MATH 1B03: Midterm 2 - VERSION 2
Instructor: Adam Van Tuyl
Date: (EARLY WRITES)
Duration: 75 min.

Name: SOLUTIONS **ID #:** ______________

Instructions:

This test paper contains 21 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 10. Scrap paper is available for rough work. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF THE INVIGILATOR.

Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The test is graded out of 20. There is no penalty for incorrect answers.

NO CALCULATORS ALLOWED.

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

The scanner that will read the answer sheets senses areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**. Your student number **MUST** be 9 digits long. If you have a student number that is 7 digits, begin your student number with 00 (two zeroes).
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked “1”.

1. Find the determinant of the following matrix:

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -9 & 8 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

(a) -14

$\boxed{\text{(b) 0}}$

(c) 10

(d) 12

(e) 2016

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2. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

compute the determinant

$$\begin{vmatrix} a+d+g & b+e+h & c+f+i \\ g & h & i \\ -3d & -3e & -3f \end{vmatrix}.$$

swap

mult row by -3

(a) -125

(b) -25

(c) -5

(d) 0

$\boxed{\text{(e) 15}}$

$= 5(-1)(-3) = 15$

3. Find the element in row 3, column 1 of the matrix adj(A) if the matrix A is:

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$C_{31} = (-1)^{3+1} \begin{vmatrix} -3 & 2 \\ 1 & 0 \end{vmatrix}$

$= \boxed{-2}$

(a) 0 (b) 3 (c) 5 (d) 4 $\boxed{\text{(e) } -2}$

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4. Which of the following statements are true?

(1) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$. FALSE

(2) If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$. TRUE

(a) (1) is false and (2) is false.

(b) (1) is true and (2) is false.

$\boxed{\text{(c) (1) is false and (2) is true.}}$

(d) (1) is true and (2) is true.

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5. Suppose that A and B are both $n \times n$ matrices, $\det(A) = 2$ and $\det(A^{-1}B) = 3$. Then $\det(BA)$ is:

(a) 3

(b) 8

$\boxed{\text{(c) } 12}$

(d) 24

(e) Cannot be determined

$\det(A^{-1})\det(B) = 3$

$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{2}$

So $\det(B) = 6$.

So $\det(B)\det(A) = 12$

6. Find a complex number z such that

$$\begin{vmatrix} 1+2i & 2+2i & 5-6i \\ 0 & z & 2016+6102i \\ 0 & 0 & 1+i \end{vmatrix} = -9-3i$$

(a) 0

(b) $-1+3i$

(c) i

$\boxed{\text{(d) } 3i}$

(e) $2016+6102i$

Need to solve

$(1+2i)z(1+i) = -9-3i$

$z = 3i$

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7. Compute $(\frac{1}{i})^{2016}$

(a) $-i$

(b) -1

(c) 0

$\boxed{\text{(d) } 1}$

(e) i

$\frac{1}{i} = -i$

$\Rightarrow (-i)^4 = 1 \Rightarrow (-i)^{2016} = \left[(-i)^4\right]^{504} = 1$

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8. Given $z_1 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ and $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$ compute $\frac{z_1}{z_2}$.

$\boxed{\text{(a) } \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)}$

(b) $4(\cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right))$

(c) $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

(d) $\cos(\pi) + i\sin(\pi)$

(e) $\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$\frac{z_1}{z_2} = \frac{2}{2}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)$

9. Solve the system of linear equations:

If you can determine y, you can get the sol^{n}. By Cramer's rule

$$\begin{aligned} ix + 2y &= 1 - 2i \\ 4x - iy &= -1 + 3i \end{aligned}$$

(a) $x = 1$, $y = 1$

(b) $x = -i$, $y = 2 - 2i$

(c) $x = i$, $y = 34 + 34i$

(d) $\boxed{x = i,\ y = 1 - i}$

(e) $x = 4 + 4i$, $y = \frac{1}{2} + \frac{1}{2}i$

$$y = \frac{\begin{vmatrix} i & 1-2i \\ 4 & -1+3i \end{vmatrix}}{\begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix}} = 1 - i$$

So answer is (d)

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10. What are the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 5 \\ 2 & 3 \end{bmatrix}$$

(a) -5 and -2

(b) -5 and 2

(c) 0 and -2

(d) $\boxed{5 \text{ and } -2}$

(e) 5 and 2

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda & -5 \\ -2 & \lambda-3 \end{vmatrix}$$

$$= \lambda(\lambda-3) - 10$$

$$= \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$$

So eigen values $\lambda = 5$, $\lambda = -2$

11. Which of the following vectors are eigenvectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2016 \\ 2016 \\ \cancel{2016} \end{bmatrix}$ 0

$$\begin{bmatrix} 2 & & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2016 \\ 2016 \\ 0 \end{bmatrix} = \begin{bmatrix} 2016 \\ 2016 \\ 0 \end{bmatrix}$$

(d) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

(e) All of them

$$A = \begin{bmatrix} & & ? & & \\ 1 & 0 & -1 & k & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}.$$

12. Suppose A is a 5×5 matrix with an eigenvector

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}$$

So $2+2+3k+3 = -2 \iff k = -3$

associated with the eigenvalue $\lambda = -2$. If the last row of A is $\begin{bmatrix} 1 & 0 & -1 & k & 3 \end{bmatrix}$, what is the value of k?

(a) -2016

$\boxed{\text{(b) -3}}$

(c) 3

(d) 4

(e) Not enough information given

13. The matrix A is a 3×3 matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Which of the following cannot be the characteristic polynomial of A.

(a) $\lambda^3 + 3\lambda^2 + 3\lambda + 4$

(b) $\lambda^3 + 6\lambda^2 + 3\lambda + 3$

(c) $\lambda^3 + 10\lambda^2 + 3\lambda + 2$

(d) $\lambda^3 + 15\lambda^2 + 3\lambda + 1$

(e) $\boxed{\lambda^3 + 21\lambda^2 + 3\lambda}$

$A\vec{x} = \vec{0}$ trivial $\Leftrightarrow$

A invertible $\Leftrightarrow$ $\lambda = 0$ is not a root of char poly.

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14. The matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

is similar to the diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

What is the matrix P that diagonalizes A?

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(b) $\boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$

(c) $\begin{bmatrix} -1 & -1 & 1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(e) None of the above

- Simply check which matrix P satisfies $AP = PD$
- alt sol^{n}, find eigenvectors for $\lambda = -1$ and $\lambda = -2$. and determine P

15. The matrix $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ is similar to a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. What is a and b?

By thm, $a \pm ib$ would be the complex eigenvalues of A.

(a) $a = 1$ and $b = 3$
(b) $a = 1$ and $b = 4 + i$.
(c) $a = 4$ and $b = 1$
(d) $a = 0$ and $b = i$
(e) $a = 5$ and $b = 2$

$$\begin{vmatrix} \lambda-5 & 2 \\ -1 & \lambda-3 \end{vmatrix} = \lambda^2 - 8\lambda + 15 + 2 = \lambda^2 - 8\lambda + 17$$

So $\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$

16. A matrix A has the following characteristic polynomial:

$$(\lambda + 1)^2(\lambda - 1)^3(\lambda - 6)^7(\lambda - 10)^2.$$

Which of the following statements are true?

(1) The matrix A is a 14×14 matrix. TRUE
(2) The geometric multiplicy of the eigenvalue $\lambda = 6$ is less than or equal to 7. TRUE

(a) (1) is false and (2) is false.
(b) (1) is true and (2) is false.
(c) (1) is false and (2) is true.
(d) (1) is true and (2) is true.

17. A row vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is called a *left eigenvector* of an $n \times n$ matrix A if there exists a nonzero scalar μ such that

$$\mathbf{v}A = \mu\mathbf{v}.$$

If $\mathbf{v}$ is a left eigenvector of A, which of the following statements must also be true:

(a) μ is an eigenvalue of A.
(b) μ is an eigenvalue of A^{-1}.
(c) $\mathbf{v}$ is an eigenvector of A.
(d) $\mathbf{v}^T$ is an eigenvector of A^T.
(e) $\frac{1}{\mu}$ is an eigenvalue of A.

$\vec{v}A = \mu\vec{v} \Rightarrow (\vec{v}A)^T = (\mu\vec{v})^T$

$\Rightarrow A^T\vec{v}^T = \mu\vec{v}^T$

$\Rightarrow \vec{v}^T$ eigenvector of A^T

18. Given the following matrices:

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad B = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.8 & 0.7 & 0.3 \\ 0 & 0 & 0.3 \end{bmatrix} \quad C = \begin{bmatrix} 0.2 & 0.4 & 0.3 & 0 \\ 0.6 & 0 & 0.3 & 0 \\ 0.2 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which matrices are stochastic matrices.

(a) A, B, and C
(b) A and B only
(c) A and C only
(d) B and C only
(e) B

19. Using the same matrices in the above question, which matrices are regular stochastic matrices.

(a) A and B only
(b) A only
(c) B only
(d) C only
(e) None of them

20. Math majors at McMaster can be broken into three types: AFM (Actuarial and Financial Math) majors, pure math majors, and statistics majors. At time t (measured in weeks), let

$$\mathbf{x}(t) = \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

describe the percentage of the three types of majors, where a_t is the percentage of AFM majors, b_t is the percentage of pure math majors, and c_t is the percentage of statistics majors.

At the end of every week, 20% of the AFM majors decide to become pure math majors, and 10% become statistics majors. At the end of every week, 10% of the pure math majors become AFM majors, 60% of the pure math majors become statistics majors, and the rest stay pure math majors. Finally, at the end of every week, 20% of the statistics majors become AFM majors, 50% become pure math majors, and the rest stay as statistics majors.

Let P be the transition matrix that drives this dynamical system, that is, $\mathbf{x}(t+1) = P\mathbf{x}(t)$. What is the trace of P?

(a) 0

(b) 0.7

(c) 1 Same as before

$\boxed{\text{(d) } 1.3}$

(e) -1

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21. What is the steady-state vector for the regular Markov chain described in the above question?

(a) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$ $\boxed{\text{(e) } \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}$

END OF TEST PAPER