MATH 4GR3 (GROUPS AND RINGS)
DUE: JANUARY 26, 2024

HOMEWORK ASSIGNMENT 1

Do all of the questions. Three questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark (a link will be sent).

When submitting to Crowdmark, *submit a pdf version* of your solution instead of a picture. I suggest using a scanner app on your phone.

Exercise 1. Let G be a group and H a subgroup of G. For any $g \in G$, prove that

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is also a subgroup of G.

Exercise 2. Determine all the the non-isomorphic abelian groups of order 2024. Justify your answer.

Hint. $2024 = 2^3 \cdot 11 \cdot 23$.

Exercise 3. Let p and q be distinct primes. Prove that the number of distinct finite abelian groups of order p^4 is the same as the number of distinct finite abelian groups of order q^4. How many distinct finite abelian groups of order $p_1^4 p_2^4 \cdots p_r^4$ are there if $p_1, \ldots, p_r$ are all distinct primes.

Exercise 4. You are given a finite abelian group G with $|G| = 16$. As well, you are told that G has an element of order 8 and two elements of order of 2. What group is G isomorphic to? Justify your answer.

Exercise 5. Let G be an abelian group where the operation is addition. Let K be a proper subgroup of G, and suppose that $d \in G \setminus K$, that is d is an element of G not in K. Show that the set

$$H = \{k + zd \mid k \in K, z \in \mathbb{Z}\}$$

is also a subgroup of G with $K \subsetneq H$.

Remark. This exercise justifies a step we used in the proof of the Fundamental Theorem of Finite Abelian Groups given in Lecture 5.

Exercise 6. Suppose G is an abelian group of order 25. You are able to ask an "oracle" about the order of a particular element in the group. What is the maximum number of times you have to ask the "oracle" for an answer to figure out the structure of G? Justify your answer.

Exercise 7. Find all composition series of $S_3 \times \mathbb{Z}_3$.

Exercise 8. Let G and H be solvable groups. Show that $G \times H$ is also solvable.

Suggested Exercises.

(1) Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 13.
(2) Practice Problems: Chapter 13 - 1-4, 6.