Extended HQFTs in dimension 2

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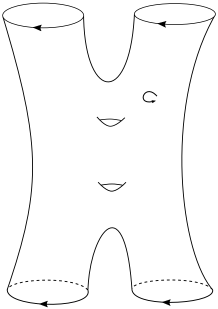
Two-dimensional topological quantum field theories

Definition (Atiyah)

A 2-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor $Z : (\mathrm{Cob}_2, \amalg, \emptyset, \sigma) \to (\mathrm{Vect}_{\mathbb{C}}, \otimes, \mathbb{C}, T)$.

Example: $Z : (\mathrm{Cob}_2, \amalg) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes)$



$$V \otimes V \longrightarrow V \otimes V$$

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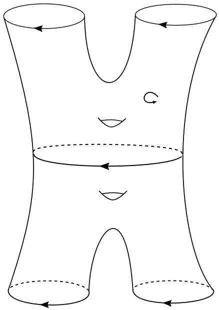


$$V \otimes V \longrightarrow V \longrightarrow V \otimes V$$

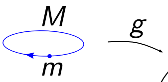
Generalizations of TQFTs

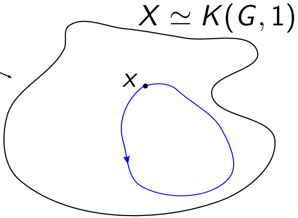
There are two different generalizations of TQFTs:

- Structured TQFTs by considering manifolds equipped with principal G-bundles.
- Extended TQFTs by considering manifolds with corners and higher categories.

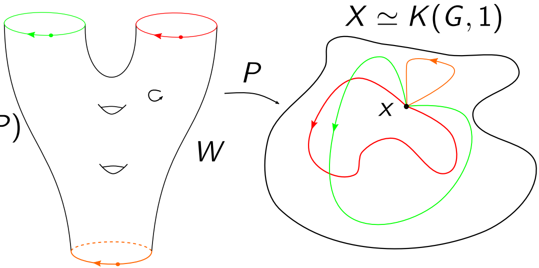
Structured TQFTs

For a discrete group G, Turaev formulated structured TQFTs by introducing X-manifolds and X-cobordisms.





X-circle is a pair (M, g) where $g \in [(M, m), (X, x)]$.



X-cobordism is a pair (W, P) where $P \in [W, X]$.

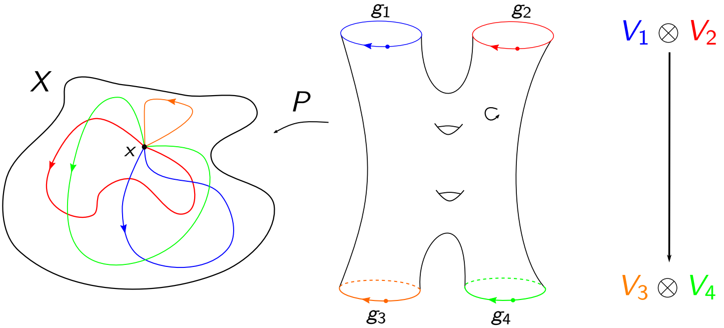
Two-dimensional homotopy quantum field theories

Definition (Turaev)

Let X be an aspherical pointed CW-complex. A 2-dimensional homotopy quantum field theory (HQFT) with target X is a symmetric monoidal functor $Z : (X\mathrm{Cob}_2, \amalg) \to (\mathrm{Vect}_{\mathbb{C}}, \otimes)$.

Example:

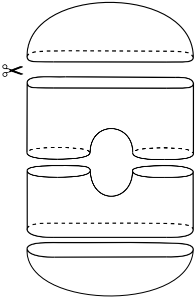
$$Z : (X\mathrm{Cob}_2, \amalg) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes)$$



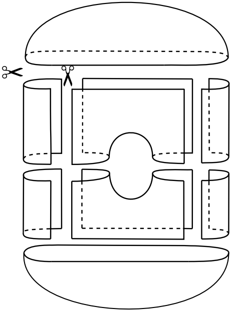
Extended TQFTs

The main motivation for extended TQFTs is to cut a cobordism along different directions and compute the invariants from the invariants of simpler pieces.

Nonextended



Extended

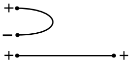
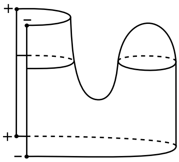


Two-dimensional Extended TQFTs

Definition (Schommer-Pries)

A 2-dimensional extended TQFT is a symmetric monoidal 2-functor

$$Z : \mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}.$$

	$\underline{\mathrm{Bord}_2}$	$\underline{\mathrm{Alg}^2_{\Bbbk}}$
Objects:		$\Bbbk$-algebras
1-morphisms:		Bimodules
2-morphisms:		Bimodule maps

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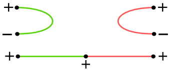
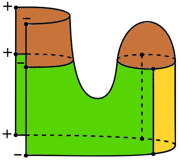
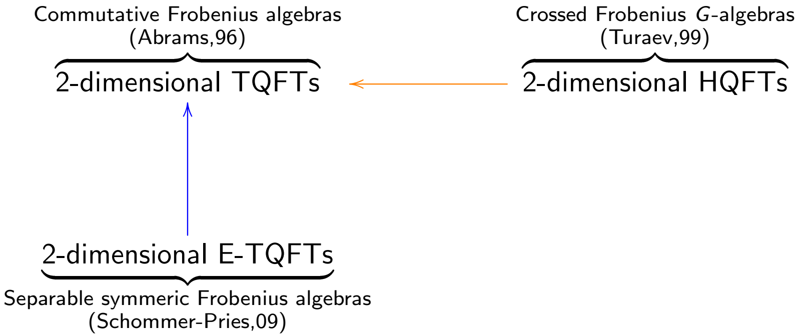
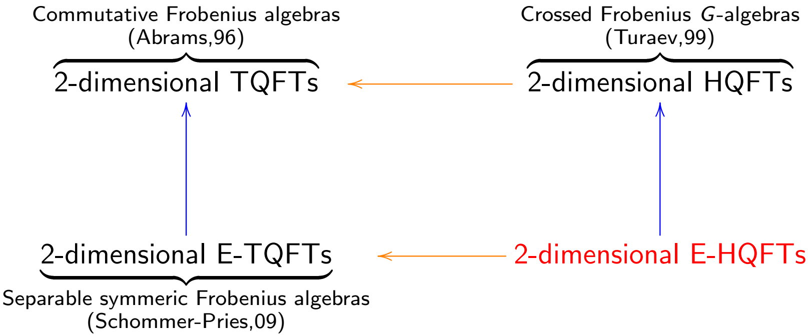
	Bord_2	$\mathrm{Alg}^2_{\Bbbk}$
Objects:	$\overset{+}{\bullet}$ $\overset{-}{\bullet}$	$\Bbbk$-algebras
1-morphisms:		${}_A M \otimes_A N_A$
2-morphisms:		${}_A R \otimes_A S_A \to {}_A M \otimes_A N_A \to {}_A T_A$



- Restriction to constant homotopy classes.
- Restriction to circles and cobordisms between circles.

The main goal



- Restriction to constant homotopy classes.
- Restriction to circles and cobordisms between circles.

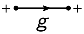
Definition

A 2-dimensional extended HQFT with target $X \simeq K(G,1)$ is a symmetric monoidal 2-functor

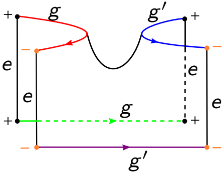
$$Z : X\mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}.$$

$$\mathrm{XBord}_2 \xrightarrow{\quad Z \quad} \mathrm{Alg}^2_{\Bbbk}$$

$A \otimes_{\Bbbk} B$ — $\Bbbk$-algebra



${}_A(M_g)_A$ — (A, A)-bimodule



$${}_{A\otimes_{\Bbbk}B}R_g \otimes_{\Bbbk} (S_{g'})_{A\otimes_{\Bbbk}B}$$
$$\downarrow$$
$${}_A(M_g)_A \otimes_{\Bbbk} {}_B(N_{g'})_B$$

$(A \otimes B, A \otimes B)$-bimodule map

The main theorem

Theorem (S.)

Every 2-dimensional E-HQFT with target $X \simeq K(G,1)$ determines a quasi-biangular G-algebra and every quasi-biangular G-algebra determines an E-HQFT. Moreover, G-graded Morita equivalent quasi-biangular G-algebras give equivalent E-HQFTs.

- Frobenius G-algebra is a G-graded $\mathbb{k}$-algebra $A = \oplus_{g \in G} A_g$ equipped with nondegenerate symmetric pairing $\beta : A \otimes A \to \mathbb{k}$.
- Quasi-biangular G-algebra A is a Frobenius G-algebra such that each A_g is both left and right rank one A_e-module and the principal component A_e is a separable algebra.

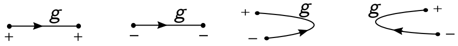
Example

$A = \oplus_{g \in G} M_k(\mathbb{k})$ where k is invertible in $\mathbb{k}$ and $\beta(A, B) = k\mathrm{Tr}(AB)$.

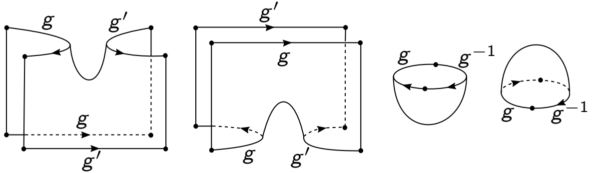
The list of generators for $X\text{Bord}_2$

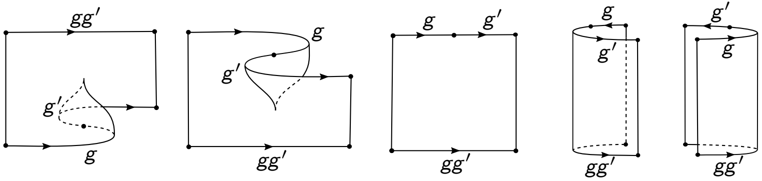
Generating Objects : $+\bullet$ $-\bullet$

Generating 1-morphisms :



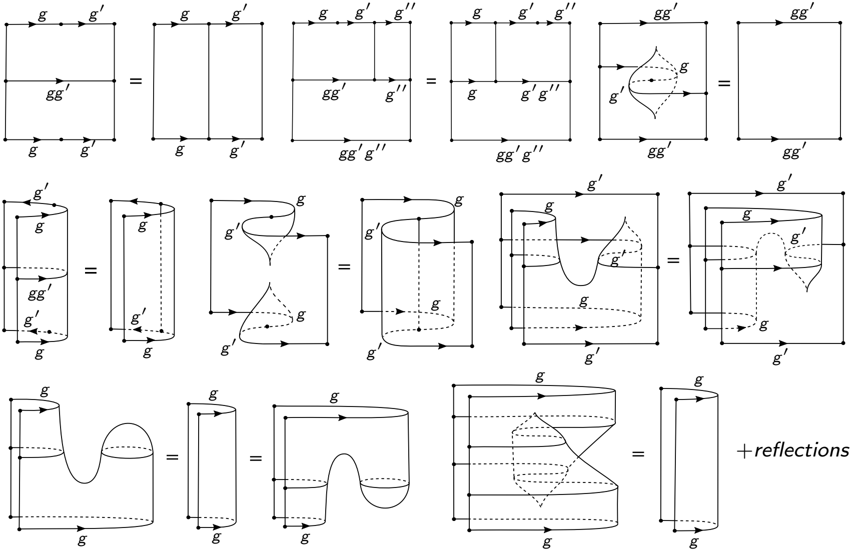
Generating 2-morphisms :





The list of relations for $X\mathrm{Bord}_2$

Relations among 2-morphisms :

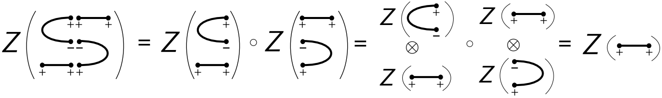


The Cobordism Hypothesis

- The cobordism hypothesis is a theorem about the classification of fully-extended (local) framed TQFTs in terms of *fully-dualizable objects* of the target higher category.

$$\text{Fully-extended framed TQFTs} \longleftrightarrow \text{Fully-dualizable objects}$$
$$Z \longmapsto Z(\bullet)$$

- Conjectured by J. Baez and J. Dolan.
- Proved by Lurie, Ayala - Francis, and Schommer-Pries.
- Why not any object but a fully-dualizable one?
  Consider a 1-dimensional framed (oriented) TQFT:



The Structured Cobordism Hypothesis

- Using (∞, n)-categories Lurie reformulated the cobordism hypothesis as the homotopy equivalence of spaces

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq (\mathcal{C}^{fd})^{\sim}.$$

- Lurie also generalized the cobordism hypothesis to manifolds equipped with structures using homotopy fixed points

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{G}, \mathcal{C}) \simeq ((\mathcal{C}^{fd})^{\sim})^{hG}.$$

- When $\Bbbk$ is algebraically closed field of characteristic zero, Davidovich computed $((\mathrm{Alg}_{\Bbbk}^{fd})^{\sim})^{h(G \times SO(2))}$ as G-equivariant algebras.

Corollary of the main theorem

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Then $(G \times SO(2))$-structured cobordism hypothesis for $\mathrm{Alg}_{\Bbbk}^2$-valued E-HFTs with target $X \simeq K(G,1)$ holds true.