MATH 1ZC3/1B03: Test 2 - Version 1
Instructors: Hildum, Lozinski, Tam
Date: March 25, 2014 - Group A
Duration: 90 min.

Name: Solutions **ID #:** ________________

Instructions:

This test paper contains 20 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 13. Pages 14 to 16 are available for rough work. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF THE INVIGILATOR.

Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL. Room for rough work has been provided in this question booklet. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The test is graded out of 20. There is no penalty for incorrect answers.

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

The scanner that will read the answer sheets senses areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**.
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked “1”.

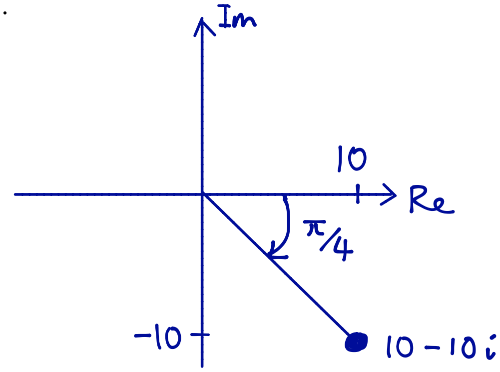
1. Express the complex number

$$10 - 10i$$

in polar form.

(a) $10\sqrt{2}e^{(\pi/4)i}$
(b) $5\sqrt{2}e^{(-\pi/2)i}$
(c) $5\sqrt{2}e^{(3\pi/4)i}$
(d) $\frac{5}{\sqrt{2}}e^{(-3\pi/2)i}$
(e) $10\sqrt{2}e^{(-\pi/4)i}$

Solution. Answer: $10\sqrt{2}e^{(-\pi/4)i}$. The modulus is $\sqrt{(10)^2 + (-10)^2} = 10\sqrt{2}$. The argument is $\arctan(-10/10) = \arctan(-1) = -\pi/4$ lying in the fourth quadrant by drawing the diagram.



---

2. Solve for z if $\bar{z}(1 + i) + (-1 + i) = 3 - 2i$.

(a) $\frac{7}{2} + 2i$
(b) $\frac{1}{2} + \frac{7}{2}i$
(c) $7 - 2i$
(d) $4 - 3i$
(e) $3 + 4i$

Solution. Answer: $\frac{1}{2} + \frac{7}{2}i$. Compute $\bar{z} = \frac{(3-2i)-(-1+i)}{1+i} = \frac{4-3i}{1+i} = \frac{(4-3i)(1-i)}{(1+i)(1-i)} = \frac{(4-3i-4i-3)}{2} = \frac{1-7i}{2}$, so that $z = \frac{1+7i}{2}$.

3. Which of the following is NOT an argument (between 0 to 2π) of a root of $z^4 = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$?

   (a) $\dfrac{\pi}{10}$

   (b) $\dfrac{3\pi}{5}$

   (c) $\dfrac{7\pi}{10}$

   (d) $\dfrac{11\pi}{10}$

   (e) $\dfrac{8\pi}{5}$

   Solution. Answer: $\frac{7\pi}{10}$. The principal argument is $\frac{1}{4}(\frac{2\pi}{5}) = \frac{\pi}{10}$, so that the arguments of the roots are $\frac{\pi}{10} + \frac{2\pi k}{4}$, where $k = 0, 1, 2, 3$. If we write $\frac{2\pi}{4}$ as $\frac{5\pi}{10}$, then the above arguments are $\frac{\pi}{10}$, $\frac{6\pi}{10} = \frac{3\pi}{5}$, $\frac{11\pi}{10}$, and $\frac{16\pi}{10} = \frac{8\pi}{5}$.

---

4. Which of the following complex numbers lies on the unit circle (a circle of radius 1)?

   (a) $1 + i$

   (b) $\dfrac{1}{3} + \dfrac{2}{3}i$

   (c) $\dfrac{3}{5} + \dfrac{4}{5}i$

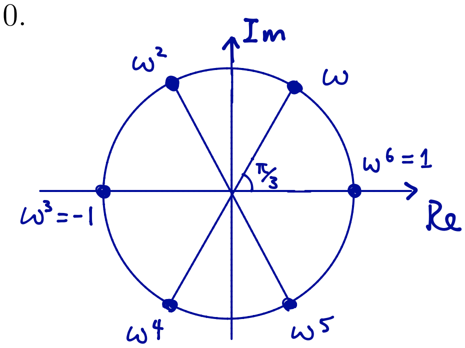
   (d) $\dfrac{4}{7} + \dfrac{5}{7}i$

   (e) $\dfrac{5}{9} + \dfrac{7}{9}i$

   Solution. Answer: $\frac{3}{5} + \frac{4}{5}i$. Notice that a complex number lies on the circle of radius 1 if its modulus r, or its modulus squared r^2, is 1. The modulus squared of a complex number $a + bi$ is $r^2 = a^2 + b^2$. The modulus squared of the above 5 options are 2, 5/9, 1, 41/49, and 74/81.

5. Let $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. Which of the following is true?
(i) $\bar{\omega}^2 = \omega^4$.
(ii) $\omega^2 = -\omega^4$.
(iii) ω is one of the cube roots of 1.

Solution. Answer: (i) only. Notice that ω has modulus $r = 1$ and has argument $\theta = \pi/3$ which is $1/6$ of a complete revolution 2π. Hence all powers of ω can be obtained by cutting the unit circle into 6 equal pieces, starting from the direction $\theta = 0$.



We check that (i) is true because $\bar{\omega}^2$ is the mirror image of $\bar{\omega}^2$, which is ω^4.
(ii) is false since $-\omega^2$ is ω^2 in the opposite direction, which is ω^5.
(iii) is false because ω is a 6^{th} root of 1 but not a cubic root of 1 (notice that $\omega^3 = -1$).

---

6. If $z_1 = 2e^{(\pi/9)i}$ and $z_2 = 10e^{(\pi/3)i}$, what is $(z_1)^3(z_2)^2$?

(a) $800(1+i)$

(b) $400\sqrt{3} + 400i$

(c) $800\pi/81$

(d) $400 - i400\sqrt{3}$

(e) -800

Solution. Answer: -800. If $z_1 = 2e^{(\pi/9)i}$, then $(z_1)^3 = 8e^{(\pi/3)i}$. If $z_2 = 10e^{(\pi/3)i}$, then $(z_2)^2 = 100e^{(2\pi/3)i}$. Therefore, $(z_1)^3(z_2)^2 = (8e^{(\pi/3)i})(100e^{(2\pi/3)i}) = 800e^{\pi i} = 800(-1) = -800$.

7. Let
$$A = \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix}.$$
What is the eigenvalue associated with the eigenvector
$$\begin{bmatrix} 1+i\sqrt{3} \\ -2 \end{bmatrix}?$$

(a) $2+i\sqrt{3}$
(b) $\sqrt{3}/2+i/2$
(c) $1/2+i\sqrt{3}/2$
(d) $\sqrt{3}+i2$
(e) $1+i\sqrt{3}$

Solution. Answer: $1+i\sqrt{3}$. We recall the definition: if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $A\mathbf{v} = \lambda\mathbf{v}$. We then put $A = \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1+i\sqrt{3} \\ -2 \end{bmatrix}$ so that

$$A\mathbf{v} = \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1+i\sqrt{3} \\ -2 \end{bmatrix} = \lambda\mathbf{v} = \lambda \begin{bmatrix} 1+i\sqrt{3} \\ -2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \# \\ -2(1+i\sqrt{3}) \end{bmatrix} = \begin{bmatrix} (1+i\sqrt{3})\lambda \\ -2\lambda \end{bmatrix} \text{ (no need to care about the first entry)}$$

Comparing the second entry on both sides, we obtain $-2(1+i\sqrt{3}) = -2\lambda$, so that $\lambda = 1+i\sqrt{3}$.

8. Given two vectors in n-space, $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} \cdot \mathbf{w} = 0$, which of the following *must* be true?

   (i) either $\mathbf{v} = 0$ or $\mathbf{w} = 0$
   (ii) $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors
   (iii) $\text{proj}_{\mathbf{v}}\mathbf{w} = \mathbf{0}$

   Solution. Answer: (ii) and (iii) only.
   (i) is false, because we can find examples of non-zero vectors which are orthogonal to each other, e.g., in 2-space, $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (1, -1)$.
   (ii) is true by definition of orthogonality.
   (iii) is true either by definition of the projection map (projection of $\mathbf{w}$ along $\mathbf{v}$ is zero if they are orthogonal), or by the formula $\text{proj}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||}\mathbf{v} = \frac{0}{||\mathbf{v}||}\mathbf{v} = \mathbf{0}$.

---

9. Find $\mathbf{v} \cdot \mathbf{w}$ if $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$.

   (a) $\mathbf{v} \cdot \mathbf{w} = -4$
   (b) $\mathbf{v} \cdot \mathbf{w} = 0$
   (c) $\mathbf{v} \cdot \mathbf{w} = -3$
   (d) $\mathbf{v} \cdot \mathbf{w} = -2\mathbf{i} - \mathbf{k}$
   (e) $\mathbf{v} \cdot \mathbf{w} = \mathbf{i} - 2\mathbf{j}$

   Solution. Answer: -3. Compute $\mathbf{v} \cdot \mathbf{w} = (-\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{k}) = (-1)(2) + (-2)(0) + (1)(-1) = -3$.

10. Let $A(1,-1,1)$, $B(0,3,2)$ and $C(-1,-2,0)$ be points in $\mathbf{R}^3$. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   (a) $\cos(\theta) = -\dfrac{1}{2\sqrt{3}}$
   (b) $\cos(\theta) = -\dfrac{3}{\sqrt{3}}$
   (c) $\cos(\theta) = 0$
   (d) $\cos(\theta) = -\dfrac{3}{\sqrt{6}}$
   (e) $\cos(\theta) = \dfrac{1}{3\sqrt{6}}$

Solution. Answer: $\cos(\theta) = -\dfrac{1}{2\sqrt{3}}$. First compute $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (0,3,2) - (1,-1,1) = (-1,4,1)$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-1,-2,0) - (1,-1,1) = (-2,-1,-1)$. We then recall the formula

$$\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}||\cos\theta, \text{ where } \theta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{w}.$$

Hence if $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$, then $\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{||\overrightarrow{AB}||\ ||\overrightarrow{AC}||} = \frac{(-1,4,1)\cdot(-2,-1,-1)}{||(-1,4,1)||\ ||(-2,-1,-1)||} = \frac{(-1)(-2)+(4)(-1)+(1)(-1)}{\sqrt{(-1)^2+(4)^2+(1)^2}\sqrt{(-2)^2+(-1)^2+(-1)^2}} = \frac{-3}{\sqrt{18}\sqrt{6}} = -\frac{1}{2\sqrt{3}}$.

---

11. Find $\mathbf{v}\times\mathbf{w}$ if $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$.

   (a) $\mathbf{v}\times\mathbf{w} = -\mathbf{i} - \mathbf{k}$
   (b) $\mathbf{v}\times\mathbf{w} = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
   (c) $\mathbf{v}\times\mathbf{w} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$
   (d) $\mathbf{v}\times\mathbf{w} = \mathbf{i} - 2\mathbf{j}$
   (e) $\mathbf{v}\times\mathbf{w} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Solution. Answer: $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Compute $\mathbf{v}\times\mathbf{w} = (-\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -2 \\ 2 & 0 \end{vmatrix}\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

12. Let $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$. Find the vector component of $\mathbf{w}$ along $\mathbf{v}$ and the component of $\mathbf{w}$ orthogonal to $\mathbf{v}$.

    (a) $(1,\ 2,\ -1)$ and $(-1,\ -2,\ 1)$
    (b) $(2,\ 4,\ -2)$ and $(0,\ -4,\ -1)$
    (c) $(1,\ 2,\ -1)$ and $(1,\ -2,\ 0)$
    (d) $(1/2,\ 1,\ -1/2)$ and $(3/2,\ -1,\ -1/2)$
    (e) $(1/2,\ 1,\ -1/2)$ and $(-1/2,\ 0,\ 1/2)$

    Solution. Answer: $(1/2,\ 1,\ -1/2)$ and $(3/2,\ -1,\ -1/2)$. Recall that the vector component of $\mathbf{w}$ along $\mathbf{v}$ is equal to

    $$\text{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{w}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}.$$

    Hence the vector component of $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$ along $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is $\frac{(2,0,-1)\cdot(-1,-2,1)}{||(-1,-2,1)||^2}(-1,-2,1) = \frac{(2)(-1)+(0)(-2)+(-1)(1)}{(-1)^2+(-2)^2+(1)^2}(-1,-2,1) = \frac{-3}{6}(-1,-2,1) = (1/2,1,-1/2)$.
    Also recall the the component of $\mathbf{w}$ orthogonal to $\mathbf{v}$ is

    $$\mathbf{w} - \text{proj}_{\mathbf{v}}(\mathbf{w}).$$

    Hence the the component of $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$ orthogonal to $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is $(2,0,-1) - (1/2,1,-1/2) = (3/2,-1,-1/2)$.

---

13. Find the equation of the plane that passes through the point $P(2,0,3)$ parallel to the vectors $\mathbf{v} = (-1,0,4)$ and $\mathbf{w} = (2,-2,4)$.

    (a) $-2x + 16z = 44$
    (b) $2x - 2y + 4z = 16$
    (c) $-x + 4z = 10$
    (d) $x - 2y + 8z = 26$
    (e) $8x + 12y + 2z = 22$

    Solution. Answer: $8x + 12y + 2z = 22$. Recall that the equation of the plane passing through a point $\mathbf{p}$ and with normal direction $\mathbf{n}$ is $(\mathbf{x}-\mathbf{p})\cdot\mathbf{n} = 0$. The normal direction of the given plane is $\mathbf{v}\times\mathbf{w} = (-1,0,4)\times(2,-2,4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 4 \\ 2 & -2 & 4 \end{vmatrix} = (8,12,2)$. Therefore, the plane containing the point $(2,0,3)$ is $((x,y,z)-(2,0,3))\cdot(8,12,2) = (x-2,y,z-3)\cdot(8,12,2) = 0$, which is simplified into $8x + 12y + 2z - 22 = 0$.

14. Find the distance between the 2 parallel planes $3x-4y+z=2$ and $6x-8y+2z=0$.

    (a) $\sqrt{3}/12$
    (b) $4/\sqrt{13}$
    (c) $2/\sqrt{26}$
    (d) $\sqrt{2}/12$
    (e) $18/\sqrt{2}$

    Solution. Answer: $2/\sqrt{26}$. Recall the distance formula between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Given planes $P_1 : 3x-4y+z=2$ and $P_2 : 6x-8y+2z=0$, we first rewrite P_1 as $P_1 : 6x-8y+2z-4=0$. We then apply the formula $\frac{|(-4)-(0)|}{\sqrt{(6)^2+(-8)^2+(2)^2}} = \frac{2}{\sqrt{(3)^2+(-4)^2+(1)^2}} = 2/\sqrt{26}$.

---

15. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be parallel vectors, and $\mathbf{v}_3$ be a vector orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. All three vectors are non-zero. Which of the following *must* be true, where t_1, t_2, and t_3 are scalars.

    (i) $\mathbf{x}=t_1\mathbf{v}_1+t_2\mathbf{v}_2$ defines the equation of a plane in $\mathbf{R}^n$
    (ii) $\mathbf{x}=t_1\mathbf{v}_2+t_2\mathbf{v}_3$ defines the equation of a plane in $\mathbf{R}^n$
    (iii) There exists a plane in $\mathbf{R}^n$ containing all three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

    Solution. Answer: (ii) and (iii) only.
    (i) is false. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, then $\mathbf{v}_2=c\mathbf{v}_1$ for some scalar c. The equation becomes $\mathbf{x}=t_1\mathbf{v}_1+t_2\mathbf{v}_2=(t_1+ct_2)\mathbf{v}_1$, which spans a line in the direction of $\mathbf{v}_1$ (or $\mathbf{v}_2$).
    (ii) is true, because two vectors in non-parallel directions can span a plane.
    (iii) is true, because the plane in (ii) contain all three vectors.

16. At time t, at an amusement park, the fraction of guests riding rides (r_t), eating food (f_t), and watching shows (s_t) is given by a state vector $\mathbf{v_t} = [r_t, f_t, s_t]^T$. The state vector at any later point in time t is determined by a discrete time dynamical system driven by the matrix A where

$$A = \begin{bmatrix} 0.9 & 0.2 & 0 \\ 0 & 0.8 & 0.6 \\ 0.1 & 0 & 0.4 \end{bmatrix}$$

In the long run steady state, what fraction of guests are watching a show?

(a) 1/5
(b) 1/10
(c) 1/3
(d) 1/8
(e) 1/6

Solution. Answer: 1/10. First notice that the matrix is a stocastic matrix (the sum of entries of a column is 1) by construction. Recall that a stocastic matrix always has eigenvalue 1, and if it has a unique eigenvector up to scaling, then it is the steady state vector. We solve directly $(A - I)\mathbf{x} = \mathbf{0}$ using row operations:

$$A - I = \begin{bmatrix} -0.1 & 0.2 & 0 \\ 0 & -0.2 & 0.6 \\ 0.1 & 0 & -0.6 \end{bmatrix} \xrightarrow{-10R_1, -10R_2, -10R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -6 \\ -1 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3+R_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -6 \\ 0 & -2 & 6 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{1/2R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is $\begin{bmatrix} r \\ f \\ s \end{bmatrix} = t \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$, where t is a parameter. The fraction of guest watching a show is $\frac{1}{6+3+1} = 1/10$.

17. A city bus is either driving a route, out of service, or parked outside a Tim Hortons. A bus that is driving a route one hour has a 1/5 probability of being out of service the next hour, and a 1/5 probability of being at a Tim's (otherwise it is still driving a route). If a bus is out of service, the next hour it will NOT be at a Tim's, but will either be driving a route, or it will still be out of service, with equal probability. A bus parked outside a Tim's will be driving a route the next hour. If you are presently sitting on a bus, riding a route, what is the probability that your bus will be parked outside a Tim Hortons two hours from now?

(a) 1/5
(b) 1/3
(c) 1/10
(d) 3/5
(e) 3/25

Solution. Answer: 3/25. The stocastic matrix in this case is

$$A = \begin{bmatrix} 3/5 & 1/2 & 1 \\ 1/5 & 1/2 & 0 \\ 1/5 & 0 & 0 \end{bmatrix},$$

where the first, second, and third row/column stand for the status of driving a route, out of service, and parked outside a Tim Hortons respectively. The probability that the bus will be outside a Tim Hortons two hours from now riding a route is given by the $(3,1)$entry of A^2, so we compute

$$(A^2)_{3,1} = (\text{3rd row}) \cdot (\text{1st column}) = (1/5, 0, 0) \cdot (3/5, 1/5, 1/5) = 3/25.$$

18. Which of the following series of Matlab commands will NOT produce the following matrix: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(a) `A = eye(3); A(1,2) = 2; A(2,3) = 3`

(b) `x = [1 1 1]; y = [2 3];  A = diag(x,0); A = diag(y,1)`

(c) `x = [2 3]; A = diag(x,1); eye(3)+A`

(d) `A = [1 2 0; 0 1 3; 0 0 1]`

(e) `eye(3) + diag([2 3],1)`

Solution. Answer: `x = [1 1 1]; y = [2 3]; A = diag(x,0); A = diag(y,1)`, whose output is $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

The code `A = diag(x,0)` assigns the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to `A`, but then the code `A = diag(y,1)` assigns $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ to `A`, replacing the previous assignment.

---

19. Which of the following sets are closed under addition?
(suggestion: consider drawing some of these sets on the complex plane)

(i) The set of complex numbers.
(ii) The set of complex numbers with modulus (absolute value) 1.
(iii)The set of complex numbers z such that either $z = 0$, or $\arg(z) = \frac{\pi}{2}$, or $\arg(z) = -\frac{\pi}{2}$.

Solution. Answer: (i) and (iii) only. (i) is closed under addition: sum of two complex numbers is also a complex number.
(ii) is not closed under addition, for example, take $z = 1$ and $w = i$, then $|z| = |w| = 1$ but $|z + w| = |1 + i| = \sqrt{2}$.
(iii) is closed under addition. All the complex numbers mentioned form the imaginary line, which can be regarded as a 1 dimensional real subspace of the complex plane.
(See the pictures on Page 14)

20. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} \oplus \mathbf{v} = (u_1 v_1, u_2 v_2)$$

$$k \odot \mathbf{u} = (ku_1, ku_2)$$

Which of the following properties is NOT satisfied?

(a) V is closed under vector addition

(b) V is closed under scalar multiplication

(c) V has a zero vector, and it is (1,1)

(d) $k \odot (\mathbf{u} \oplus \mathbf{v}) = (k \odot \mathbf{u}) \oplus (k \odot \mathbf{v})$

(e) $1 \odot \mathbf{u} = \mathbf{u}$

Solution. 'V is closed under vector addition' is satisfied, because $\mathbf{u} \oplus \mathbf{v} = (u_1 v_1, u_2 v_2)$ is also an ordered pair of real numbers.

'V is closed under scalar multiplication' is satisfied, because $k \odot \mathbf{u} = (ku_1, ku_2)$ is also an ordered pair of real numbers.

'V has a zero vector, and it is (1,1)' is satisfied. We check that if the zero vector $\mathbf{0} = (u, v)$, then $\mathbf{0} \oplus (x, y) = (u, v) \oplus (x, y) = (x, y)$ for all (x, y). But $(u, v) \oplus (x, y) = (ux, vy)$ by defined. Hence $(ux, vy) = (x, y)$ for all (x, y), which implies that $u = v = 1$ and $(u, v) = (1, 1)$ is the zero vector.

'$k \odot (\mathbf{u} \oplus \mathbf{v}) = (k \odot \mathbf{u}) \oplus (k \odot \mathbf{v})$' is NOT satisfied, because the left side is

$$k \odot (\mathbf{u} \oplus \mathbf{v}) = k \odot (u_1 v_1, u_2 v_2) = (ku_1 v_1, ku_2 v_2),$$

but right side is

$$(k \odot \mathbf{u}) \oplus (k \odot \mathbf{v}) = (ku_1, ku_2) \oplus (kv_1, kv_2) = (ku_1 kv_1, ku_2 kv_2),$$

which is not equal to the right side in general.

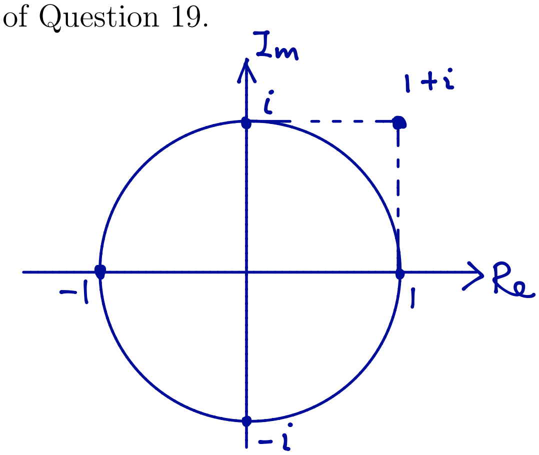
'$1 \odot \mathbf{u} = \mathbf{u}$' is satisfied, since $1 \odot \mathbf{u} = (1u_1, 1u_2) = (u_1, u_2) = \mathbf{u}$.

END OF TEST QUESTIONS

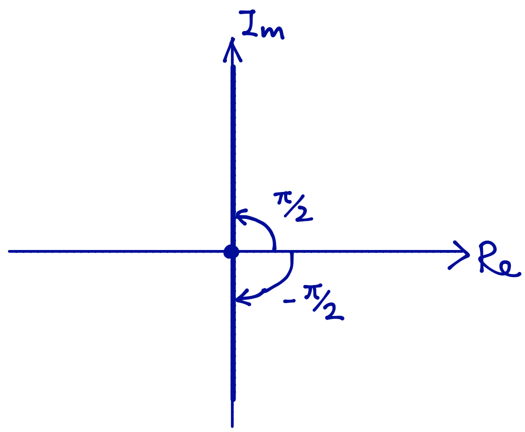
Extra page for rough work. DO NOT DETACH!

Pictures of Question 19.

(ii)



(iii)



Extra page for rough work. DO NOT DETACH!

Extra page for rough work. DO NOT DETACH!

END OF TEST PAPER