Survey on low-dimensional HQFTs
Quantum Symmetries Student Seminar

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Functorial approach to TQFTs

\[
\left( \text{Cob}_n, \sqcup, \phi, \sigma \right)
\]

**Objects:** Closed oriented \((n - 1)\)-dimensional manifolds

**Morphisms:** Oriented \(n\)-dimensional cobordisms up to orientation preserving diffeomorphisms relative to boundary

\[
\left( \text{Vect}_C, \otimes, \phi, T \right)
\]

**Objects:** Finite dimensional complex vector spaces

**Morphisms:** Linear transformations

Let \(M_1\) and \(M_2\) be \((n-1)\)-dim closed oriented manifolds.

Composition of morphisms for \(n=2\)

Symmetric braiding for \(n=2\)

Different choices of collars yield different but diffeomorphic (equivalent) smooth struct. on manifolds are either abstract manifolds or embedded in \(\mathbb{R}^{N(n)}\)}
Functorial approach to TQFTs

\textbf{Objects:} Closed oriented \((n-1)\)-dimensional manifolds

\textbf{Morphisms:} Oriented \(n\)-dimensional cobordisms up to orientation preserving diffeomorphisms relative to boundary

\begin{align*}
\textbf{Cob}_n & \quad \textbf{Vect}_\mathbb{C} \\
\textbf{Objects:} & \quad \text{Finite dimensional complex vector spaces} \\
\textbf{Morphisms:} & \quad \text{Linear transformations}
\end{align*}

\textbf{Definition (Atiyah)}

An \(n\)-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor \(Z : (\text{Cob}_n, \amalg) \rightarrow (\text{Vect}_\mathbb{C}, \otimes)\).

produce manifold invariants which behave well under gluing.

numerical
**Manifolds with maps to** $K(G, 1)$-**space and category** $\text{XCob}_n$

**The main idea of HQFTs with** $K(G, 1)$-**targets:**

Equip manifolds and cobordisms with homotopy classes of maps to a fixed $K(G, 1)$-space

**Objects:** Closed **pointed** oriented $(n - 1)$-dimensional manifolds equipped with homotopy classes of maps to $(X, x)$

**Morphisms:** Equivalence classes of oriented $n$-dimensional cobordisms equipped with homotopy classes of maps to $X$
Homotopy Quantum Field Theories

The main idea of HQFTs with $K(G, 1)$-targets:

Equip manifolds and cobordisms with homotopy classes of maps to a fixed $K(G, 1)$-space

$\text{X Cob}_n$

**Objects:** Closed *pointed* oriented $(n - 1)$-dimensional manifolds equipped with homotopy classes of maps to $(X, x)$

**Morphisms:** Equivalence classes of oriented $n$-dimensional cobordisms equipped with homotopy classes of maps to $X$

**Definition (Turaev)**

An $n$-dimensional homotopy quantum field theory with target $X$ is a symmetric monoidal functor $Z : (\text{X Cob}_n, \sqcup) \rightarrow (\text{Vect}_\mathbb{C}, \otimes)$. An invariant of manifolds with homotopy classes

\[ n \geq 2 \]
1-dimensional HQFTs

Generators for the skeleton of $\text{XCob}_1$

Objects: $\hat{\tau} \mapsto \mathcal{Z}(\hat{\tau}) = V \quad \mathcal{Z}(\hat{\imath}) = W$

Morphisms: $g \mapsto \rho_g$ indexed by $g \in G$

$\Rightarrow \rho_g \circ \rho_h = \rho_{gh}$ for $g = e \Rightarrow \text{id} \times \text{id}$

$\text{e is the identity elt. of } G$

Relations for the skeleton of $\text{XCob}_1$

$\Rightarrow \eta$ is nondegenerate

$\Rightarrow W \cong V^* \Rightarrow \eta = \text{ev}$

A 1-dimensional HQFT with $K(G, 1)$-target determines a finite $\dim.$ $\mathcal{C}$-repres. $\rho$ of $G$.

$\mathcal{Z}(\hat{\imath}) = \chi^G_{\rho}(g) \in \mathcal{C}$
2-dimensional HQFTs with $K(G, 1)$-targets

Generators for the skeleton of $\text{X Cob}_2$

Objects: $\{ g \}$ for $g \in G$

$\mathcal{Z}(\mathbf{g}) = A_g$

Morphisms:

$A_g$ from $g h h^{-1}$ to $A_{h g h^{-1}}$

$A_{g h}$ from $g h$ to $A_{h g h^{-1}}$

Relations for the skeleton of $\text{X Cob}_2$

$A \cong \bigoplus_{g \in G} A_g$

$\varphi : G \to \text{Aut}(A)$

$\Rightarrow$ + many other relations

Nondegenerate symmetric bilinear form

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2-dimensional HQFTs

**Definition**

A crossed Frobenius $G$-algebra is a triple $(A = \bigoplus_{g \in G} A_g, \eta, \varphi)$ where

- $A = \bigoplus_{g \in G} A_g$ is a $G$-graded associative $k$-algebra i.e. $A_g A_h \subseteq A_{gh}$,
- $\eta : A \otimes A \to k$ is a nondegenerate symmetric bilinear form with $\eta|_{A_g \otimes A_h} = 0$ for $gh \neq e$,
- $\varphi : G \to \text{Aut}(A)$ with $ab = \varphi_g(b)a$ for all $a \in A_g, b \in A_h$ and $g, h \in G$.

**Theorem (Turaev)**

There is an equivalence of categories $\text{Func}(\text{XCob}_2, \text{Vect}_\mathbb{C}) \simeq \text{GFrob}_\mathbb{C}$. 

\[ \eta_h = \begin{cases} g & \text{if } h = g^{-1} \cr g^{-1} & \text{if } h = g \end{cases} \] 

\[ \varphi_h = \begin{cases} g & \text{if } h = g^{-1} \cr g^{-1} & \text{if } h = g \end{cases} \]
Note on 2-dimensional extended HQFTs

Question: Can we combine 1- and 2-dimensional HQFTs considering X-surfaces as cobordisms between cobordisms similar to the notion of morphisms between morphisms in higher category theory? Yes!

Objects:

1-morphisms:

2-morphisms:

Every \( Z: \text{Bord}_2 \to \text{Alg}_{\mathbb{k}} \)

gives a \( G \)-graded algebra

\( A = \bigoplus_{g \in G} A_g \)

and non-deg sym. bilinear form \( \eta: A \otimes A \to \mathbb{k} \)
such that

- Each \( A_g \) is rank 1 \((A_e,A_e)\)-bimodule
- \( A_e \) is semisimple \( \mathbb{k} \)-algebra.

\[ A \otimes_{\mathbb{k}} B \overset{R_g \otimes_{\mathbb{k}} (S_{g'})A \otimes_{\mathbb{k}} B}{\Rightarrow} \text{bimodule map} \]

\[ A(M_g)A \otimes_{\mathbb{k}} B(N_{g'})B \]
Some works on 2-dimensional HQFTs

- For a finite group $G$, $G$-equivariant TQFTs were initially studied by Dijkgraaf-Witten, Freed, and Quinn.

- Unoriented 2-dimensional HQFTs were studied by Tagami and Kapustin-Turzillo.

- 2-dimensional HQFTs with $K(A, 2)$-targets were studied by Brightwell-Turner and Rodrigues.

- 2-dimensional HQFTs with arbitrary targets were studied by Staic-Turaev.

- The connection between 2-dimensional HQFTs and flat gerbes were studied by Rodrigues and Bunke-Turner-Willerton.

- 2-dimensional extended HQFTs with arbitrary targets is a work on progress.
3-dimensional HQFTs

State-sum HQFT
- studied by Turaev-Virelizier generalizing the work of Turaev-Viro-Barrett-Westbury on state-sum TQFTs.
- utilises triangulations of 3-manifolds, more generally skeletons of 3-manifolds.
- constructed from a spherical $G$-fusion category.

Surgery HQFT
- studied by Turaev-Virelizier generalizing the work of Reshetikhin-Turaev on surgery TQFT.
- utilises surgery representation of 3-manifolds.
- constructed from a $G$-modular category.
Spherical G-fusion categories

A **G-graded category** is a $\mathbb{C}$-additive monoidal category $\mathcal{C}$ endowed with $\mathbb{C}$-additive full-subcategories $\{\mathcal{C}_g\}_{g \in G}$ such that

1. each object $U \in \mathcal{C}$ splits as $\bigoplus_{i=1}^{n} U_{g_i}$ where $U_{g_i} \in \mathcal{C}_{g_i}$,
2. if $U \in \mathcal{C}_g$ and $V \in \mathcal{C}_h$, then $U \otimes V \in \mathcal{C}_{gh}$,
3. if $U \in \mathcal{C}_g$ and $V \in \mathcal{C}_h$, then $\text{Hom}_{\mathcal{C}}(U, V) = 0$,
4. the unit object $1 \in \mathcal{C}_e$.

A **G-graded category** $\mathcal{C}$ is **spherical** if it is spherical as a monoidal category.

A **G-graded category** is **pre-fusion** if it is prefusion as a monoidal category. In a pre-fusion $G$-category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ each object of $\mathcal{C}_g$ is a finite direct sum of simple objects of $\mathcal{C}_g$.

A **G-fusion category** is a pre-fusion $G$-category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that the set of isomorphism classes of simple objects of $\mathcal{C}_g$ is finite and nonempty for all $g \in G$. 

Diagram calculus for pivotal cats:

- $\text{id}_X = \begin{array}{c} X \\ X \end{array}$
- $\text{id}_{X^*} = \begin{array}{c} X^* \\ X \end{array}$
- $\text{ev}_X = \begin{array}{c} X \\ Y \end{array}$
- $\text{coev}_X = \begin{array}{c} X^* \\ X \end{array}$
- $\text{tr}_{eg}(f) = \begin{array}{c} X \\ \circ \end{array}$
- $\text{tr}_{rg}(f) = \begin{array}{c} X \\ \circ \end{array}$
3-dimensional state-sum HQFTs

- Let $C$ be a spherical $G$-fusion category with a set $I = \bigsqcup_{g \in G} I_g$ of simple objects.
- A $C$-colored graph in $S^2$ is an oriented graph graph in $S^2$ whose edges are labeled with an object of $C$. Vertices are colored as follows:

For a given $C$-colored graph $\Gamma$, using graphical calculus we have $F_e(\Gamma) : \otimes H_v \to \text{End}_c(1) \simeq 1_k$
- Let $M$ be a closed oriented 3-manifold equipped with a homotopy class $p \in [M, X]$ and let $T$ be a triangulation of $M$.
- A $G$-labeling is an assignment $\ell : T^2 \to G$ such that $\prod_{i=1}^n \ell(b_i) = e$ around any edge.
- A $C$-coloring is an assignment $c : T^2 \to I$ such that $c(r) \in I_{\ell(r)}$ for all 2-faces $r$ of $T$. 

\begin{align*}
\text{Hom}(1, i \otimes j \otimes k) &= H_v \\
\text{Hom}(1, c(1) \otimes c(2) \otimes c(3) \otimes c(4)) &= H_c(e) \\
\text{Hom}(1, c(1)^* \otimes c(2) \otimes c(3) \otimes c(4)^*) &= H_c(e^{op})
\end{align*}
3-dimensional state-sum HQFTs

For a given coloring \( c \) the number \(|c| \in \mathbb{k}\) is defined as follows:

At each vertex of \( T \) take a ball \( B^3 \) centered at the vertex \( \partial B^3 \cap T^2 \)
gives a \( c \)-colored graph on sphere \( \partial B^3 \)

- vertices of \( \Gamma \): edges of \( T^2 \)
- edges of \( \Gamma \): interior of \( T^2 \)

\[ |c| = \bigotimes_{v \in \partial T} \bigotimes_{e \in T^2} \mathbb{F}(\Gamma_v) \in \mathbb{k} \]

Theorem (Turaev, Virelizier)

Let \((M, p)\) be a 3-dimensional closed \( X \)-cobordism with a triangulation \( T \) and let \( C \) be a spherical \( G \)-fusion category with a set \( I = \coprod_{g \in G} I_g \) of simple objects. The number

\[ |M|_C = (\dim(C_e))^{-|T^3|} \sum_{c} \left( \prod_{r \in T^2} \dim c(r) \right) |c| \in \mathbb{k} \]

is a topological invariant of \( M \) where \(|T^3|\) is the number of 3-simplices.
**G-modular categories**

- A **G-crossed category** is a G-graded category $\mathcal{C}$ equipped with a *crossing*; a monoidal functor $\varphi : \overline{G} \to \text{Aut}(\mathcal{C})$ such that $\varphi_g(\mathcal{C}_h) \subseteq \mathcal{C}_{g^{-1}h}g$ for all $g, h \in G$.

- A **G-braided category** is a G-crossed category $(\mathcal{C}, \varphi)$ endowed with a **G-braiding** $\tau$ i.e. the family of isomorphisms $\{\tau_{X,Y} : X \otimes Y \to Y \otimes \varphi_{|Y|}(X)\}_{X,Y \in \mathcal{C}, Y \text{is homogeneous}}$ satisfying certain conditions.

- A **twist** of $\mathcal{C}$ is the family of isomorphisms $\theta = \{\theta_X : X \to \varphi_{|X|}(X)\}_{X \text{ is homogeneous}}$.

- A **G-ribbon category** is a pivotal G-graded category $\mathcal{C}$ such that its crossing $\varphi$ is pivotal and its twist $\theta$ is self-dual.

- A **G-modular category** is a G-ribbon G-fusion category whose $S$-matrix is invertible.

- A **G-modular category** is a G-ribbon G-fusion category whose neutral component is a modular tensor category.
3-dimensional surgery HQFTs

Let $I = \Pi_{g \in G} I_g$ be a representative set of simple objects of a $G$-modular category $C$.

- $D = \sqrt{\dim(C)} = \sqrt{\sum_{i \in I} \dim_l(i) \dim_r(i)} \in \mathbb{k}^*$.
- $\Delta_\sim = \sum_{i \in I} \nu_i^{-1}(\dim(i))^2 \in \mathbb{k}$ where $\theta_i = \nu_i \text{id}_i$ for the twist $\theta_i : i \to i$ morphism.
- $L(C) = \bigoplus_{g \in G} L_g$ with $L_g = \bigoplus_{X \in \text{Ob}(C_g)} \text{End}_C(X)/\sim$ is the fusion $\mathbb{k}$-algebra where $f \circ g \sim g \circ f$ for $f : X \to Y$ and $g : Y \to X$ in $C_g$.
- $\omega_C^g = \sum_{i \in I_g} \dim(i) \langle \text{id}_i \rangle \in L_g$ and $\omega_C = (\omega_C^g)_{g \in G}$.
- $\Lambda : \otimes_{r=1}^n L_g \to \mathbb{k}$ is an $n$-form defined as follows:

Figure is taken from 3d surgery HQFT paper of Turaev-Virelizier
3-dimensional surgery HQFTs

Theorem (Turaev, Virelizier)

Let \((M, p)\) be a 3-dimensional closed \(X\)-cobordism obtained as a surgery on \(S^3\) along a framed link \(\ell\) with \(#\ell\) components. Let \(C\) be \(G\)-modular category with rank \(D\). Then the number

\[
\tau_C(M) = \Delta_{\sigma(\ell)} D^{-\sigma(\ell) - #\ell - 1} F(\ell, p, \omega_C) \in \mathbb{k}
\]

is a homeomorphism invariant \(X\)-cobordism \((M, p)\) where \(\sigma(\ell)\) is the signature of the compact oriented 4-manifold \(B_\ell\) with \(\partial B_\ell = M\), obtained from \(B^4\) by attaching 2-handles along tubular neighborhoods of the components of \(\ell\) in \(\partial B^4\).
**Graded center: the connection between two approaches**

The $G$-center $\mathcal{Z}_G(C)$ of a $G$-graded category over $\mathbb{k}$ is the category obtained as the (left) center of $C$ relative to its neutral component $C_e$. That is,

- the objects of $\mathcal{Z}_G(C)$ are (left) half braidings of $C$ relative to $C_e$; pairs $(A, \sigma)$ where $A \in \text{Ob}(C)$ and $\sigma = \{\sigma_X : A \otimes X \to X \otimes A\}_{X \in C_e}$ satisfying $\sigma_X \otimes Y (\text{id}_X \otimes \sigma_Y)(\sigma_X \otimes \text{id}_Y)$,

- morphisms $\text{Hom}((A, \sigma), (A', \sigma'))$ is a morphism $f : A \to A'$ such that $(\text{id}_X \otimes f)\sigma_X = \sigma'_X(f \otimes \text{id}_X)$ for all $X \in \text{Ob}(C_e)$.

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**Theorem (Turaev, Virelizier)**

*If $C$ is an additive spherical $G$-fusion category over an algebraically closed field such that $\dim(C_e) \neq 0$, then $\mathcal{Z}_G(C)$ is a $G$-modular category.*

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**Theorem (Turaev, Virelizier)**

*For any additive spherical $G$-fusion category $C = \bigoplus_{g \in G} C_g$ over an algebraically closed field $\mathbb{k}$ with $\dim(C_e) \neq 0$, the state-sum HQFT $|\cdot|_C$ and the surgery HQFT $\tau_{\mathcal{Z}_G(C)}$ are isomorphic.*
Some works on 3-dimensional HQFTs and $G$-tensor categories

- For a finite group $G$, $G$-equivariant 3-dimensional TQFTs were initially studied by Dijkgraaf-Witten and Freed-Quinn.

- Extended 3-dimensional HQFTs were studied by Schweigert-Woike, Müller-Woike, and Maier-Nikolaus-Schweigert.

- Braided crossed $G$-categories were studied by Müger.

- Modular $G$-tensor categories were studied by Maier-Nikolaus-Schweigert, A. Krillov, and Turaev-Virelizier.

- Generalization of Kuperberg and Hennings invariants to 3-dimensional closed $X$-manifolds were studied by Virelizier.
Thank You for Your Attention!