MATH 1B03: Midterm 2 - VERSION 1
Instructor: Adam Van Tuyl
Date: Thursday, November 10, 2016 7:00PM
Duration: 75 min.

Name: SOLUTIONS ID #: ______________

Instructions:

This test paper contains 21 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 10. Scrap paper is available for rough work. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF THE INVIGILATOR.

Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The test is graded out of ~~20~~ 21. There is no penalty for incorrect answers.

NO CALCULATORS ALLOWED.

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

The scanner that will read the answer sheets senses areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**. Your student number **MUST** be 9 digits long. If you have a student number that is 7 digits, begin your student number with 00 (two zeroes).
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked "1".

1. Find the determinant of the following matrix:

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -9 & 8 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

(a) $\boxed{-14}$
(b) 0
(c) 10
(d) 12
(e) 2016

A triangular matrix, so determinant is product of diagonal entries

$\Rightarrow \det(A) = 2 \cdot 1 \cdot (-1) \cdot 1 \cdot 7 = -14$

---

2. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

compute the determinant

$$\begin{vmatrix} a+d+g & b+e+h & c+f+i \\ g & h & i \\ 5d & 5e & 5f \end{vmatrix}.$$

(a) -125
(b) $\boxed{-25}$
(c) -5
(d) 0
(e) 15

we have the following row operations

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow[\text{+row 3}]{\text{row 1+row 2}} \begin{bmatrix} a+d+g & b+e+h & c+f+i \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\xrightarrow[\text{2 + 3}]{\text{swap row}} \begin{bmatrix} a+d+g & b+e+h & c+f+i \\ g & h & i \\ d & e & f \end{bmatrix} \xrightarrow[\text{by 5}]{\text{mult row 3}} \begin{bmatrix} a+d+g & b+e+h & c+f+i \\ g & h & i \\ 5d & 5e & 5f \end{bmatrix}$$

So determinant of new matrix $= (1)\cdot(-1)\cdot(5)$ det of old matrix $= -25$

McMaster University Math 1B03 Fall 2016 Page 2 of 10

3. Find the element in row 3, column 1 of the matrix adj(A) if the matrix A is:

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

element in row 3 column 1 of adj (A) is

$C_{13} = (-1)^{1+3} A_{13}$

$= (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = 5$

(a) 0 (b) 3 $\boxed{\text{(c) 5}}$ (d) 4 (e) 2

---

4. Which of the following statements are true?

   (1) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$. FALSE

   (2) If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$. TRUE

   (a) (1) is false and (2) is false.

   (b) (1) is true and (2) is false.

   $\boxed{\text{(c) (1) is false and (2) is true.}}$

   (d) (1) is true and (2) is true.

---

5. Suppose that A and B are both $n \times n$ matrices, $\det(A) = 2$ and $\det(A^{-1}B) = 6$. Then $\det(BA)$ is:

   $6 = \det(A^{-1}B) = \det(A^{-1})\det(B)$

   Also $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{2}$.

   So $6 = \frac{1}{2}\det(B) \Rightarrow \det(B) = 12$.

   So $\det(BA) = \det(B)\det(A) = 12 \cdot 2 = 24$

   (a) 3

   (b) 8

   (c) 12

   $\boxed{\text{(d) 24}}$

   (e) Cannot be determined

6. Find a complex number z such that

$$\begin{vmatrix} 1 & 2+2i & 5-6i \\ 0 & z & 2016+6102i \\ 0 & 0 & 1-i \end{vmatrix} = 2+4i$$

(a) 0

$\boxed{\text{(b) } -1+3i}$

(c) i

(d) $3i$

(e) $2016+6102i$

Triangular matrix so

$1 \cdot z(1-i) = 2+4i$

$\Rightarrow z = \frac{2+4i}{1-i} \cdot \frac{1+i}{1+i} = \frac{2+2i+4i-4}{1+1} = \frac{-2+6i}{2} = -1+3i$

---

7. Compute $(\frac{1}{i})^{2016}$

(a) $-i$

(b) -1

(c) 0

$\boxed{\text{(d) } 1}$

(e) i

$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{1}$

Note $(-i)^4 = (-i)(-i)(-i)(-i) = 1$

So $(-i)^{2016} = [(-i)^4]^{504} = 1^{504} = 1$

---

8. Given $z_1 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ and $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$ compute $\frac{z_1}{z_2}$.

$\boxed{\text{(a) } \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)}$

(b) $4(\cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right))$

(c) $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

(d) $\cos(\pi) + i\sin(\pi)$

(e) $\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$\frac{z_1}{z_2} = \frac{2}{2}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)$

$= \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$

9. Solve the system of linear equations:

$$\begin{aligned} ix + 2y &= 1 - 2i \\ 4x - iy &= -1 + 3i \end{aligned}$$

Note: you can "brute force" it by trying all answers. Note, if we can solve for y, we can figure out the answer. By Cramer's rule

(a) $x = 1$, $y = 1$
(b) $x = -i$, $y = 2 - 2i$
(c) $x = i$, $y = 34 + 34i$
$\boxed{\text{(d) } x = i,\ y = 1 - i}$
(e) $x = 4 + 4i$, $y = \frac{1}{2} + \frac{1}{2}i$

$$y = \frac{\begin{vmatrix} i & 1-2i \\ 4 & -1+3i \end{vmatrix}}{\begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix}} = \frac{(-i-3)-4+8i}{-7} = \frac{-7+7i}{-7} = 1-i$$

So answer is probably (d)

You can then check it is!

---

10. What are the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

(a) 1 and 2
(b) 1 and -2
(c) 0 and -2
(d) -1 and 2
$\boxed{\text{(e) -1 and -2}}$

$$\lambda I_2 - A = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$

$$\det(\lambda I_2 - A) = \lambda(\lambda+3) + 2 = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$$

$$\det(\lambda I_2 - A) = 0 \iff \lambda = -2 \text{ or } -1$$

11. Which of the following vectors are eigenvectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2016 \\ 2016 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 0 \\ 2016 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(e) All of them

For each vector, compute $A\vec{x}$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2016 \\ 2016 \\ 0 \end{bmatrix} = \begin{bmatrix} 2016 \\ 2016 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2016 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4032 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 2016 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

---

12. Suppose A is a 5×5 matrix with an eigenvector

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} & & ? & & \\ 1 & 0 & -1 & k & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}$$

So

$2 + 2 + 3k + 3 = -2$

$\iff k = -3$

associated with the eigenvalue $\lambda = -2$. If the last row of A is $\begin{bmatrix} 1 & 0 & -1 & k & 3 \end{bmatrix}$, what is the value of k?

(a) -2016

(b) -3

(c) 3

(d) 4

(e) Not enough information given

13. The matrix A is a 3×3 matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Which of the following cannot be the characteristic polynomial of A.

(a) $\lambda^3 + 3\lambda^2 + 3\lambda + 4$

(b) $\lambda^3 + 6\lambda^2 + 3\lambda + 3$

(c) $\lambda^3 + 10\lambda^2 + 3\lambda + 2$

(d) $\lambda^3 + 15\lambda^2 + 3\lambda + 1$

(e) $\boxed{\lambda^3 + 21\lambda^2 + 3\lambda}$

$A\vec{x} = \vec{0}$ has only trivial sol^n $\iff$ A invertible $\iff$ $\lambda = 0$ is not an eigenvalue.

Since $\lambda = 0$ is a root of (e), $\lambda^3 + 21\lambda^2 + 3\lambda$ cannot be the char poly of A

---

14. The matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

is similar to the diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Can "brute force" your way to a sol^n by checking if $AP = PD$ for any matrix below.

What is the matrix P that diagonalizes A?

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(b) $\boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$ ← answer

(c) $\begin{bmatrix} -1 & -1 & 1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(e) None of the above

15. The matrix $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ is similar to a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. What is a and b?

(a) $a = 1$ and $b = 3$

(b) $a = 1$ and $b = 4 + i$.

(c) $a = 4$ and $b = 1$

(d) $a = 0$ and $b = i$

(e) $a = 5$ and $b = 2$

$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda-5 & 2 \\ -1 & \lambda-3 \end{vmatrix}$

$= (\lambda-5)(\lambda-3)+2$

$= \lambda^2 - 8\lambda + 15 + 2$

$= \lambda^2 - 8\lambda + 17 = 0$

So $\lambda = \dfrac{8 \pm \sqrt{64 - 17\cdot 4}}{2} = 4 \pm i$

So A is similar to $\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

---

16. A matrix A has the following characteristic polynomial:

$$(\lambda + 1)^2(\lambda - 1)^3(\lambda - 6)^7(\lambda - 10)^2.$$

Which of the following statements are true?

(1) The matrix A is a 4×4 matrix. FALSE (It has size 14×14)

(2) The geometric multiplicy of the eigenvalue $\lambda = 6$ is greater than 7. FALSE

geo mult ≤ 7

(a) (1) is false and (2) is false.

(b) (1) is true and (2) is false.

(c) (1) is false and (2) is true.

(d) (1) is true and (2) is true.

---

17. A row vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is called a *left eigenvector* of an $n \times n$ matrix A if there exists a nonzero scalar μ such that

$$\mathbf{v}A = \mu\mathbf{v}.$$

If $\mathbf{v}$ is a left eigenvector of A, which of the following statements must also be true:

(a) μ is an eigenvalue of A.

(b) μ is an eigenvalue of A^{-1}.

(c) $\mathbf{v}$ is an eigenvector of A.

(d) $\mathbf{v}^T$ is an eigenvector of A^T.

(e) $\frac{1}{\mu}$ is an eigenvalue of A.

$\vec{v}A = \mu\vec{v} \Rightarrow (\vec{v}A)^T = (\mu\vec{v})^T$

$\Rightarrow A^T\vec{v}^T = \mu\vec{v}^T$

$\Rightarrow \vec{v}^T$ is eigenvector of A^T

18. Given the following matrices:

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad B = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.8 & -0.4 & -0.3 \\ 0 & 0.3 & 0.3 \end{bmatrix} \quad C = \begin{bmatrix} 0.2 & 0.4 & 0.3 & 0 \\ 0.6 & 0 & 0.3 & 0 \\ 0.2 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

not allowed negative entries

which matrices are stochastic matrices.

(a) A, B, and C
(b) A and B only
(c) A and C only
(d) B and C only
(e) B

---

19. Using the same matrices in the above question, which matrices are regular stochastic matrices.

(a) A and B only
(b) A only
(c) B only
(d) C only
(e) None of them

For any power of C, C^t will always have a zero entry. So it is not regular.

Note, if $*$ = non zero or zero and O is a zero entry, then

$$\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

I.e., can never have a non zero spot

20. Math majors at McMaster can be broken into three types: AFM (Actuarial and Financial Math) majors, pure math majors, and statistics majors. At time t (measured in weeks), let

$$\mathbf{x}(t) = \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

describe the percentage of the three types of majors, where a_t is the percentage of AFM majors, b_t is the percentage of pure math majors, and c_t is the percentage of statistics majors.

At the end of every week, 20% of the AFM majors decide to become pure math majors, and 10% become statistics majors. At the end of every week, 10% of the pure math majors become AFM majors, 60% of the pure math majors become statistics majors, and the rest stay pure math majors. Finally, at the end of every week, 20% of the statistics majors become AFM majors, 50% become pure math majors, and the rest stay as statistics majors.

Let P be the transition matrix that drives this dynamical system, that is, $\mathbf{x}(t+1) = P\mathbf{x}(t)$. What is the trace of P?

(a) 0

(b) 0.7

(c) 1

$\boxed{\text{(d) } 1.3}$

(e) -1

$$P = \begin{array}{c} \\ \text{AFM} \\ \text{Pure} \\ \text{Stats} \end{array} \begin{array}{c} \begin{array}{ccc} \text{AFM} & \text{Pure} & \text{Stats} \end{array} \\ \begin{bmatrix} 0.70 & 0.10 & 0.20 \\ 0.20 & 0.30 & 0.50 \\ 0.10 & 0.60 & 0.30 \end{bmatrix} \end{array}$$

trace of P
$= 0.70 + 0.30 + 0.30$
$= 1.3$

---

21. What is the steady-state vector for the regular Markov chain described in the above question?

(a) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$ $\boxed{\text{(e) } \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}$

We want an eigenvector of P associated to $\lambda = 1$ that is also a probability vector. (b) & (c) are not prob vectors

For (a), $P\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ For (d) $P\begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix}$, For (e) $\underbrace{P\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}}_{\text{answer}} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

END OF TEST PAPER