree monomial g, with $\deg(g) = d_k - l$, such that $u = vg$ with $\gcd(g,v) = 1$. Since $v \in G_{n-d_k}(\hat{I})$, then $\frac{x_1\cdots x_n}{v} \in G_{d_k}(I)$. But since $\gcd(g,v) = 1$, then dividing by g, we obtain

$$\frac{x_1\cdots x_n}{vg} = \frac{x_1\cdots x_n}{u} \in \sqcap^{d_k-l}(G_{d_k}(I)) \subseteq \bigcup_{d_i>l}\left(\sqcap^{d_i-l}G_{d_i}(I)\right).$$

From this, we see that

$$\psi\left(\frac{x_1\cdots x_n}{u}\right) = \frac{x_1\cdots x_n}{\frac{x_1\cdots x_n}{u}} = u.$$

We therefore have that ψ is a well-defined bijective map. Since we have a bijection between the two finite sets $\bigcup_{d_i>l}\left(\sqcap^{d_i-l}G_{d_i}(I)\right)$ and $\bigcup_{d_i>l}\left(\sqcup^{d_i-l}G_{n-d_i}(\hat{I})\right)$, we can conclude that

$$\left|\bigcup_{d_i>l}\left(\sqcap^{d_i-l}G_{d_i}(I)\right)\right| = \left|\bigcup_{d_i>l}\left(\sqcup^{d_i-l}G_{n-d_i}(\hat{I})\right)\right|.$$

Once again, since $\hat{\hat{I}} = I$, then we have that

$$\left|\bigcup_{d_i>l}\left(\sqcap^{d_i-l}G_{d_i}(\hat{I})\right)\right| = \left|\bigcup_{d_i>l}\left(\sqcup^{d_i-l}G_{n-d_i}(I)\right)\right|,$$

thus completing the proof. □

THEOREM 5.19. *Let $I \subseteq R = k[x_1,\ldots,x_n]$ be a squarefree monomial ideal. Then I is an f-ideal if and only if $\hat{I}$ is an f-ideal.*

PROOF. ($\Rightarrow$) Let $I \subseteq R = k[x_1,\ldots,x_n]$ be an f-ideal and let $G_{d_i}(I)$ be the subset of minimal generators of I of degree d_i. We first observe that $|G_l(\hat{I})| = |G_{n-l}(I)|$ for each $l \in \{1,\ldots,n\}$, and since I is an f-ideal, then by Theorem 4.20, we know that for each

$l \in \{1, \ldots, n\}$, we have that

$$\left|G_l(\hat{I})\right| = |G_{n-l}(I)|$$
$$= \frac{1}{2}\left(\binom{n}{n-l} - \left|\bigcup_{d_i > n-l}(\sqcap^{d_i-(n-l)}(G_{d_i}(I)))\right| - \left|\bigcup_{d_i < n-l}(\sqcup^{(n-l)-d_i}(G_{d_i}(I)))\right|\right).$$

Using the fact that $\binom{n}{n-l} = \frac{n!}{(n-l)!(n-(n-l))!} = \frac{n!}{l!(n-l)!} = \binom{n}{l}$ and rearranging indices, the above expression is equal to

$$\frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{l > n-d_i}(\sqcap^{l-(n-d_i)}(G_{d_i}(I)))\right| - \left|\bigcup_{l < n-d_i}(\sqcup^{(n-d_i)-l}(G_{d_i}(I)))\right|\right).$$

But since $d_i = n - (n - d_i)$, then Lemma 5.18 implies that

$$\left|\bigcup_{l > n-d_i}(\sqcap^{l-(n-d_i)}(G_{d_i}(I)))\right| = \left|\bigcup_{l > n-d_i}(\sqcup^{l-(n-d_i)}(G_{n-d_i}(\hat{I})))\right|,$$

and

$$\left|\bigcup_{l < n-d_i}(\sqcup^{(n-d_i)-l}(G_{d_i}(I)))\right| = \left|\bigcup_{l < n-d_i}(\sqcap^{(n-d_i)-l}(G_{n-d_i}(\hat{I})))\right|.$$

Therefore, the above formula yields

$$\left|G_l(\hat{I})\right| = \frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{l > n-d_i}(\sqcup^{l-(n-d_i)}(G_{n-d_i}(\hat{I})))\right| - \left|\bigcup_{l < n-d_i}(\sqcap^{(n-d_i)-l}(G_{n-d_i}(\hat{I})))\right|\right).$$

Since we know this formula holds for all degrees and all l, we set $e_i = n - d_i$ and substitute into the formula above to obtain

$$\left|G_l(\hat{I})\right| = \frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{e_i > l}(\sqcap^{e_i-l}(G_{e_i}(\hat{I})))\right| - \left|\bigcup_{e_i < l}(\sqcup^{l-e_i}(G_{e_i}(\hat{I})))\right|\right).$$

We have therefore shown that for each $l \in \{1, \ldots, n\}$,

$$\left|G_l(\hat{I})\right| = \frac{1}{2}\left(\binom{n}{l} - \left|\bigcup_{e_i > l}(\sqcap^{e_i-l}(G_{e_i}(\hat{I})))\right| - \left|\bigcup_{e_i < l}(\sqcup^{l-e_i}(G_{e_i}(\hat{I})))\right|\right),$$

and thus, by Theorem 4.20, $\hat{I}$ must be an f-ideal.

($\Leftarrow$) If $\hat{I}$ is an f-ideal, then it follows from above that $\hat{\hat{I}}$ is an f-ideal. But $\hat{\hat{I}} = I$, therefore I is an f-ideal, thus completing the proof. $\square$

CHAPTER 6

Results on f-Ideals

In this chapter, we discuss some of the implications of Theorem 5.19. We also perform a raw count of all f-ideals generated in degree d for various n and d.

1. Implication of Complementary f-Ideals

This section will summarize some of the implications associated to Theorem 5.19. We present various corollaries.

Corollary 6.1. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in degree d with minimal generating set $G(I) = \{g_1, \ldots, g_p\}$, and let $\hat{I}$ be the complement of I. Then $\hat{I}$ is an f-ideal if and only if $G(I)$ is $(n,d)^{th}$ perfect and $|G(I)| = \frac{1}{2}\binom{n}{d}$.*

Proof. Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in degree d with minimal generating set $G(I) = \{g_1, \ldots, g_p\}$. Then by Theorem 5.19, $\hat{I}$ is an f-ideal if and only if I is an f-ideal, but by Theorem 4.6, I is an f-ideal if and only if $G(I)$ is $(n,d)^{th}$ perfect and $|G(I)| = \frac{1}{2}\binom{n}{d}$. □

Corollary 6.2. *Let $V(n,d)$ be the set of all f-ideals generated in degree d in $R = k[x_1, \ldots, x_n]$ and let $|V(n,d)|$ denote the cardinality of $V(n,d)$. Then*

$$|V(n,d)| = |V(n,n-d)|.$$

Proof. For each f-ideal generated in degree d in $R = k[x_1, \ldots, x_n]$, Theorem 5.19 implies that there is a complementary f-ideal generated in degree $n-d$ also living in $R = k[x_1, \ldots, x_n]$. Because of this one-to-one correspondence, it follows that $|V(n,d)| = |V(n,n-d)|$. □

Corollary 6.3. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be an f-ideal. Then I cannot be generated by a squarefree monomial of degree $n-1$.*

Proof. Let $I \subseteq R = k[x_1, \ldots, x_n]$ be an f-ideal and assume that I contains at least one generator of degree $n-1$. Then by Theorem 5.19, there must be a complementary f-ideal which contains a generator of degree 1. But by Lemma 5.3, this cannot happen. From this we conclude that I cannot contain a generator of degree $n-1$. □

Guo, Wu, and Liu [**9**],[**10**] also provide several algorithms to construct f-ideals of various types. In particular, working in $R = k[x_1, \ldots, x_n]$, they break their algorithms into three cases, namely, when $d = 2$, when $d > 2$ and $n = d+2$, and lastly when

$n > d + 2$. These algorithms are seen in [**9**] on page 4, and in [**10**] as Algorithm 4.1, and 4.3, respectively.

As a result of Theorem 5.19, the algorithms mentioned above can therefore be used to construct complementary f-ideals of degree $n - d$, for any given case. In other words, for each f-ideal that can be constructed using one of the algorithms above, Theorem 5.19 can produce another f-ideal.

2. Counting f-Ideals

In this section, we perform a raw count of all f-ideals generated in degree d for a given n inside $R = k[x_1, \dots, x_n]$, where the minimal generating set contains p squarefree monomials. For example, if we let $I = \langle x_1x_2, x_2x_4, x_2x_5, x_3x_4 \rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5]$, then $n = 5$, $d = 2$, and $p = 4$.

By Theorem 3.14, we know that for each combination of n and d, there is only one value of p that will yield an f-ideal generated in degree d. Below, we calculate which values we need to check, simply to alleviate computing combinations that surely will not work. In order to calculate p, recall that $p = \frac{1}{2}\binom{n}{d}$.

n \ d	2	3	4	5	6	7	8
2	0.5*	-	-	-	-	-	-
3	1.5*	0.5*	-	-	-	-	-
4	3	2	0.5*	-	-	-	-
5	5	5	2.5*	0.5*	-	-	-
6	7.5*	10	7.5*	3	0.5*	-	-
7	10.5*	17.5*	17.5*	10.5*	3.5*	0.5*	-
8	14	28	35	28	14	4	0.5*

TABLE 1. Calculation of p for specific combinations of n and d. An asterisk (i.e. 17.5*) indicates that p is not divisible by 2 and thus cannot yield an f-ideal.

With the p values computed for each combination of n and d, we now use Macaulay2 to compute the number of f-ideals generated in degree d, with $2 \leqslant d \leqslant 8$, for each $n \in \{2, \dots, 8\}$. These results are displayed in Table 2. Moreover, Theorem 3.14 implies that any combination of n and d that produce a p value not divisible by 2 cannot yield an f-ideal. Thus, in Table 2, we have automatically filled in those cells with a zero.

$n \backslash d$	2	3	4	5	6	7	8
2	0	-	-	-	-	-	-
3	0	0	-	-	-	-	-
4	12	0	0	-	-	-	-
5	72	72	0	0	-	-	-
6	0	48,494	0	0	0	-	-
7	0	0	0	0	0	0	-
8	$29,680^{\dagger}$	?	?	?	$29,680^{\ddagger}$	0	0

TABLE 2. Number of f-ideals in the polynomial ring of n variables, generated by p squarefree monomials of degree d. † represents a number computed using Theorem 4.12, and ‡ represents a number computed using Corollary 6.2

Working in the polynomial ring $k[x_1, \ldots, x_8]$, we were unable to compute the number of f-ideals generated in degree 3, 4, and 5, due to restraints on the computation power of the computers we used.

In addition to the computations for f-ideals generated in degree d inside polynomials rings $R = k[x_1, \ldots, x_n]$ for $n \in \{2, \ldots, 8\}$, we used Macaulay2 to compute all f-ideals generated in mixed and unmixed degree in $R = k[x_1, x_2, x_3, x_4]$. Lastly, we used Macaulay2 to verify that all 72 f-ideals generated in degree 2 in $R = k[x_1, x_2, x_3, x_4, x_5]$ were complements of the 72 f-ideals generated in degree 3 in R.

CHAPTER 7

Conclusion

Throughout this project, we examined f-ideals and summarized many known results about these ideals. We focused mainly on the work done in [**1**], [**3**], [**9**], and [**10**]. In particular, we looked at the classification of f-ideals generated in degree 2 given by Abbasi et al. [**1**], as well as a generalization of this result given by Anwar et al. [**3**]. In addition to our study of these results, we also summarized a different approach to the study of f-ideals, given by Guo, Wu, and Liu in [**9**] and [**10**].

Using the generalized Newton complementary dual of an ideal, first introduced by Costa and Simis [**4**] and expanded upon by Ansaldi et al. [**2**], together with the results mentioned above, we were ultimately able to discover our main result, namely, Theorem 5.19.

For future work, we would be interested in examining the relationship between the primary decompositions of I and $\hat{I}$. In particular, we found that given an unmixed f-ideal I, its complement $\hat{I}$ will not necessarily be unmixed.

Example 7.1. Consider the f-ideal from Example 3.11:

$$I = \langle x_1x_2, x_2x_5, x_3x_4, x_3x_5, x_4x_5\rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5].$$

We know from Example 3.11 that I has the following primary decomposition:

$$I = \langle x_1, x_3, x_5\rangle \cap \langle x_1, x_4, x_5\rangle \cap \langle x_2, x_3, x_4\rangle \cap \langle x_2, x_3, x_5\rangle \cap \langle x_2, x_4, x_5\rangle.$$

Furthermore, we know the complement of I, namely, $\hat{I}$, from Example 5.14. We can compute the primary decomposition of $\hat{I}$ to obtain

$$\begin{aligned}\hat{I} &= \langle x_3x_4x_5, x_1x_3x_4, x_1x_2x_5, x_1x_2x_4, x_1x_2x_3\rangle \\ &= \langle x_1, x_3\rangle \cap \langle x_1, x_4\rangle \cap \langle x_1, x_5\rangle \cap \langle x_2, x_3\rangle \cap \langle x_2, x_4\rangle \cap \langle x_3, x_4, x_5\rangle.\end{aligned}$$

From above, it is clear that I is unmixed, but $\hat{I}$ is not, and hence I unmixed does not imply that $\hat{I}$ is unmixed.

The above problem would serve as a good point of departure for future investigation. In general, we end with several questions:

Question 7.2. *How are the primary decompositions of I and $\hat{I}$ related?*

Question 7.3. *How many f-ideals generated in mixed degree exist in $k[x_1, \ldots, x_n]$ for various n?*

Question 7.4. *Are there instances in which $I = \hat{I}$?*

APPENDIX A

Macaulay2 Code

Below is the Macaulay2 code used to compute all f-ideals generated in degree 2 in the polynomial ring $R = k[x_1, x_2, x_3, x_4]$:

```
loadPackage "SimplicialComplexes"
loadPackage "EdgeIdeals"
loadPackage "SimplicialDecomposability"

-- This computes all f-ideals of degree 2 in the polynomial ring of
-- 4 variables generated by 3 generators. It outputs a list of all such
-- f-ideals as "lp". There are 12 of these ideals.


-- **************** n = 4 ******** Degree 2 ********************************


R = QQ[x_1,x_2,x_3,x_4]


lprime = {x_1,x_2,x_3,x_4}

l = {};
for i from 0 to #lprime-2 do (
    for j from i+1 to #lprime-1 do (
     l =append(l,lprime_i*lprime_j)
        )
      )
print l

t = subsets(l,3) -- subsets of size 3 - this can be altered - p value

lp = {};

for j from 0 to #t-1 do (

   print {t_j}; -- list of facets

   print "************ Information for Facet Complex *************";

   facetcomplex = simplicialComplex(t_j); -- Facet Complex of I

   nffacetideal = monomialIdeal(facetcomplex); -- Stanley-Reisner ideal of Facet Complex

   fvecfacet = fVector(facetcomplex);

   print fvecfacet;

   print "********** Information for Non-Face Complex *************";
```

```
   facetideal = monomialIdeal(t_j); -- Facet ideal of facet complex

   nonfacecomplex = simplicialComplex(facetideal); -- non face complex of I

   nfnonfaceideal = monomialIdeal(nonfacecomplex); -- stanley riesner ideal of nonface complex

   fvecnonface = fVector(nonfacecomplex);

   print fvecnonface;

   print "*****************************************************************";
   print "Same f-vector:";
   << fvecfacet === fvecnonface;
   print " ";
   if fvecfacet === fvecnonface then print t_j;
   if fvecfacet === fvecnonface then lp=append(lp,t_j);
   print "*****************************************************************"

   );

print lp
```

The set of f-ideals obtained using the code above is given below:

$I = \langle x_1x_2, x_1x_4, x_2x_3\rangle$ $I = \langle x_1x_3, x_1x_4, x_2x_3\rangle$ $I = \langle x_1x_2, x_1x_3, x_2x_4\rangle$
$I = \langle x_1x_3, x_1x_4, x_2x_4\rangle$ $I = \langle x_1x_3, x_2x_3, x_2x_4\rangle$ $I = \langle x_1x_4, x_2x_3, x_2x_4\rangle$
$I = \langle x_1x_2, x_1x_3, x_3x_4\rangle$ $I = \langle x_1x_2, x_1x_4, x_3x_4\rangle$ $I = \langle x_1x_2, x_2x_3, x_3x_4\rangle$
$I = \langle x_1x_4, x_2x_3, x_3x_4\rangle$ $I = \langle x_1x_2, x_2x_4, x_3x_4\rangle$ $I = \langle x_1x_3, x_2x_4, x_3x_4\rangle$

The code below can be used to determine if a given squarefree monomial ideal I is an f-ideal, and determine whether its complement, $\hat{I}$, is an f-ideal.

```
-- This script determines whether a given ideal I is an f-ideal
-- and also whether its complement is an f-ideal.

loadPackage "SimplicialComplexes"
loadPackage "EdgeIdeals"
loadPackage "SimplicialDecomposability"


R = QQ[x_1,x_2,x_3,x_4,x_5]

mx = x_1*x_2*x_3*x_4*x_5

l = {x_1*x_2*x_3,x_3*x_4*x_5,x_1*x_4,x_2*x_5} -- input ideal here


lcomp = {mx//l_0,mx//l_1,mx//l_2,mx//l_3}

t = l;

   print {t}; -- list of facets

   print "************ Information for Facet Complex *************";
```

```
   facetcomplex = simplicialComplex(t)

   nffacetideal = monomialIdeal(facetcomplex);

   fvecfacet = fVector(facetcomplex);

   print fvecfacet;

   print "********** Information for Non-Face Complex *************";

   facetideal = monomialIdeal(t);

   nonfacecomplex = simplicialComplex(facetideal)

   nfnonfaceideal = monomialIdeal(nonfacecomplex);

   fvecnonface = fVector(nonfacecomplex);

   print "*****************************************************************";
   print "Same f-vector:";
   << fvecfacet === fvecnonface;
   print " ";
   print "*****************************************************************"

s = lcomp;

   print {s}; -- list of facets

   print "************ Information for Facet Complex *************";

   facetcomplex = simplicialComplex(s)

   nffacetideal = monomialIdeal(facetcomplex);

   fvecfacet = fVector(facetcomplex);

   print fvecfacet;

   print "********** Information for Non-Face Complex *************";

   facetideal = monomialIdeal(s);

   nonfacecomplex = simplicialComplex(facetideal)

   nfnonfaceideal = monomialIdeal(nonfacecomplex);

   fvecnonface = fVector(nonfacecomplex);

   print fvecnonface;

   print "*****************************************************************";
   print "Same f-vector:";
   << fvecfacet === fvecnonface;
   print " ";
   print "*****************************************************************"
```

Bibliography

[1] G. Q. Abbasi, S. Ahmad, I. Anwar and W. A. Baig, f-Ideals of Degree 2, Algebra Colloquium, **19** (2012) 921-926.

[2] K. Ansaldi, K. Lin, Y. Shen, Generalized Newton Complementary Duals of Monomial Ideals, Preprint (2017), arXiv:1702.00519v1.

[3] I. Anwar, H. Mahmood, M. A. Binyamin and M. K. Zafar, On the Characterization of f-Ideals, Communications in Algebra, **42** (2014) 3736-3741.

[4] B. Costa and A. Simis, New constructions of Cremona maps, Math. Res. Lett. **20** (2013), 629-645.

[5] S. Faridi, The Facet Ideal of a Simplicial Complex, Manuscripta Mathematica, **109** (2002) 159-174.

[6] C. A. Francisco, J. Mermin, and J. Schweig, A survey of Stanley-Reisner theory. *Connections Between Algebra, Combinatorics, and Geometry*, Springer Proc. Math. Stat., Springer, New York, **76** (2014) 209-234.

[7] D. R. Grayson, M. E. Stillman, Macaulay2, a software system for research in algebraic geometry. `http://www.math.uiuc.edu/Macaulay2/`

[8] J. Guo, T. Wu, Q. Liu, Perfect Sets and f-Ideals, Preprint (2013), arXiv:1312.0324v1.

[9] J. Guo, T. Wu, Q. Liu, F-Ideals and f-Graphs, Communications in Algebra, **45** (2016) 3207-3220.

[10] J. Guo, T. Wu, On the $(n,d)^{th}$ f-Ideals, J. Korean Math. Soc., **52** (2015) 685-697.

[11] J. Herzog, T. Hibi, *Monomial Ideals*, Springer, London, **260** (2011).

[12] R. H. Villarreal. Cohen-Macaulay Graphs. Manuscripta Math., **66** (1990) 277-293.

[13] R. H. Villarreal, *Monomial Algebras*, CRC Press **8** (2001).