THESIS WORK

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1. Introduction

In algebra, finding generators and relations of an algebraic object often provides a better understanding of the object and its representations. Likewise, in topology, decomposition of a manifold into simple pieces is often useful in the study of manifold invariants which behave well under cut and paste operations. Topological (quantum) field theories (TFTs) produce such invariants, which makes the construction and the classification of TFTs an interesting problem in topology.

Atiyah [At] axiomatized TFTs as symmetric monoidal functors from cobordism categories to the category of vector spaces. Using this formulation, TFTs were generalized in two different directions. The first generalization, so-called *extended TFTs*, is motivated by the idea of cutting and pasting manifolds in several directions. Accordingly, the study of extended TFTs involves manifolds with corners and higher categories. The second generalization of TFTs is obtained by considering manifolds endowed with principal G-bundles. When G is a discrete group, such theories were introduced by Turaev [Tu] who named them *homotopy (quantum) field theories* (HFTs).

Since TFTs are functors on cobordism categories, finding generators and relations of cobordism categories is the key step toward the construction and classification of TFTs. In my thesis, I combine two generalizations of TFTs described above; more precisely, I define 2-dimensional extended homotopy field theories and classify them by giving a specific presentation of the corresponding cobordism bicategory in terms of generators and relations.

In dimension 2, Schommer-Pries [Sc] classified extended TFTs and Turaev [Tu] classified HFTs. To define 2*-dimensional extended HFTs*, I introduce a 2-dimensional G-equivariant cobordism bicategory which contains both the Schommer-Pries' cobordism bicategory [Sc] and the Turaev's G-equivariant cobordism category as subcategories. Using the formalism of HFTs, I generalize the methods introduced in [Sc] and provide a list of generators and relations for this new bicategory. Using this list and a coherence theorem for symmetric monoidal 2-functors, I classify 2-dimensional extended HFTs.

Conventions. Throughout the paper, G is a fixed discrete group with identity element $e \in G$ and X is a pointed aspherical CW complex with $\pi_1(X, x) = G$ where x is the distinguished point of X.

2. Definitions

In the following definition, we use pointed manifolds and cobordisms endowed with relative homotopy classes of maps to $X \simeq BG$ to describe the principal G-bundles on manifolds.

Definition 2.1. *The oriented* 2*-dimensional* G*-equivariant cobordism bicategory* $X\text{Bord}_2$ has

(i) compact oriented 0-dimensional manifolds as objects,

(ii) triples (M, T, g) as 1-morphisms where M is an oriented cobordism between compact oriented 0-manifolds, $T \subset M$ is a finite set with $\partial M \subseteq T$, and $g \in [(M, T), (X, \{x\})]$ is a relative homotopy class,

–The fact that homotopy classes are relative to points allows us to label 1-morphisms with group elements. Figure 1 shows two 1-morphisms having the same source and target objects. In order to define 2-morphisms of $X\text{Bord}_2$ we need to recall certain surfaces with corners. A $\langle 2\rangle$-*surface* is

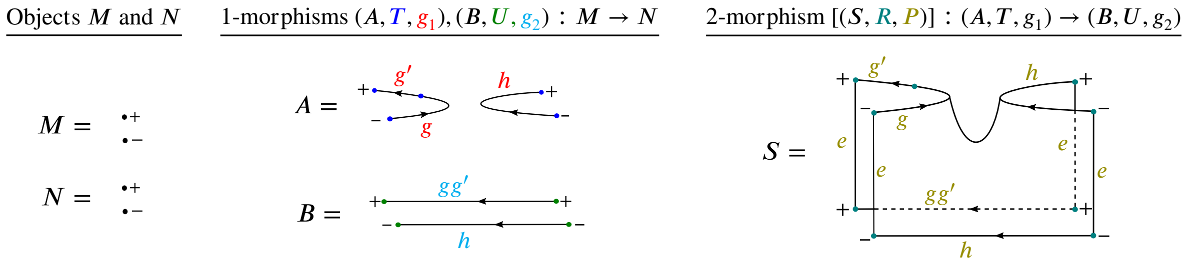

FIGURE 1. Examples of objects, 1-morphisms, and 2-morphisms in $X\mathrm{Bord}_2$

a compact surface with corners S equipped with two submanifolds with faces $\partial_h S$ and $\partial_v S$ called *horizontal* and *vertical faces* respectively such that $\partial S = \partial_h S \cup \partial_v S$ and $\partial_h S \cap \partial_v S$ is either empty or a face of both. A $\langle 2 \rangle$-surface S is said to be *cobordism type* if the vertical boundary $\partial_v S$ is topologically trivial (see S in Figure 1) –

(iii) isomorphism classes of triples (S, R, P) as 2-morphisms where S is an oriented cobordism type $\langle 2 \rangle$-surface, $R \subset \partial S$ is a finite subset, and $P \in [(S, R), (X, \{x\})]$ is a relative homotopy class such that $\partial_h S \cap \partial_v S \subset R$, $\partial_v S \cap R = \partial_h S \cap \partial_v S$, and the restriction P to each component of vertical face is the constant homotopy class. Two such triples (S, R, P) and (S', R', P') are isomorphic if there exists a diffeomorphism $F : S \to S'$ whose restriction to ∂S is identity with $F(R) = R'$ and $P = P' \circ [F] \in [(S, R), (X, \{x\})]$.

Note that the source and target 1-morphisms of a 2-morphism are given by the specific connected components of the horizontal face (see Figure 1). Also, the disjoint union operation ($\amalg$) turns $X\mathrm{Bord}_2$ into a symmetric monoidal bicategory.

Definition 2.2. For a symmetric monoidal bicategory $(\mathscr{C}, \otimes)$, a *$\mathscr{C}$-valued oriented 2-dimensional extended homotopy field theory (E-HFTs) with target $X \simeq K(G, 1)$* is a symmetric monoidal 2-functor

$$Z : (X\mathrm{Bord}_2, \amalg) \to (\mathscr{C}, \otimes).$$

3. Main Results

For any symmetric monoidal bicategory $\mathscr{C}$, the classification of $\mathscr{C}$-valued E-HFTs with target X is formulated below in terms of equivalence of two bicategories. The first bicategory $\mathsf{E\text{-}HFT}(X, \mathscr{C})$ is a functor bicategory which has $\mathscr{C}$-valued oriented 2-dimensional E-HFTs with target X as objects, symmetric monoidal natural transformations between such 2-functors as 1-morphisms, and symmetric monoidal modifications between such transformations as 2-morphisms (see [Sc]).

To describe the second bicategory $\mathbb{XP}(\mathscr{C})$, we first give a presentation $\mathbb{XP}$ of $X\mathrm{Bord}_2$ by generators and relations. A presentation consists of four sets; namely generating objects, generating 1-morphisms, generating 2-morphisms, and generating relations among 2-morphisms. Each presentation $\mathbb{P}$ gives rise to a freely generated symmetric monoidal bicategory $\mathsf{F}(\mathbb{P})$ and $\mathbb{P}$ is called a presentation of $\mathscr{C}$ if there exists a symmetric monoidal equivalence $\mathsf{F}(\mathbb{P}) \to \mathscr{C}$. Figure 2 shows one, denoted by $\mathbb{XP}$, for $X\mathrm{Bord}_2$.

For the presentation $\mathbb{XP}$ given in Figure 2 the objects of $\mathbb{XP}(\mathscr{C})$ are images of the generators of $\mathbb{XP}$ subject to the relations, the 1-morphisms are natural maps taking generating objects to 1-morphisms of $\mathscr{C}$ and generating 1-morphisms to 2-morphisms of $\mathscr{C}$, and the 2-morphisms are natural maps taking generators to 2-morphisms of $\mathscr{C}$.

Theorem 3.1. *Let $\mathbb{XP}$ be the presentation of the bicategory $X\mathrm{Bord}_2$ given in Figure 2. Then for any symmetric monoidal bicategory $\mathscr{C}$ there is an equivalence of bicategories $\mathsf{E\text{-}HFT}(X, \mathscr{C}) \simeq \mathbb{XP}(\mathscr{C})$.*

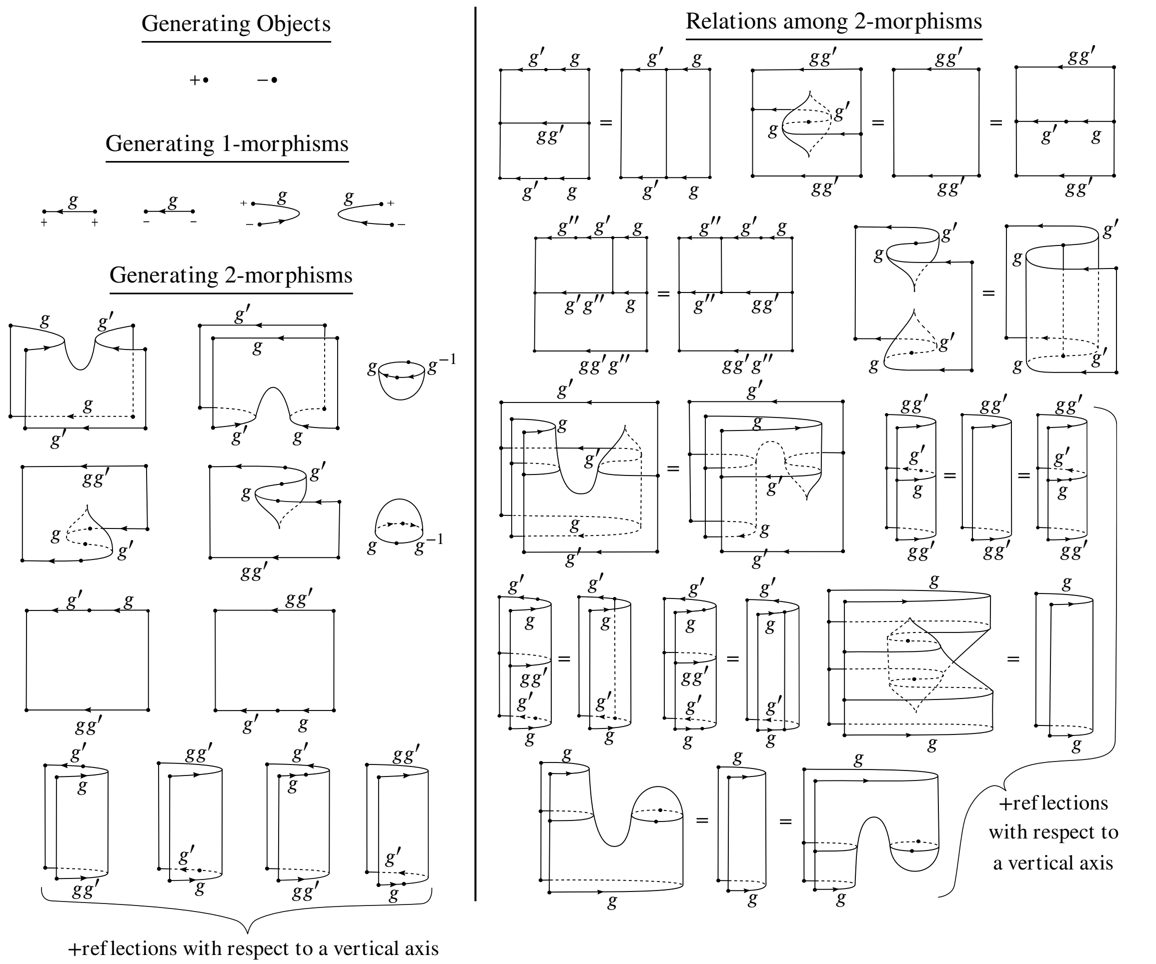

FIGURE 2. Generating objects, 1-morphisms, 2-morphisms, and relations among 2-morphisms

Next, for any commutative ring $\Bbbk$, we study $\mathrm{Alg}^2_{\Bbbk}$-valued E-HFTs where the symmetric monoidal bicategory $(\mathrm{Alg}^2_{\Bbbk}, \otimes_{\Bbbk})$ has $\Bbbk$-algebras as objects, bimodules as 1-morphisms, and bimodule maps as 2-morphisms. The following notions are the ingredients of the next theorem. A strongly graded G-algebra is a G-graded associative $\Bbbk$-algebra $A = \oplus_{g\in G} A_g$ with unity such that $A_g A_{g'} = A_{gg'}$ for all $g, g' \in G$. The opposite G-algebra is $A^{op} = \oplus_{g\in G} A_{g^{-1}}$ where the order of multiplication is reversed.

A Frobenius algebra is a pair (A, η) where $A = \oplus_{g\in G} A_g$ is a G-algebra such that each A_g is a finitely generated projective $\Bbbk$-module and $\eta : \oplus_{g\in G} A_g \otimes A_{g^{-1}} \to \Bbbk$ is a nondegenerate bilinear form satisfying $\eta(ab, c) = \eta(a, bc)$ for any $a, b, c \in A$. A quasi-biangular G-algebra is a strongly graded Frobenius G-algebra (A, η_z) such that A_e is a separable algebra and η_z is a trace given by $\eta_z(a, b) = \mathrm{Tr}(\mu_{abz} : A_e \to A_e)$ where μ_{abz} is the mutliplication by abz map and z is a fixed central element in A_e. Lastly, we need G-graded Morita contexts between G-algebras which were introduced by Boisen [Bo]. I introduce the notion of compatibility for a graded Morita context between Frobenius G-algebras as a condition about preserving the Frobenius structures (see [Sö]).

Theorem 3.2. *Any $\mathrm{Alg}^2_{\Bbbk}$-valued oriented 2-dimensional E-HFT with target $X \simeq K(G, 1)$ determines a triple (A, B, ζ) where A and B are quasi-biangular G-algebras, and ζ is a compatible G-graded Morita context between A and B^{op}. Moreover, any such triple (A, B, ζ) is realized by an oriented 2-dimensional E-HFT with target X.*

Every 2-dimensional E-HFT gives rise to a nonextended one by restricting $Z : X\mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}$ to full subcategory $X\mathrm{Cob}_2$ consisting of circles and cobordisms between them. The relation between these nonextended HFTs and Turaev's classification [Tu] of (nonextended) 2-dimensional HFTs in terms of crossed Frobenius G-algebras is given by the notion of a G-center (see [Sö]).

Corollary 3.2.1. *Let $Z : XBord_2 \to Alg^2_{\Bbbk}$ be an oriented E-HFT giving (A, B, ζ). Then, the nonextended oriented HFT obtained from Z by restricting to $XCob_2$ is the nonextended oriented HFT associated to the G-center of the quasi-biangular G-algebra (A, η_A).*

By studying the 1- and 2-morphisms of $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ I obtain the following result.

Theorem 3.3. *Let $Frob^G$ be the bicategory of quasi-biangular G-algebras, compatible G-graded Morita contexts, and equivalences of such Morita contexts. Then, there exists an equivalence of bicategories E-HFT$(X, Alg^2_{\Bbbk}) \simeq Frob^G$.*

4. Applications

In their seminal paper, J. Baez and J. Dolan [BD] stated a stabilization hypothesis, a tangle hypothesis, and a cobordism hypothesis. These hypotheses are interconnected and originate from the connections between topology and higher categories. For brevity, here we only focus on the cobordism hypothesis which is now a theorem due to Lurie [Lu] and Ayala-Francis [AF]. This theorem gives a classification of fully-extended[1] framed[2] TFTs in terms of specific objects in the target higher category called *fully-dualizable objects*:

$$\text{Fully-extended framed TFTs} \xleftrightarrow{\text{Bijection}} \text{Fully-dualizable objects.}$$

Lurie [Lu] reformulated the cobordism hypothesis using (∞, n)-categories and provided a sketch of a proof. His formalism replaces the bijection with the weak homotopy equivalence between the space of fully-extended framed TFTs and the space of fully-dualizable objects. Moreover, Lurie [Lu] generalized the cobordism hypothesis to manifolds equipped with (topological) structures like principal Γ-bundles for a group Γ. This generalization uses the orthogonal group action on the space of fully-dualizable objects and homotopy fixed points:

$$\begin{array}{ccc}
\text{Space of fully-extended } \Gamma\text{-structured TFTs} & \xleftrightarrow{\text{Weak homotopy equivalence}} & \text{Space of homotopy } \Gamma\text{-fixed points} \\
{\scriptstyle \text{When } \Gamma=\{e\}} \Big\downarrow & & \Big\downarrow {\scriptstyle \text{When } \Gamma=\{e\}} \\
\text{Space of fully-extended framed TFTs} & \xleftrightarrow[\text{Weak homotopy equivalence}]{} & \text{Space of fully-dualizable objects}
\end{array}$$

In this approach, 2-dimensional E-HFTs with target $X \simeq K(G, 1)$ correspond to $(G \times SO(2))$-structured 2-dimensional E-TFTs. When $\Bbbk$ is an algebraically closed field of characteristic zero, Davidovich [Da] computed homotopy $(G \times SO(2))$-fixed points in $\mathrm{Alg}^2_{\Bbbk}$ as quasi-biangular G-algebras. Then, the comparison of the classification result stated above with Davidovich's result on homotopy $(G \times SO(2))$-fixed points yields a new proof of the $(G \times SO(2))$-structured cobordism hypothesis.

Remark. There are similar theorems and their applications in the unoriented setting (see [Sö]).

[1]Fully-extended means the extended cobordism higher category has 0-dimensional manifolds as objects.

[2]Framed TFTs are defined using manifolds and cobordisms carrying a framing.

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