Université de Lille

Survey on low-dimensional HQFTs

Quantum Symmetries Student Seminar

Kursat Sozer
kursat.sozer@univ-lille.fr

November 20, 2020

Functorial approach to TQFTs

$(\underline{\mathbf{Cob}_n}, \sqcup, \emptyset, \sigma)$

Objects: Closed oriented $(n-1)$-dimensional manifolds

Morphisms: Oriented n-dimensional cobordisms up to orientation preserving diffeomorphisms relative to boundary

$(\underline{\mathbf{Vect}_{\mathbb{C}}}, \otimes_{\mathbb{C}}, \mathbb{C}, T)$

Objects: Finite dimensional complex vector spaces

Morphisms: Linear transformations

manifolds are either abstract manifolds or embedded in $\mathbb{R}^{N(n)\gg 0}$

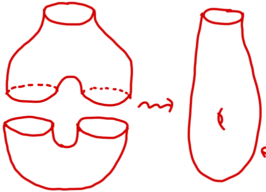
Let M_1 and M_2 be $(n-1)$-dim closed oriented manifolds

between M_1 & M_2 is a quintuple $(W, M_1, M_2, \psi_1, \psi_2)$

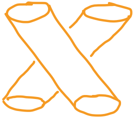
compact oriented n-dim manifold with $\partial W = \overline{\psi_1(M_1)} \sqcup \psi_2(M_2)$

$\psi_i : M_i \hookrightarrow \partial W$ orientation preserving embedding

Composition of morphisms for $n=2$



$\boxed{\text{Symmetric}}$ braiding for $n=2$



different choices of collars yield different but diffeomorphic (equivalent) smooth struct. on

Functorial approach to TQFTs

$\mathbf{Cob}_n$

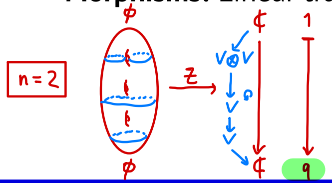
Objects: Closed oriented $(n-1)$-dimensional manifolds

Morphisms: Oriented n-dimensional cobordisms up to orientation preserving diffeomorphisms relative to boundary

$\mathbf{Vect}_{\mathbb{C}}$

Objects: Finite dimensional complex vector spaces

Morphisms: Linear transformations



Definition (Atiyah)

An n-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor $Z : (\mathrm{Cob}_n, \amalg) \to (\mathrm{Vect}_{\mathbb{C}}, \otimes)$.

produce numerical manifold invariants which behave well under gluing.

Manifolds with maps to $K(G, 1)$-space and category XCob$_n$

discrete group

The main idea of HQFTs with $K(G, 1)$-targets:

X with a basepoint $x \in X$

Equip manifolds and cobordisms with homotopy classes of maps to a fixed $K(G, 1)$-space

a model for classifying space BG

XCob$_n$

each connected component is equipped with a point.

Objects: Closed $\boxed{\textit{pointed}}$ oriented $(n-1)$-dimensional manifolds equipped with relative homotopy classes of maps to (X, x)

$(X\mathrm{Cob}_n, \sqcup, \emptyset, \sigma)$ sym. mon. cat.

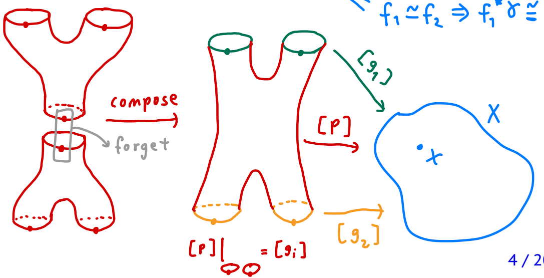
manifolds & cobordisms equipped with isom. classes of principal G-bundles

$$\begin{tikzcd} P \arrow[d] & EG \arrow[d, "\gamma"] \\ M \arrow[r, "\exists f"] & BG \end{tikzcd}$$

$P \cong f^*\gamma$

$f_1 \simeq f_2 \Rightarrow f_1^*\gamma \cong f_2^*\gamma$

Morphisms: Equivalence classes of oriented n-dimensional cobordisms equipped with homotopy classes of maps to X



$\mathrm{Cob}_n \hookrightarrow X\mathrm{Cob}_n$

- add basepoints
- consider trivial homotopy classes

if Y is contractible target $\cong K(*,1)$-space then $Y\mathrm{Cob}_n \simeq \mathrm{Cob}_n$.

Homotopy Quantum Field Theories

The main idea of HQFTs with $K(G,1)$-targets:

Equip manifolds and cobordisms with homotopy classes of maps to a fixed $K(G,1)$-space

$\mathbf{XCob}_n$

Objects: Closed *pointed* oriented $(n-1)$-dimensional manifolds equipped with homotopy classes of maps to (X,x)

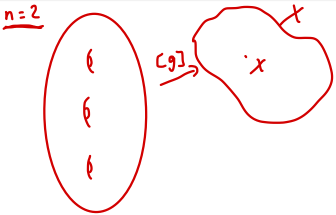
Morphisms: Equivalence classes of oriented n-dimensional cobordisms equipped with homotopy classes of maps to X

Definition (Turaev)

An n-dimensional homotopy quantum field theory with target X is a symmetric monoidal functor $\simeq K(G,1)$

$Z : (\mathrm{XCob}_n, \amalg) \to (\mathrm{Vect}_{\mathbb{C}}, \otimes)$.

invariant of manifolds with homotopy classes

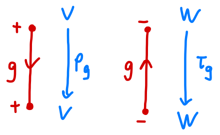


1-dimensional HQFTs

Generators (w.r.t composition & disj. union) for the skeleton of $XCob_1$

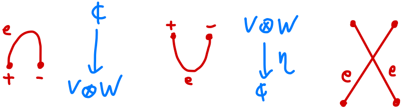
Objects: $+$ $\quad$ $-$ $\qquad Z(+) = V \qquad Z(-) = W$

Morphisms:



→ indexed by $g \in G$

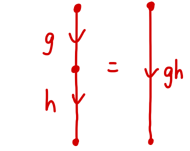
for $g = e \Rightarrow id_+$ & id_-



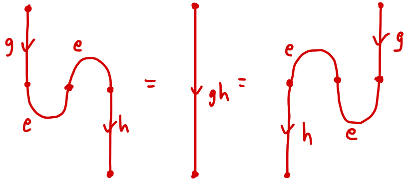
all possible signs

e is the identity elt. of G

Relations for the skeleton of $XCob_1$



$\Rightarrow \rho_h \circ \rho_g = \rho_{gh}$ $\quad \boxed{\rho : G \to GL(V)}$



$\Rightarrow \eta$ is nondeg.

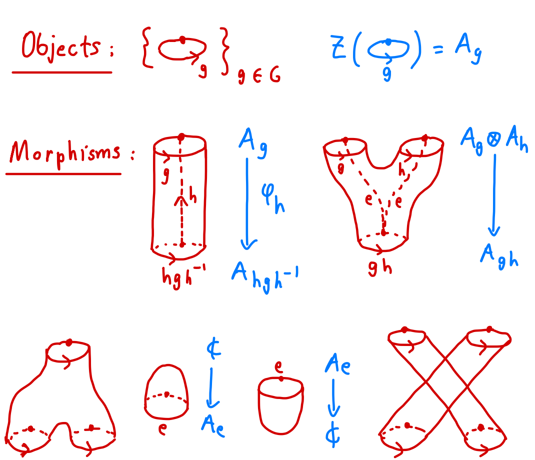
$\rightsquigarrow W \simeq V^* \rightsquigarrow \eta = ev$

A 1-dimensional HQFT $Z : XCob_1 \to Vect_{\mathbb{C}}$ with $K(G,1)$-target determines a finite dim. $\mathbb{C}$-repres. ρ of G.

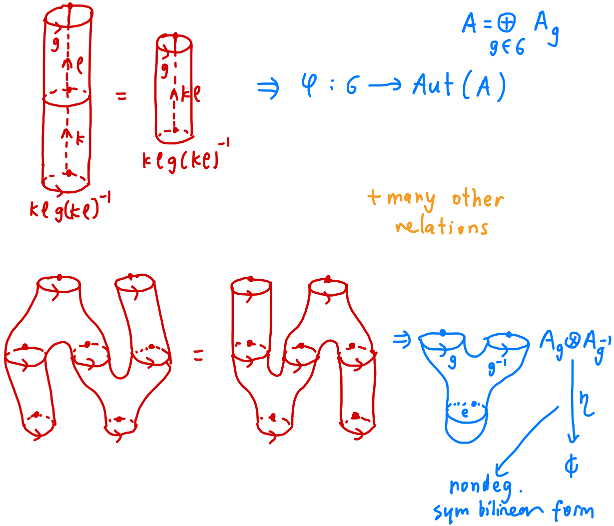
$Z(\overset{g}{\bigcirc}) = \chi_\rho(g) \in \mathbb{C}$

2-dimensional HQFTs with $K(G,1)$-targets

Generators for the skeleton of XCob_2



Relations for the skeleton of XCob_2



2-dimensional HQFTs

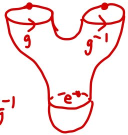
Definition

A crossed Frobenius G-algebra is a triple $(A = \oplus_{g\in G} A_g, \eta, \varphi)$ where

- $A = \oplus_{g\in G} A_g$ is a G-graded associative $\Bbbk$-algebra i.e. $A_g A_h \subseteq A_{gh}$,
- $\eta : A \otimes A \to \Bbbk$ is a nondegenerate symmetric bilinear form with $\eta|_{A_g \otimes A_h} = 0$ for $gh \neq e$,
- $\boxed{\varphi : G \to \mathrm{Aut}(A)}$ with $\overbrace{ab}^{A_{gh}} = \overbrace{\varphi_g(b)a}^{A_{ghg^{-1}g} = A_{gh}}$ for all $a \in A_g$, $b \in A_h$ and $g, h \in G$.

φ-crossed commutativity

$\varphi_h =$ 

$\eta|_{A_g \otimes A_{g^{-1}}} =$ 

sym mon functors & mon. nat. trans.

cat. of crossed Frob G-alg and their morphisms

Theorem (Turaev)

There is an equivalence of categories $Func(XCob_2, Vect_{\mathbb{C}}) \simeq GFrob_{\mathbb{C}}$.

Note on 2-dimensional extended HQFTs

Question: Can we combine 1- and 2-dimensional HQFTs considering X-surfaces as cobordisms between cobordisms similar to the notion of morphisms between morphisms in higher category theory? Yes!

sym. mon. 2-functor

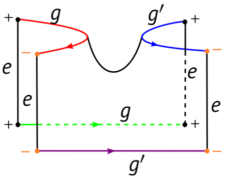
$$\mathrm{XBord}_2 \xrightarrow{\mathcal{Z}} \mathrm{Alg}^2_{\Bbbk}$$

field char$(\Bbbk)=0$

Objects: $\overset{+}{\bullet}$ $\overset{-}{\bullet}$ $\qquad A \otimes_{\Bbbk} B$ $\Bbbk$-alg

1-morphisms: $+\bullet \xrightarrow{g} \bullet+$ $\qquad {}_A(M_g)_A$ bimodule

2-morphisms:



$$\begin{array}{c} {}_{A\otimes_{\Bbbk}B}R_g \otimes_{\Bbbk} (S_{g'})_{A\otimes_{\Bbbk}B} \\ \big\downarrow \\ {}_A(M_g)_A \otimes_{\Bbbk} {}_B(N_{g'})_B \end{array}$$

bimodule map

Every $\mathcal{Z}: X\mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}$ gives a G-graded algebra $A = \bigoplus_{g\in G} A_g$ and nondeg sym. bilinear form $\eta: A\otimes A \to \Bbbk$ such that

- Each A_g is rank 1 (A_e, A_e)-bimodule
- A_e is semisimple $\Bbbk$-algebra.

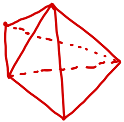
Some works on 2-dimensional HQFTs

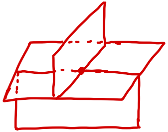
- For a finite group G, G-equivariant TQFTs were initially studied by Dijkgraaf-Witten, Freed, and Quinn.
- Unoriented 2-dimensional HQFTs were studied by Tagami and Kapustin-Turzillo.
- 2-dimensional HQFTs with $K(A,2)$-targets were studied by Brightwell-Turner and Rodrigues.
- 2-dimensional HQFTs with arbitrary targets were studied by Staic-Turaev.
- The connection between 2-dimensional HQFTs and flat gerbes were studied by Rodrigues and Bunke-Turner-Willerton.
- 2-dimensional extended HQFTs with arbitrary targets is a work on progress.

3-dimensional HQFTs

State-sum HQFT

- studied by Turaev-Virelizier generalizing the work of Turaev-Viro-Barrett-Westbury on state-sum TQFTs.
- utilises triangulations of 3-manifolds, more generally skeletons of 3-manifolds.

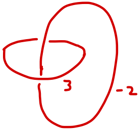


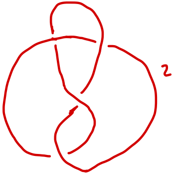


- constructed from a spherical G-fusion category.

Surgery HQFT

- studied by Turaev-Virelizier generalizing the work of Reshetikhin-Turaev on surgery TQFT.
- utilises surgery representation of 3-manifolds.





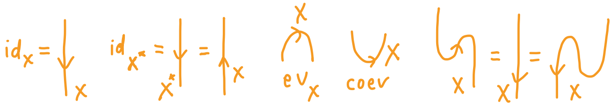
- constructed from a G-modular category

Spherical G-fusion categories

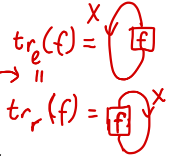
A *G-graded category* is a $\mathbb{C}$-additive monoidal category $\mathcal{C}$ endowed with $\mathbb{C}$-additive full-subcategories $\{\mathcal{C}_g\}_{g\in G}$ such that

1. each object $U \in \mathcal{C}$ splits as $\oplus_{i=1}^{n} U_{g_i}$ where $U_{g_i} \in \mathcal{C}_{g_i}$,
2. if $U \in \mathcal{C}_g$ and $V \in \mathcal{C}_h$, then $U \otimes V \in \mathcal{C}_{gh}$, $\rightsquigarrow$ $U \in \mathcal{C}_g$, $U^* \in \mathcal{C}_{g^{-1}}$
3. if $U \in \mathcal{C}_g$ and $V \in \mathcal{C}_h$, then $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$,
4. the unit object $1 \in \mathcal{C}_e$.

Diagram calculus for pivotal cats



spherical cat. $\longrightarrow$



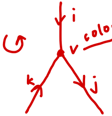
- A G-graded category $\mathcal{C}$ is *spherical* if it is spherical as a monoidal category.
- A G-graded category is *pre-fusion* if it is prefusion as a monoidal category. In a pre-fusion G-category $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ each object of $\mathcal{C}_g$ is a finite direct sum of simple objects of $\mathcal{C}_g$.
- A **G-fusion category** is a pre-fusion G-category $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ such that the set of isomorphism classes of simple objects of $\mathcal{C}_g$ is finite and nonempty for all $g \in G$.

3-dimensional state-sum HQFTs

- Let $\mathcal{C}$ be a spherical G-fusion category with a set $I = \coprod_{g \in G} I_g$ of simple objects.

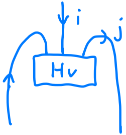
  isom. classes of

- A $\mathcal{C}$-colored graph in S^2 is an oriented graph graph in S^2 whose edges are labeled with an object of $\mathcal{C}$. Vertices are colored as follows:



$v \xrightarrow{\text{color}} \mathrm{Hom}(1, i \otimes j^* \otimes k) = H_v$

because graph is turned into by isotopy



For a given $\mathcal{C}$-colored graph Γ, using graphical calculus we have $\mathbb{F}_{\mathcal{C}}(\Gamma) : \underbrace{\bigotimes_{v \in \Gamma}}_{\text{all vertices}} H_v \to \mathrm{End}_{\mathcal{C}}(1) \simeq \Bbbk$

$\mathbb{F}_{\mathcal{C}}(\Gamma) \in \left(\bigotimes_{v \in \Gamma} H_v \right)^*$

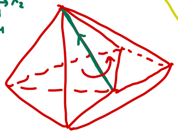
- Let M be a closed oriented 3-manifold equipped with a homotopy class $p \in [M, X]$ and let T be a triangulation of M.

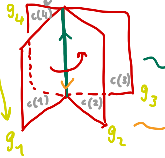
  2d faces of T

  identity elt of G

- A $\boxed{G\text{-labeling}}$ is an assignment $\ell : T^2 \to G$ such that $\prod_{i=1}^{n} \ell(b_i) = e$ around any edge.
- A $\boxed{\mathcal{C}\text{-coloring}}$ is an assignment $c : T^2 \to I$ such that $c(r) \in I_{\ell(r)}$ for all 2-faces r of T.

orient. of M





$g_1 g_2 g_3 g_4 = e$

due to orientation

oriented edge

$\rightsquigarrow \mathrm{Hom}(1, c(1) \otimes c(2)^* \otimes c(3) \otimes c(4)^*) = H_c(e)$

$\rightsquigarrow \mathrm{Hom}(1, c(1)^* \otimes c(2) \otimes c(3)^* \otimes c(4)) = H_c(e^{op})$

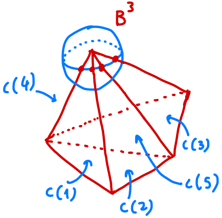
opposite or. edge

contraction homom.

$*_c : \underbrace{\bigotimes_{e \in \Gamma}}_{\text{oriented edges}} H_c(e)^* \to \Bbbk$

3-dimensional state-sum HQFTs

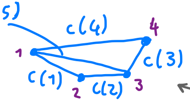
- For a given coloring c the number $|c| \in \mathbb{k}$ is defined as follows:



At each vertex of T take a ball B^3 centered at the vertex

$\partial B^3 \cap T^2$ gives a $\mathcal{C}$-colored graph Γ on sphere ∂B^3

- vertices of Γ: edges of T^2
- edges of Γ: interior of T^2



$$|c| = *_c\Big(\underbrace{\bigotimes_{v}}_{\text{over all vertices of } T} \mathbb{F}(\Gamma_v)\Big) \in \mathbb{k}$$

Note that $\bigotimes_{i=1}^{4} \overbrace{H_{v_i}}^{\text{color of vertices}} = \underbrace{\bigotimes_{e}}_{\text{edges incident to the corresponding vertex } v \text{ of } T} H_{c(e)}$

Theorem (Turaev, Virelizier)

Let (M, p) be a 3-dimensional closed X-cobordism with a triangulation T and let $\mathcal{C}$ be a spherical G-fusion category with a set $I = \coprod_{g \in G} I_g$ of isom. classes of simple objects. The number

$$|M|_{\mathcal{C}} = (\dim(\mathcal{C}_e))^{-|T^3|} \underbrace{\sum_{c}}_{\text{over all colorings}} \left(\prod_{r \in T^2} \dim c(r) \right) |c| \in \mathbb{k}$$

is a topological invariant of M where $|T^3|$ is the number of 3-simplices.

G-modular categories

$\boxed{g\otimes h = gh}$

discrete mon. cat.

$\varphi(g)$

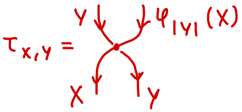
- A *G-crossed category* is a G-graded category $\mathcal{C}$ equipped with a *crossing*; a monoidal functor $\varphi : \overline{G} \to \mathrm{Aut}(\mathcal{C})$ such that $\varphi_g(\mathcal{C}_h) \subseteq \mathcal{C}_{g^{-1}hg}$ for all $g, h \in G$.

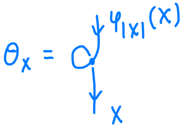
monoidal auto-equiv & mon nat. isoms.

- A *G-braided category* is a G-crossed category $(\mathcal{C}, \varphi)$ endowed with a *G-braiding* τ i.e. the family of isomorphisms $\{\tau_{X,Y} : X \otimes Y \to Y \otimes \varphi_{|Y|}(X)\}_{X,Y \in \mathcal{C}, Y \text{is homogeneous}}$ satisfying certain conditions.

$Y \in \mathcal{C}_g$ for some $g \in G$

$|Y| = g$





- A *twist of $\mathcal{C}$* is the family of isomorphisms $\theta = \{\theta_X : X \to \varphi_{|X|}(X)\}_{X \text{ is homogeneous}}$.
- A G-ribbon category is a pivotal G-graded category $\mathcal{C}$ such that its crossing φ is pivotal and its twist θ is self-dual.
- A G-modular category is a G-ribbon G-fusion category whose S-matrix is invertible.
- A G-modular category is a G-ribbon G-fusion category whose neutral component is a modular tensor category.

3-dimensional surgery HQFTs

Let $I = \amalg_{g\in G} I_g$ be a representative set of isom. classes of simple objects of a G-modular category $\mathcal{C}$.

- $\mathcal{D} = \sqrt{\dim(\mathcal{C}_e)} = \sqrt{\sum_{i\in I_e} \dim_l(i)\dim_r(i)} \in \Bbbk^*$.
- $\Delta_- = \sum_{i\in I_e} \nu_i^{-1}(\dim(i))^2 \in \Bbbk$ where $\theta_i = \nu_i \mathrm{id}_i$ for the twist $\theta_i : i \to i$ morphism.
- $L(\mathcal{C}) = \oplus_{g\in G} L_g$ with $L_g = \oplus_{X\in \mathrm{Ob}(\mathcal{C}_g)} \mathrm{End}_{\mathcal{C}}(X)/\sim$ is the fusion $\Bbbk$-algebra where $f\circ g \sim g\circ f$ for $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}_g$.
- $\omega_{\mathcal{C}}^g = \sum_{i\in I_g} \dim(i)\langle \mathrm{id}_i\rangle \in L_g$ and $\omega_{\mathcal{C}} = (\omega_{\mathcal{C}}^g)_{g\in G}$.
- $\Lambda : \otimes_{r=1}^n L_{g_r} \to \Bbbk$ is an n-form defined as follows:

$\theta_i = \nu_i$

$L(\mathcal{C})$ is a free $\Bbbk$-module with basis $\{\langle i\rangle\}_{i\in I}$

$[\gamma \ell_e \gamma'^{-1}] = \mu_\gamma \in \pi_1(\overline{\Omega}, z)$ complement

$(M, p\in[M,X])$ $p : \pi_1(\overline{\Omega}, z) \to G$

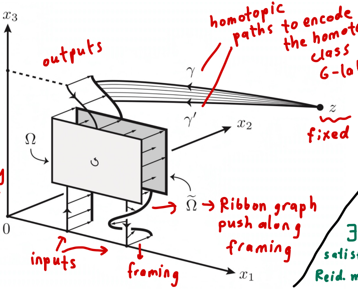
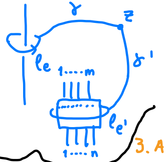


Figure is taken from 3d surgery HQFT paper of Turaev-Virelizier



$\mathcal{C}$-coloring: 1. Assign $[\gamma] \mapsto u_\gamma \in \mathcal{C}_{g(\mu_\gamma)}$ and isom. $u_{\beta\gamma} \to \varphi_{g(\beta^{-1})}(u_\gamma)$ $\forall \beta \in \pi_1(\overline{\Omega})$ satisfying certain conditions

2. Assign $[\gamma'] \mapsto v_{\gamma'} \in \mathrm{Hom}_{\mathcal{C}}(\otimes_{i=1}^m u_{\gamma_i}^{\varepsilon_i}, \otimes_{j=1}^n u_{\gamma_j}^{\varepsilon_j})$

$\varepsilon_i, \varepsilon_j = \mp 1$ depends on orientation s.t. $X^{+1} = X$, $X^{-1} = X^*$

3. Assign $\mathcal{C}$-colors to crossings using τ & θ

$\exists!$ functor $\underbrace{D_{\mathcal{C}}}_{\text{cat. of } \mathcal{C}\text{-colored diagrams}} \xrightarrow{F} \mathcal{C}$ satisfying colored Reid. moves

for $\mathcal{C}$-colored links F induces $\Lambda : \otimes_i L_{g_i} \to \Bbbk = \mathrm{End}_{\mathcal{C}}(1)$

3-dimensional surgery HQFTs

Theorem (Turaev, Virelizier)

Let (M, p) be a 3-dimensional closed X-cobordism obtained as a surgery on S^3 along a framed link ℓ with $\#\ell$ components. Let $\mathcal{C}$ be G-modular category with rank $\mathcal{D}$. Then the number

$$\tau_{\mathcal{C}}(M) = \Delta_{-}^{\sigma(\ell)} \mathcal{D}^{-\sigma(\ell)-\#\ell-1} F(\ell, p, \omega_{\mathcal{C}}) \in \Bbbk$$

is a homeomorphism invariant X-cobordism (M, p) where $\sigma(\ell)$ is the signature of the compact oriented 4-manifold B_ℓ with $\partial B_\ell = M$, obtained from B^4 by attaching 2-handles along tubular neighborhoods of the components of ℓ in ∂B^4.

Graded center: the connection between two approaches

The G-center $\mathcal{Z}_G(\mathcal{C})$ of a G-graded category $\mathcal{C}$ over $\mathbb{k}$ is the category obtained as the (left) center of $\mathcal{C}$ relative to its neutral component $\mathcal{C}_e$. That is,

- the objects of $\mathcal{Z}_G(\mathcal{C})$ are (left) half braidings of $\mathcal{C}$ relative to $\mathcal{C}_e$; pairs (A, σ) where $A \in \mathrm{Ob}(\mathcal{C})$ and $\sigma = \{\sigma_X : A \otimes X \to X \otimes A\}_{X \in \mathcal{C}_e}$ satisfying $\sigma_{X \otimes Y}(\mathrm{id}_X \otimes \sigma_Y)(\sigma_X \otimes \mathrm{id}_Y)$,
- morphisms $\mathrm{Hom}((A,\sigma),(A',\sigma'))$ is a morphism $f : A \to A'$ such that $(\mathrm{id}_X \otimes f)\sigma_X = \sigma'_X(f \otimes \mathrm{id}_X)$ for all $X \in \mathrm{Ob}(\mathcal{C}_e)$.

Theorem (Turaev, Virelizier)

If $\mathcal{C}$ is an additive spherical G-fusion category over an algebraically closed field such that $\dim(\mathcal{C}_e) \neq 0$, *then $\mathcal{Z}_G(\mathcal{C})$ is a G-modular category.*

Theorem (Turaev, Virelizier)

For any additive spherical G-fusion category $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ over an algebraically closed field $\mathbb{k}$ with $\dim(\mathcal{C}_e) \neq 0$, *the state-sum HQFT $|\cdot|_{\mathcal{C}}$ and the surgery HQFT $\tau_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic.*

Some works on 3-dimensional HQFTs and G-tensor categories

- For a finite group G, G-equivariant 3-dimensional TQFTs were initially studied by Dijkgraaf-Witten and Freed-Quinn.
- Extended 3-dimensional HQFTs were studied by Schweigert-Woike, Müller-Woike, and Maier-Nikolaus-Schweigert.
- Braided crossed G-categories were studied by Müger.
- Modular G-tensor categories were studied by Maier-Nikolaus-Schweigert, A. Krillov, and Turaev-Virelizier.
- Generalization of Kuperberg and Hennings invariants to 3-dimensional closed X-manifolds were studied by Virelizier.

Thank You for Your Attention!