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Strong A_∞-weights and scaling invariant Besov capacities

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Dedicated to the memory of my mother Dorina
and of my advisor Juha Heinonen

Abstract

This article studies strong A_∞-weights and Besov capacities as well as their relationship to Hausdorff measures. It is shown that in the Euclidean space $\mathbb{R}^n$ with $n \geq 2$, whenever $n-1 < s \leq n$, a function u yields a strong A_∞-weight of the form $w = e^{nu}$ if the distributional gradient ∇u has sufficiently small $|| \cdot ||_{\mathcal{L}^{s,n-s}}(\mathbb{R}^n;\mathbb{R}^n)$-norm. Similarly, it is proved that if $2 \leq n < p < \infty$, then $w = e^{nu}$ is a strong A_∞-weight whenever the Besov B_p-seminorm $[u]_{B_p(\mathbb{R}^n)}$ of u is sufficiently small.

Lower estimates of the Besov B_p-capacities are obtained in terms of the Hausdorff content associated with gauge functions h satisfying the condition $\int_0^1 h(t)^{p'-1} \frac{dt}{t} < \infty$.

1. Introduction

In this paper we study sufficient conditions under which one would get strong A_∞-weights in $\mathbb{R}^n$. We also study Besov capacity. We explore how Hausdorff measures and this capacity are related.

A doubling measure μ on $\mathbb{R}^n$ is a Radon measure for which there exists a constant $C > 1$ such that

$$0 < \mu(2B) \leq C\mu(B)$$

for all balls B. Throughout this paper λB represents the ball concentric with B with radius λ times the radius of B for every $\lambda > 0$. To every

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doubling measure μ on $\mathbb{R}^n$ with density w (that is, $\mu(A) = \int_A w(x)dx$ for all measurable sets $A \subset \mathbb{R}^n$), we can associate a quasidistance on $\mathbb{R}^n$ defined by

$$\delta_\mu(x, y) = \mu(B_{x,y})^{\frac{1}{n}}, \tag{1.1}$$

where $B_{x,y}$ is the smallest closed ball which contains the points x and y. To say that $\delta_\mu(x, y)$ is a quasidistance means by definition that it is non-negative and symmetric, that it vanishes if and only if $x = y$, and that it satisfies

$$\delta_\mu(x, z) \leq C(\delta_\mu(x, y) + \delta_\mu(y, z))$$

for some $C \geq 1$ and all $x, y, z \in \mathbb{R}^n$. If the above inequality was satisfied with $C = 1$, then the quasidistance $\delta_\mu(x, y)$ would in fact be a distance function.

A weight w is said to be an A_∞-*weight* if there exist constants $C \geq 1$ and $q > 1$ such that

$$\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}} \leq C\frac{1}{|B|}\int_B w(x)dx$$

for all balls $B \subset \mathbb{R}^n$. Here $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$ whenever E is measurable. See for example [15, Chapter 4] for a discussion about A_∞-weights.

To say that w is a *strong* A_∞-*weight* means, by definition, that w is an A_∞-weight and that the quasidistance δ_μ is comparable to a distance δ'_μ, namely there exists a distance function δ'_μ on $\mathbb{R}^n$ and a constant $C > 0$ such that

$$C^{-1}\delta_\mu(x, y) \leq \delta'_\mu(x, y) \leq C\delta_\mu(x, y). \tag{1.2}$$

Here μ is the measure on $\mathbb{R}^n$ with density w.

Strong A_∞-weights were introduced in the early 90's by Semmes and David [9], [30] when trying to identify the subclass of A_∞-weights that are comparable to the Jacobian determinants of quasiconformal mappings. See [10], [19], and [31].

Strong A_∞-weights also provide examples of admissible weights in the sense of [20]. In particular, our results give new such examples. See [3].

Bonk and Lang proved in [5] that if μ is a signed Radon measure on $\mathbb{R}^2$ such that $\mu^+(\mathbb{R}^2) < 2\pi$ and $\mu^-(\mathbb{R}^2) < \infty$, then $(\mathbb{R}^2, \tilde{D}_\mu)$ is bi-Lipschitz equivalent to $\mathbb{R}^2$ endowed with the Euclidean metric, where

$$\tilde{D}_\mu(x, y) = \inf\left\{\int_\alpha e^u ds : \alpha \text{ analytic curve connecting } x, y\right\},$$

the function u is a solution of $-\Delta u = \mu$ with $|\nabla u| \in L^2(\mathbb{R}^2)$, and $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. In particular, it is proved that $w = e^{2u}$ is comparable to the Jacobian of a quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, which implies that w is a strong A_∞-weight.

Here we prove a weaker result in $\mathbb{R}^n, n \geq 2$, related to the one from [5]. One of our results states that A_∞-weights of the form $w = e^{nu}$ are strong A_∞-weights if u is a locally integrable function with distributional gradient ∇u in the Morrey space $\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)$ with small $||\cdot||_{\mathcal{L}^{s,n-s}}(\mathbb{R}^n;\mathbb{R}^n)$-norm for some $s \in (n-1, n]$.

We recall that for $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the *Morrey space* $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is defined to be the linear space of measurable functions $u \in L^1_{loc}(\mathbb{R}^n)$ such that

$$||u||_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} \left(r^{-\lambda} \int_{B(x,r)} |u(y)|^p dy \right)^{1/p} < \infty.$$

In particular, $\mathcal{L}^{n,0}(\mathbb{R}^n) = L^n(\mathbb{R}^n)$. We refer to [17, p. 65] for more information about Morrey spaces and their use in the theory of partial differential equations. One notices that the weak Lebesgue space $L^{n,\infty}(\mathbb{R}^n)$ is contained in $\mathcal{L}^{s,n-s}(\mathbb{R}^n)$ for every $s \in [1, n)$. Indeed, it can be shown that for every $s \in [1, n)$, there exists a constant $C = C(n, s)$ such that

$$||u||_{\mathcal{L}^{s,n-s}(\mathbb{R}^n)} \leq C||u||_{L^{n,\infty}(\mathbb{R}^n)}. \tag{1.3}$$

Similarly we can define the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n;\mathbb{R}^m)$ for vector-valued measurable functions. It follows from the Poincaré inequality that for every $s \in [1, n]$, there exists a constant $C = C(n, s) > 0$ such that

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} \leq C||\nabla u||_{\mathcal{L}^{s,n-s}}(\mathbb{R}^n;\mathbb{R}^n), \tag{1.4}$$

where $[u]_{\mathrm{BMO}(\mathbb{R}^n)}$ is the *bounded mean oscillation* seminorm that measures the oscillation of u on balls in $\mathbb{R}^n$, given by

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n} \sup_{r>0} \frac{1}{|B(a,r)|} \int_{B(a,r)} |u(x) - u_{B(a,r)}|\, dx.$$

Here and throughout this paper u_E denotes the average of u on the measurable set $E \subset \mathbb{R}^n$ whenever $0 < |E| < \infty$.

As a consequence of our result, (see [18], [11]), we can obtain strong A_∞-weights of the form $w = e^{nu}$, where u is a distributional solution of

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = \mu$$

whenever μ is a signed Radon measure with small total variation. Indeed, it follows from the results of [18] and [11] that every distributional solution u of the previous equation has the property that the weak L^n-norm of the distributional gradient of u is controlled by $(|\mu|(\mathbb{R}^n))^{\frac{1}{n-1}}$.

In [4, Theorem 3.1] the authors prove that if u belongs to the Bessel potential space $L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$, $0 < \alpha < n$, then $w = e^{nu}$ is a strong A_∞-weight with data depending only on α, n, and the $L^{\alpha,\frac{n}{\alpha}}$-norm of u. We prove a result similar to [4, Theorem 3.1]. This result yields strong A_∞-weights of the form $w = e^{nu}$ when u has small Besov B_p-seminorm, $2 \le n < p < \infty$.

We define

$$B_p(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : ||u||_{B_p(\mathbb{R}^n)} < \infty\},$$

where

$$||u||_{B_p(\mathbb{R}^n)} = ||u||_{L^p(\mathbb{R}^n)} + [u]_{B_p(\mathbb{R}^n)} \tag{1.5}$$

with

$$[u]_{B_p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{2n}} \, dx \, dy \right)^{1/p}. \tag{1.6}$$

It is known that $L^{\frac{n}{p},p}(\mathbb{R}^n) \subset B_p(\mathbb{R}^n)$ for every $p \in (n, \infty)$, so our result generalizes [4, Theorem 3.1] to Besov B_p spaces.

Besov spaces have recently been used in the study of quasiconformal mappings in metric spaces and in geometric group theory. See [6] and [7].

Capacities associated with Besov spaces were studied by Netrusov in [26] and [27] and by Adams and Hurri-Syrjänen in [2]. Bourdon in [6] studied Besov B_p-capacity in metric settings.

We develop a theory of Besov B_p-capacity on $\mathbb{R}^n$ and we prove that this capacity is a Choquet set function. We also relate Hausdorff measure and Besov capacity. Some of the ideas used here follow [22], [23], [7], and [6].

This is part of my PhD thesis at the University of Michigan under the guidance of Professor Juha Heinonen.

2. Scaling invariant Besov spaces

In this section we prove some basic properties of the scaling invariant Besov spaces $B_p(\mathbb{R}^n)$ and their closed subspaces $B_p(\Omega)$ and $B_p^0(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set.

The expressions $||u||_{B_p(\mathbb{R}^n)}$ and $[u]_{B_p(\mathbb{R}^n)}$ from (1.5) and (1.6) are called the *Besov norm* and the *Besov seminorm* of u respectively. We have

$$[u]_{B_p(\mathbb{R}^n)} = 0 \text{ if and only if } u \text{ is constant a.e.} \tag{2.1}$$

We know that $B_p(\mathbb{R}^n)$ is a reflexive Banach space and moreover, $\mathcal{S}$ is dense in $B_p(\mathbb{R}^n)$ where $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the Schwartz class. See [1, Theorem 4.1.3] and [28, Chapter 3]. It is easy to see that every Lipschitz function with compact support belongs to $B_p(\mathbb{R}^n)$. We also note that $[u_{(\alpha)}]_{B_p(\mathbb{R}^n)} = [u]_{B_p(\mathbb{R}^n)}$, where $u_{(\alpha)}(x) = u(\alpha x)$ for $\alpha > 0$.

If $u \in L^1_{loc}(\mathbb{R}^n)$ is such that $[u]_{B_p(\mathbb{R}^n)} < \infty$, then $u \in \mathrm{BMO}(\mathbb{R}^n)$. Indeed, it is easy to see that

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} \leq C(n,p)[u]_{B_p(\mathbb{R}^n)}. \tag{2.2}$$

For an open set $\Omega \subset \mathbb{R}^n$ we define

$$B_p(\Omega) = \{u \in B_p(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

For a function $u \in B_p(\Omega)$ we let $||u||_{B_p(\Omega)} = ||u||_{B_p(\mathbb{R}^n)}$.

We notice that $B_p(\Omega)$ is a closed subspace of $B_p(\mathbb{R}^n)$ with respect to the Besov norm, hence it is itself a reflexive space.

We define $B_p^0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $B_p(\mathbb{R}^n)$. Since $C_0^\infty(\Omega) \subset B_p(\Omega)$, it follows that $B_p^0(\Omega) \subset B_p(\Omega)$, so we can say that $B_p^0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $B_p(\Omega)$.

Lemma 2.1. *$B_p(\Omega)$ is closed under truncations. In particular, bounded functions in $B_p(\Omega)$ are dense in $B_p(\Omega)$.*

Proof. It is easy to show that $v_\lambda \in B_p(\Omega)$ for every $\lambda \geq 0$, where $v_\lambda = \min(v, \lambda)$. Indeed, we have $||v_\lambda||_{L^p(\mathbb{R}^n)} \leq ||v||_{L^p(\mathbb{R}^n)}$ and $[v_\lambda]_{B_p(\Omega)} \leq [v]_{B_p(\Omega)}$.

To prove the second assertion, for positive integers k we define the function v_k by $v_k = \max(-k, \min(v,k))$. From the first assertion it follows that $v_k \in B_p(\Omega)$ with $||v_k||_{B_p(\Omega)} \leq ||v||_{B_p(\Omega)}$. Furthermore, we have $|v_k(x)| \leq |v(x)|$ for every $x \in \mathbb{R}^n$ and from the Lebesgue Dominated Convergence Theorem it follows that $||v_k - v||_{L^p(\Omega)} \to 0$. We also notice that $|v_k(x) - v_k(y)| \leq |v(x) - v(y)|$ for every $x, y \in \mathbb{R}^n$ and since $|(v_k(x) - v_k(y)) - (v(x) - v(y))| \to 0$ for almost every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, it follows from the Lebesgue Dominated Convergence Theorem that $[v_k - v]_{B_p(\Omega)} \to 0$ as $k \to \infty$. ■

For a measurable function $u : \Omega \to \mathbb{R}$, we let $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$.

Lemma 2.2. *If $u_j \to u$ in $B_p(\Omega)$ and $v_j \to v$ in $B_p(\Omega)$, then*

$$\min(u_j, v_j) \to \min(u, v) \quad \textit{in} \quad B_p(\Omega).$$

Proof. It suffices to show that if u_j converges to u in $B_p(\Omega)$, then u_j^+ converges to u^+ in $B_p(\Omega)$. By Lemma 2.1, we have that $u^+ \in B_p(\Omega)$ whenever $u \in B_p(\Omega)$. We can assume without loss of generality that u_j and u are 0 everywhere in $\mathbb{R}^n \setminus \Omega$ and that $u_j \to u$ pointwise a.e. in $\mathbb{R}^n$. Since

$$|u_j^+(x) - u^+(x)| \leq |u_j(x) - u(x)| \tag{2.3}$$

for every $x \in \mathbb{R}^n$, it is clear that $u_j^+ \to u^+$ in $L^p(\Omega)$.

For every $x, y \in \mathbb{R}^n$ we have

$$|(u_j^+(x) - u^+(x)) - (u_j^+(y) - u^+(y))| \leq |u_j^+(x) - u^+(x)| + |u_j^+(y) - u^+(y)|$$

which, together with (2.3) and the fact that $u_j \to u$ pointwise a.e. in $\mathbb{R}^n$, implies that

$$|(u_j^+(x) - u^+(x)) - (u_j^+(y) - u^+(y))| \to 0 \tag{2.4}$$

for a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ as $j \to \infty$. We also notice that

$$\begin{aligned} |(u_j^+(x) - u^+(x)) - (u_j^+(y) - u^+(y))| &\leq |u_j^+(x) - u_j^+(y)| + |u^+(x) - u^+(y)| \\ &\leq |u_j(x) - u_j(y)| + |u(x) - u(y)| \end{aligned}$$

for every $x, y \in \mathbb{R}^n$ which implies, via the triangle inequality, that

$$\begin{aligned} |(u_j^+(x) - u^+(x)) - (u_j^+(y) - u^+(y))| &\leq |(u_j(x) - u(x)) - (u_j(y) - u(y))| \\ &\quad + 2\,|u(x) - u(y)| \end{aligned}$$

for every $x, y \in \mathbb{R}^n$. We notice that the above inequality implies

$$[u_j^+ - u^+]_{B_p(\Omega)} \leq [u_j - u]_{B_p(\Omega)} + 2[u]_{B_p(\Omega)} \tag{2.5}$$

for every integer $j \geq 1$. From (2.4) and the inequality preceding (2.5) it follows, via a general version of Lebesgue Dominated Convergence Theorem (see [14, p. 57, Exercise 20]) that

$$[u_j^+ - u^+]_{B_p(\Omega)} \to 0$$

as $j \to \infty$. This, together with the fact that $||u_j^+ - u^+||_{L^p(\mathbb{R}^n)} \to 0$ as $j \to \infty$, implies that $||u_j^+ - u^+||_{B_p(\Omega)} \to 0$ as $j \to \infty$, which proves our claim. ■

Next we show that the space $B_p^0(\Omega)$ is a lattice.

Lemma 2.3. *If $u, v \in B_p^0(\Omega)$, then $\min(u, v)$ and $\max(u, v)$ are in $B_p^0(\Omega)$. Moreover, if $u \in B_p^0(\Omega)$ is nonnegative, then there exists a sequence of nonnegative functions $\varphi_j \in C_0^\infty(\Omega)$ converging to u in $B_p(\Omega)$.*

Proof. It is enough to show, due to Lemma 2.2, that u^+ is in $B_p^0(\Omega)$ whenever u is in $C_0^\infty(\Omega)$. Let $\eta \in C_0^\infty(B(0,1))$, $0 \leq \eta \leq 1$, be a mollifier. For every $\varepsilon > 0$, we define η_ε by $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\frac{x}{\varepsilon})$. We notice that $\eta_\varepsilon * u^+ \in C_0^\infty(\Omega)$ for $\varepsilon < \varepsilon_0 = \operatorname{dist}(\operatorname{supp} u, \partial\Omega)$. We know that $\eta_\varepsilon * u^+ \to u^+$ uniformly on $\mathbb{R}^n$ as $\varepsilon \to 0$. We also know that $||\eta_\varepsilon * u^+||_{L^p(\mathbb{R}^n)} \leq ||u^+||_{L^p(\mathbb{R}^n)}$ and that $[\eta_\varepsilon * u^+]_{B_p(\mathbb{R}^n)} \leq [u^+]_{B_p(\mathbb{R}^n)}$ for every $\varepsilon > 0$. Then $\eta_\varepsilon * u^+, 0 < \varepsilon < \varepsilon_0$ is a sequence of nonnegative functions in $C_0^\infty(\Omega)$, bounded in $B_p^0(\Omega)$, and converging to u^+ uniformly on $\mathbb{R}^n$ as $\varepsilon \to 0$. The convexity and reflexivity of $B_p^0(\Omega)$ together with Mazur's lemma [32, p. 120] imply that $u^+ \in B_p^0(\Omega)$ and that there exists a sequence of nonnegative functions φ_j in $C_0^\infty(\Omega)$ converging to u^+ in $B_p(\Omega)$. This finishes the proof. ■

Proposition 2.4. *$C_0^\infty(\mathbb{R}^n)$ is dense in $B_p(\mathbb{R}^n)$.*

Proof. Let $u \in B_p(\mathbb{R}^n)$. Without loss of generality we can assume that u is in $\mathcal{S}$, and in particular bounded. For every integer $k \geq 2$, we define $\varphi_k : \mathbb{R}^n \to \mathbb{R}$,

$$\varphi_k(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq k \\ \left(\ln \frac{k^2}{|x|}\right)/\ln k & \text{if } k < |x| \leq k^2 \\ 0 & \text{if } |x| > k^2. \end{cases}$$

Then $\varphi_k \in B_p(\mathbb{R}^n)$ and moreover, $[\varphi_k]^p_{B_p(\mathbb{R}^n)} \leq C(\ln k)^{1-p}$. (See (3.4).) Let $\eta \in C_0^\infty(B(0,1))$ be a mollifier. Let $\widetilde{\varphi}_k = \eta * \varphi_k$. Then $\widetilde{\varphi}_k \in C_0^\infty(\mathbb{R}^n)$ and

$$[\widetilde{\varphi}_k]_{B_p(\mathbb{R}^n)} \leq [\varphi_k]_{B_p(\mathbb{R}^n)} \leq C(\ln k)^{1-p}.$$

Moreover $\widetilde{\varphi}_k(x) = 1$ for $|x| \leq k-1$ and $\widetilde{\varphi}_k(x) = 0$ for $|x| \geq k^2+1$.

Let $u_k = u\widetilde{\varphi}_k$. Then $u_k \in C_0^\infty(\mathbb{R}^n)$ and

$$||u-u_k||_{L^p(\mathbb{R}^n)} \leq ||u\chi_{\mathbb{R}^n\setminus B(0,k-1)}||_{L^p(\mathbb{R}^n)} \to 0 \text{ as } k \to \infty.$$

We also have

$$[u-u_k]_{B_p(\mathbb{R}^n)} \leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(1-\widetilde{\varphi}_k(y))^p|u(x)-u(y)|^p}{|x-y|^{2n}}\,dx\,dy\right)^{1/p} + ||u||_{L^\infty(\mathbb{R}^n)}[\widetilde{\varphi}_k]_{B_p(\mathbb{R}^n)} \to 0$$

as $k \to \infty$. This completes the proof. ∎

Lemma 2.5. *Let φ be a Lipschitz function with compact support in $\mathbb{R}^n$. If $u \in B_p(\mathbb{R}^n)$, then $u\varphi \in B_p(\mathbb{R}^n)$ with*

$$||u\varphi||_{B_p(\mathbb{R}^n)} \leq C\,||u||_{B_p(\mathbb{R}^n)},$$

where C depends on n, p, the Lipschitz constant of φ, and the diameter of supp φ.

Proof. Let R be the diameter of supp φ. We choose $x_0 \in$ supp φ such that supp $\varphi \subset \overline{B}$, where $B = B(x_0, R)$. Let $L > 0$ be a constant such that $|\varphi(x)-\varphi(y)| \leq L|x-y|$ for every $x, y \in \mathbb{R}^n$. Then note that $||\varphi||_{L^\infty(\mathbb{R}^n)} \leq LR$. We also notice that

$$||u\varphi||_{L^p(\mathbb{R}^n)} \leq ||\varphi||_{L^\infty(\mathbb{R}^n)}\,||u||_{L^p(\mathbb{R}^n)},$$

hence $u\varphi \in L^p(\mathbb{R}^n)$. For every $x, y \in \mathbb{R}^n$ we have

$$|u(x)\varphi(x)-u(y)\varphi(y)| \leq |u(x)-u(y)|\,|\varphi(x)| + |u(y)|\,|\varphi(x)-\varphi(y)|.$$

Therefore, if we denote

$$I = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{2n}} dx\,dy,$$

we have

$$[u\varphi]_{B_p(\mathbb{R}^n)} \le ||\varphi||_{L^\infty(\mathbb{R}^n)}[u]_{B_p(\mathbb{R}^n)} + I^{1/p}. \tag{2.6}$$

We notice that $I = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \int_{2B}\int_{2B} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{2n}} dx\,dy \\ I_2 &= \int_{2B}\int_{\mathbb{R}^n\setminus 2B} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{2n}} dx\,dy \\ I_3 &= \int_{\mathbb{R}^n\setminus 2B}\int_{2B} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{2n}} dx\,dy. \end{aligned}$$

From the definition of I_1 we have, since φ is Lipschitz with constant L,

$$I_1 \le \int_{2B}\int_{2B} \frac{L^p\,|u(y)|^p}{|x-y|^{2n-p}} dx\,dy \tag{2.7}$$

$$= L^p \int_{2B} |u(y)|^p \left(\int_{2B} |x-y|^{p-2n} dx\right) dy. \tag{2.8}$$

We have

$$\int_{2B} |x-y|^{p-2n} dx \le C(n,p) R^{p-n} \tag{2.9}$$

for every $y \in 2B$, where we recall that R is the radius of B. From (2.7) and (2.9) we get

$$I_1 \le C(n,p)\, L^p\, R^{p-n} \int_{2B} |u(y)|^p dy \le C(n,p)\, L^p\, R^{p-n}\, ||u||^p_{L^p(\mathbb{R}^n)}. \tag{2.10}$$

Since φ is supported in B, it follows from the definition of I_2 that in fact

$$I_2 = \int_B \int_{\mathbb{R}^n\setminus 2B} \frac{|u(y)|^p\,|\varphi(y)|^p}{|x-y|^{2n}} dx\,dy.$$

Hence

$$I_2 \le ||\varphi||^p_{L^\infty(\mathbb{R}^n)} \int_B \int_{\mathbb{R}^n\setminus 2B} \frac{|u(y)|^p}{|x-y|^{2n}} dx\,dy$$

and since $|x-y| \geq \frac{|x-x_0|}{2}$ whenever $x \in \mathbb{R}^n \setminus 2B$ and $y \in B$, we get

$$I_2 \leq 2^{2n}\, ||\varphi||^p_{L^\infty(\mathbb{R}^n)} \int_B |u(y)|^p\, dy \int_{\mathbb{R}^n \setminus 2B} \frac{1}{|x-x_0|^{2n}} dx.$$

Hence

$$\begin{aligned} I_2 &\leq C(n)\, ||\varphi||^p_{L^\infty(\mathbb{R}^n)}\, R^{-n} \int_B |u(y)|^p dy \\ &\leq C(n)\, ||\varphi||^p_{L^\infty(\mathbb{R}^n)}\, R^{-n}\, ||u||^p_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{2.11}$$

We notice that

$$\begin{aligned} I_3 &\leq 2^{p-1}\left(I_2 + \int_{\mathbb{R}^n \setminus 2B} \int_{2B} \frac{|u(x)-u(y)|^p\, |\varphi(x)-\varphi(y)|^p}{|x-y|^{2n}} dx\, dy \right) \\ &\leq 2^{p-1} I_2 + 2^{p-1}\, ||\varphi||^p_{L^\infty(\mathbb{R}^n)}\, [u]^p_{B_p(\mathbb{R}^n)}. \end{aligned} \tag{2.12}$$

From (2.6), (2.10), (2.11), (2.12), and the fact that $I = I_1 + I_2 + I_3$, we get that $u\varphi \in B_p(\mathbb{R}^n)$ with

$$||u\varphi||_{B_p(\mathbb{R}^n)} \leq C||u||_{B_p(\mathbb{R}^n)}, \tag{2.13}$$

where the constant C is as required. This finishes the proof. ■

Lemma 2.6. *Let φ be a Lipschitz function with compact support in $\mathbb{R}^n$. Suppose u_k is a sequence in $B_p(\mathbb{R}^n)$ converging to u in $B_p(\mathbb{R}^n)$. Then $u_k\varphi$ converges to $u\varphi$ in $B_p(\mathbb{R}^n)$.*

Proof. From Lemma 2.5, we have that $u_k\varphi \in B_p(\mathbb{R}^n)$ for every $k \geq 1$ and $u\varphi \in B_p(\mathbb{R}^n)$. Moreover, Lemma 2.5 implies

$$||u_k\varphi - u\varphi||_{B_p(\mathbb{R}^n)} \leq C||u_k - u||_{B_p(\mathbb{R}^n)} \tag{2.14}$$

for every $k \geq 1$ and since $u_k \to u$ in $B_p(\mathbb{R}^n)$, it follows that $u_k\varphi \to u\varphi$ in $B_p(\mathbb{R}^n)$. This finishes the proof. ■

Remark 2.7. Let $\Omega, \widetilde{\Omega}$ be bounded and open subsets of $\mathbb{R}^n$ such that $\Omega \subset\subset \widetilde{\Omega}$. Suppose that φ is a function in $C_0^\infty(\widetilde{\Omega})$ satisfying

$$0 \leq \varphi \leq 1,\ \varphi = 1 \text{ in } \Omega \text{ and } ||\nabla\varphi||_{L^\infty(\mathbb{R}^n)} \leq \frac{C(n)}{\operatorname{dist}(\Omega, \partial\widetilde{\Omega})}. \tag{2.15}$$

By doing an argument very similar to the one from Lemma 2.5, one can show that $u\varphi \in B_p(\widetilde{\Omega})$ whenever $u \in B_p(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\widetilde{\Omega})$ satisfies (2.15). Moreover, in this case

$$||u\varphi||_{B_p(\widetilde{\Omega})} \leq C||u||_{B_p(\mathbb{R}^n)}$$

for all $u \in B_p(\mathbb{R}^n)$ and the constant $C > 0$ can be chosen to depend only on n, p, $\operatorname{dist}(\Omega, \partial\widetilde{\Omega})$ and diam $\widetilde{\Omega}$.

Remark 2.8. It is easy to see that $u\varphi \in B_p(\mathbb{R}^n)$ whenever u, φ are bounded functions in $B_p(\mathbb{R}^n)$. Moreover,

$$||u\varphi||_{L^p(\mathbb{R}^n)} \leq \min(||u||_{L^\infty(\mathbb{R}^n)}||\varphi||_{L^p(\mathbb{R}^n)},\, ||\varphi||_{L^\infty(\mathbb{R}^n)}||u||_{L^p(\mathbb{R}^n)})$$

and

$$[u\varphi]_{B_p(\mathbb{R}^n)} \leq ||u||_{L^\infty(\mathbb{R}^n)}[\varphi]_{B_p(\mathbb{R}^n)} + ||\varphi||_{L^\infty(\mathbb{R}^n)}[u]_{B_p(\mathbb{R}^n)}.$$

Lemma 2.9. *Let $B = B(x_0, R) \subset \mathbb{R}^n$. Let $\eta \in C_0^\infty(2B)$ such that $0 \leq \eta \leq 1$, that $\eta = 1$ on B, and that $||\nabla\eta||_{L^\infty(2B)} < \frac{2}{R}$. Then there exists a constant $C = C(n, p)$ such that*

$$[\eta(v - v_B)]_{B_p(\mathbb{R}^n)} \leq C[v]_{B_p(\mathbb{R}^n)}$$

whenever $v \in L^1_{loc}(\mathbb{R}^n)$ with $[v]_{B_p(\mathbb{R}^n)} < \infty$.

Proof. Let $v \in L^1_{loc}(\mathbb{R}^n)$ such that $[v]_{B_p(\mathbb{R}^n)} < \infty$. Then $v \in L^p_{loc}(\mathbb{R}^n)$ and this implies, since $\eta \in C_0^\infty(2B)$, that $\eta(v - v_B) \in L^p(\mathbb{R}^n)$. We repeat to some extent the argument of Lemma 2.5 with $\varphi = \eta$, and $u = v - v_B$. We can choose $L = \frac{2}{R}$ and we notice that $||\eta||_{L^\infty(\mathbb{R}^n)} = 1$. By repeating the argument from Lemma 2.5, we get

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{|x - y|^{2n}} dx\, dy\right)^{1/p} \leq [u]_{B_p(\mathbb{R}^n)} + I^{1/p}, \tag{2.16}$$

where

$$I = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(y) - v_B|^p\, |\eta(x) - \eta(y)|^p}{|x - y|^{2n}} dx\, dy.$$

Like in Lemma 2.5, we notice that $I = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \int_{4B}\int_{4B} \frac{|v(y) - v_B|^p\, |\eta(x) - \eta(y)|^p}{|x - y|^{2n}} dx\, dy \\
I_2 &= \int_{4B}\int_{\mathbb{R}^n \setminus 4B} \frac{|v(y) - v_B|^p\, |\eta(x) - \eta(y)|^p}{|x - y|^{2n}} dx\, dy \\
I_3 &= \int_{\mathbb{R}^n \setminus 4B}\int_{4B} \frac{|v(y) - v_B|^p\, |\eta(x) - \eta(y)|^p}{|x - y|^{2n}} dx\, dy.
\end{aligned}$$

As in (2.10), we get

$$I_1 \leq C(n, p) \left(\frac{2}{R}\right)^p R^{p-n} \int_{4B} |v(y) - v_B|^p dy \leq C(n, p)\, [v]^p_{B_p(\mathbb{R}^n)}. \tag{2.17}$$

Since η is supported in $2B$, it follows from the definition of I_2 that in fact

$$I_2 = \int_{2B}\int_{\mathbb{R}^n \setminus 4B} \frac{|v(y) - v_B|^p\, |\eta(y)|^p}{|x - y|^{2n}} dx\, dy.$$

Hence

$$I_2 \leq \int_{2B} \int_{\mathbb{R}^n \setminus 4B} \frac{|v(y) - v_B|^p}{|x-y|^{2n}} dx\, dy$$

and since $|x-y| \geq \frac{|x-x_0|}{2}$ whenever $x \in \mathbb{R}^n \setminus 4B$ and $y \in 2B$, we get

$$I_2 \leq 2^{2n} \int_{\mathbb{R}^n \setminus 4B} \frac{1}{|x-x_0|^{2n}} dx \int_{2B} |v(y) - v_B|^p\, dy.$$

Hence

$$\begin{aligned} I_2 &\leq C(n)\, R^{-n} \int_{2B} |v(y) - v_B|^p dy \\ &\leq C(n,p)\, [v]^p_{B_p(\mathbb{R}^n)}. \end{aligned} \tag{2.18}$$

We notice that

$$\begin{aligned} I_3 &\leq 2^{p-1} \left(I_2 + \int_{\mathbb{R}^n \setminus 4B} \int_{4B} \frac{|v(x) - v(y)|^p\, |\eta(x) - \eta(y)|^p}{|x-y|^{2n}} dx\, dy \right) \\ &\leq 2^{p-1} I_2 + 2^{p-1}\, [v]^p_{B_p(\mathbb{R}^n)}. \end{aligned} \tag{2.19}$$

From (2.16), (2.17), (2.18), (2.19) and the fact that $I = I_1 + I_2 + I_3$, we have that $\eta(v - v_B) \in B_p(\mathbb{R}^n)$ with

$$[\eta(v - v_B)]_{B_p(\mathbb{R}^n)} \leq C(n,p)\, [u]_{B_p(\mathbb{R}^n)} = C(n,p)\, [v]_{B_p(\mathbb{R}^n)}. \tag{2.20}$$

■

Lemma 2.10. *Let $v \in B_p(\Omega)$.*

(i) *If* supp $v \subset\subset \Omega$, then $v \in B_p^0(\Omega)$.

(ii) If $u \in B_p^0(\Omega)$ and if $0 \leq v \leq u$ in $\mathbb{R}^n$, then $v \in B_p^0(\Omega)$.

Proof. For the proof of (i), let $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on the support of v. If a sequence $v_j \in C_0^\infty(\mathbb{R}^n)$ converges to v in $B_p(\mathbb{R}^n)$, then from Lemma 2.6 we see that $\psi v_j \in C_0^\infty(\Omega)$ converges to $\psi v = v$ in $B_p(\mathbb{R}^n)$, therefore $v \in B_p^0(\Omega)$.

As to assertion (ii), let $\varphi_j \in C_0^\infty(\Omega)$ be an approximating sequence for $u \in B_p^0(\Omega)$. From Lemma 2.3 we can assume that the functions φ_j are nonnegative. We can assume without loss of generality that $v = u = 0$ everywhere on $\mathbb{R}^n \setminus \Omega$. Then $\min(v, \varphi_j)$ has as support a compact subset of Ω and hence belongs to $B_p^0(\Omega)$. Moreover, since $\min(v, \varphi_j)$ converges to $\min(u, v) = v$ in $B_p(\Omega)$ (see Lemma 2.2), we have $v \in B_p^0(\Omega)$. ■

Lemma 2.11. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is bounded. Let* $u \in B_p(\Omega)$ *be such that* $u = 0$ *on* $\mathbb{R}^n \setminus \Omega$ *and* $\lim_{x \to y, x \in \Omega} u(x) = 0$ *for all* $y \in \partial\Omega$*. Then* $u \in B_p^0(\Omega)$*.*

Proof. Recalling that $u = u^+ - u^-$, we may assume that u is nonnegative. The function $u_\varepsilon = \max(u - \varepsilon, 0)$ is in $B_p(\Omega)$ for $\varepsilon > 0$ and has compact support in Ω. Thus $u_\varepsilon \in B_p^0(\Omega)$, $||u_\varepsilon||_{B_p(\Omega)} \leq ||u||_{B_p(\Omega)}$ for every $\varepsilon > 0$ and $u_\varepsilon \to u$ both in $L^p(\mathbb{R}^n)$ and pointwise as $\varepsilon \to 0$. The convexity and reflexivity of $B_p^0(\Omega)$ together with Mazur's lemma imply that $u \in B_p^0(\Omega)$. ■

3. Relative Besov capacity

In this section we establish a general theory of the relative Besov capacity and study how this capacity is related to Hausdorff measures.

For $E \subset \Omega$ we define

$$BA(E, \Omega) = \{u \in B_p^0(\Omega) : u \geq 1 \text{ on a neighborhood of } E\}.$$

We call $BA(E, \Omega)$ the set of *admissible functions for the condenser* (E, Ω). The *relative Besov p-capacity* of the pair (E, Ω) is denoted by

$$\mathrm{cap}_{B_p}(E, \Omega) = \inf\{[u]^p_{B_p(\Omega)} : u \in BA(E, \Omega)\}.$$

If $BA(E, \Omega) = \emptyset$, we set $\mathrm{cap}_{B_p}(E, \Omega) = \infty$.

Since $B_p^0(\Omega)$ is closed under truncations from below by 0 and from above by 1 and since these truncations do not increase the Besov p-seminorm, we may restrict ourselves to those admissible functions u for which $0 \leq u \leq 1$.

3.1. Basic properties of the relative Besov capacity

A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the relative Besov p-capacity.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set. The set function* $E \mapsto \mathrm{cap}_{B_p}(E, \Omega)$*,* $E \subset \Omega$*, enjoys the following properties:*

(i) *If* $E_1 \subset E_2$*, then* $\mathrm{cap}_{B_p}(E_1, \Omega) \leq \mathrm{cap}_{B_p}(E_2, \Omega)$*.*

(ii) *If* $\Omega_1 \subset \Omega_2$ *are open and bounded and* $E \subset \Omega_1$*, then*

$$\mathrm{cap}_{B_p}(E, \Omega_2) \leq \mathrm{cap}_{B_p}(E, \Omega_1).$$

(iii) $\mathrm{cap}_{B_p}(E, \Omega) = \inf\{\mathrm{cap}_{B_p}(U, \Omega) : E \subset U \subset \Omega, \ U \text{ open}\}$*.*

(iv) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^\infty K_i$, then*

$$\operatorname{cap}_{B_p}(K,\Omega) = \lim_{i\to\infty} \operatorname{cap}_{B_p}(K_i,\Omega).$$

(v) *If $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_{i=1}^\infty E_i \subset \Omega$, then*

$$\operatorname{cap}_{B_p}(E,\Omega) = \lim_{i\to\infty} \operatorname{cap}_{B_p}(E_i,\Omega).$$

(vi) *If $E = \bigcup_{i=1}^\infty E_i \subset \Omega$, then*

$$\operatorname{cap}_{B_p}(E,\Omega) \le \sum_{i=1}^\infty \operatorname{cap}_{B_p}(E_i,\Omega).$$

Proof. Properties (i), (ii) and (iii) are immediate consequences of the definition.

The proof of (iv), (v) and (vi) follows [22] and [23].

(iv) We notice that by monotonicity we have

$$\operatorname{cap}_{B_p}(K,\Omega) \le \lim_{i\to\infty} \operatorname{cap}_{B_p}(K_i,\Omega).$$

On the other hand let $U \subset \Omega$ be an open set containing K. By the compactness of the sets K_i and K, we have that $K_i \subset U$ for all sufficiently large i. Therefore

$$\lim_{i\to\infty} \operatorname{cap}_{B_p}(K_i,\Omega) \le \operatorname{cap}_{B_p}(U,\Omega),$$

and we obtain the claim from (iii) by taking the infimum over all such open sets U.

(v) Monotonicity yields

$$\lim_{i\to\infty} \operatorname{cap}_{B_p}(E_i,\Omega) \le \operatorname{cap}_{B_p}(E,\Omega).$$

To prove the opposite inequality, we may assume without loss of generality that $\lim_{i\to\infty} \operatorname{cap}_{B_p}(E_i,\Omega) < \infty$. Let $\varepsilon > 0$ be fixed. For every $i = 1, 2, \ldots$ we choose $u_i \in BA(E_i,\Omega)$, $0 \le u_i \le 1$, such that

$$[u_i]^p_{B_p(\Omega)} < \operatorname{cap}_{B_p}(E_i,\Omega) + \varepsilon. \tag{3.1}$$

Since Ω is bounded, it follows that u_i is a bounded sequence in $B^0_p(\Omega)$ and hence there exists a subsequence, which we denote again by u_i such that $u_i \to u$ weakly in $B^0_p(\Omega)$ as $i \to \infty$. Using Mazur's lemma we obtain a sequence v_i of convex combinations of u_i such that $v_i \in BA(E_i,\Omega)$, $v_i \to u$ in $B^0_p(\Omega)$, $v_i \to u$ a.e. This sequence can be found in the following way. Let i_0

be fixed. Since every subsequence of u_i converges to u weakly in $B_p^0(\Omega)$, we may use the Mazur lemma for the subsequence $u_i, i \geq i_0$. We obtain a finite convex combination of the functions $u_i, i \geq i_0$ as close to u as we want in $B_p^0(\Omega)$. For every $i = i_0, i_0+1, \ldots$ there is an open neighborhood O_i of E_{i_0} such that $u_i = 1$ in O_i. The intersection of finitely many open neighborhoods of E_{i_0} is an open neighborhood of E_{i_0}. Therefore, v_{i_0} equals 1 in an open neighborhood U_{i_0} of E_{i_0}. Moreover, since for every $i = 1, 2, \ldots$ we have

$$[u_i]_{B_p(\Omega)}^p < \mathrm{cap}_{B_p}(E_i, \Omega) + \varepsilon \leq \lim_{j\to\infty} \mathrm{cap}_{B_p}(E_j, \Omega) + \varepsilon,$$

we obtain from the convexity of the B_p-seminorm and (3.1) that

$$[v_i]_{B_p(\Omega)}^p \leq \lim_{j\to\infty} \mathrm{cap}_{B_p}(E_j, \Omega) + \varepsilon \tag{3.2}$$

for every $i = 1, 2, \ldots$. Passing to subsequences if necessary, we may assume that for every $i = 1, 2, \ldots$ we have

$$||v_{i+1} - v_i||_{B_p(\Omega)} \leq 2^{-i}. \tag{3.3}$$

For $j = 1, 2, \ldots$ we set

$$w_j = \sup_{i\geq j} v_i.$$

It is easy to see that $w_j = \lim_{k\to\infty} w_{j,k}$ pointwise a.e., where $w_{j,k}$ is defined for every $k \geq j$ by

$$w_{j,k} = \sup_{k\geq i\geq j} v_i.$$

We notice that $w_{j,k} \in BA(E_j, \Omega)$. Moreover,

$$w_{j,k} \leq v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i|$$

pointwise in $\mathbb{R}^n$ and

$$\begin{aligned} |w_{j,k}(x) - w_{j,k}(y)| &\leq \sup_{i\geq j} |v_i(x) - v_i(y)| \leq |v_j(x) - v_j(y)| \\ &+ \sum_{i=j}^{\infty} |(v_{i+1}(x) - v_i(x)) - (v_{i+1}(y) - v_i(y))| \end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and every $k \geq j$. The convexity and reflexivity of $B_p^0(\Omega)$ together with Mazur's lemma and formula (3.3) imply that $w_j \in B_p^0(\Omega)$ with

$$w_j \leq v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i|$$

pointwise a.e. in $\mathbb{R}^n$ and

$$\begin{aligned}|w_j(x)-w_j(y)| &\le \sup_{i\ge j}|v_i(x)-v_i(y)|\le |v_j(x)-v_j(y)|\\ &\qquad +\sum_{i=j}^{\infty}|(v_{i+1}(x)-v_i(x))-(v_{i+1}(y)-v_i(y))|\end{aligned}$$

for a.e. $x,y\in\mathbb{R}^n$. It is easy to see that $w_j=1$ in a neighborhood of E and this shows, since $w_j\in B_p^0(\Omega)$, that in fact $w_j\in BA(E,\Omega)$ and hence $\mathrm{cap}_{B_p}(E,\Omega)\le[w_j]^p_{B_p(\Omega)}$. We notice that

$$[w_j]_{B_p(\Omega)}\le[v_j]_{B_p(\Omega)}+\sum_{i=j}^{\infty}[v_{i+1}-v_i]_{B_p(\Omega)}\le[v_j]_{B_p(\Omega)}+2^{-j+1}$$

for every $j\ge 1$. Therefore, for all sufficiently large j we have from (3.2) that

$$\mathrm{cap}_{B_p}(E,\Omega)\le[w_j]^p_{B_p(\Omega)}\le\lim_{i\to\infty}\mathrm{cap}_{B_p}(E_i,\Omega)+2\varepsilon.$$

By letting $\varepsilon\to 0$, we get the converse inequality so (v) is proved.

(vi) To prove the countable subadditivity, we need to prove the finite subadditivity first. It is enough to prove this for two sets because then the general finite case follows by induction. So let E_1 and E_2 be two subsets of Ω. We can assume without loss of generality that $\mathrm{cap}_{B_p}(E_1,\Omega)+\mathrm{cap}_{B_p}(E_2,\Omega)<\infty$. Let $u_i\in BA(E_i,\Omega)$ such that $0\le u_i\le 1$ and $[u_i]^p_{B_p(\Omega)}<\mathrm{cap}_{B_p}(E_i,\Omega)+\varepsilon$ for $i=1,2$. Then $u=\max(u_1,u_2)$ belongs to $BA(E_1\cup E_2,\Omega)$ and since $|u(x)-u(y)|\le\max(|u_1(x)-u_1(y)|,|u_2(x)-u_2(y)|)$ for all $x,y\in\mathbb{R}^n$, it follows that

$$\begin{aligned}\mathrm{cap}_{B_p}(E_1\cup E_2,\Omega) &\le [u]^p_{B_p(\Omega)}\le[u_1]^p_{B_p(\Omega)}+[u_2]^p_{B_p(\Omega)}\\ &\le \mathrm{cap}_{B_p}(E_1,\Omega)+\mathrm{cap}_{B_p}(E_2,\Omega)+2\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$ we complete the proof in the case of two sets, and hence the general finite case.

The general case follows from the finite case together with (v). The theorem is proved. ■

A set function that satisfies properties (i), (iv), (v) and (vi) is called a *Choquet capacity* (relative to Ω). We may thus invoke an important capacitability theorem of Choquet and state the following result. See [12, Appendix II].

Theorem 3.2. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$. The set function $E \mapsto \mathrm{cap}_{B_p}(E, \Omega)$, $E \subset \Omega$, is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic) subsets E of Ω are capacitable, i.e.,*

$$\mathrm{cap}_{B_p}(E, \Omega) = \sup\{\mathrm{cap}_{B_p}(K, \Omega) : K \subset E \text{ compact}\}$$

whenever $E \subset \Omega$ is analytic.

Remark 3.3. If K is a compact subset of the bounded and open set $\Omega \subset \mathbb{R}^n$, we get the same Besov B_p-capacity for (K, Ω) if we restrict ourselves to a smaller set, namely

$$BW(K, \Omega) = \{u \in C_0^\infty(\Omega) : u = 1 \text{ in a neighborhood of } K\}.$$

Indeed, let $u \in BA(K, \Omega)$; we may clearly assume that $u = 1$ in a neighborhood $U \subset\subset \Omega$ of K. Then we choose a cut-off function $\eta \in C^\infty(\mathbb{R}^n)$, $0 \le \eta \le 1$ such that $\eta = 1$ in $\mathbb{R}^n \setminus U$ and $\eta = 0$ in a neighborhood $\widetilde{U}$ of K, $\widetilde{U} \subset\subset U$. Now, if $\varphi_j \in C_0^\infty(\Omega)$ is a sequence converging to u in $B_p^0(\Omega)$, then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $BW(K, \Omega)$ which converges to $1 - \eta(1 - u)$ in $B_p^0(\Omega)$. (See Lemma 2.6.) But $1 - \eta(1 - u) = u$. This establishes the assertion, since $BW(K, \Omega) \subset BA(K, \Omega)$. In fact, it is easy to see that if $K \subset \Omega$ is compact we get the same Besov B_p-capacity if we consider

$$\widetilde{BW}(K, \Omega) = \{u \in C_0^\infty(\Omega) : u = 1 \text{ on } K\}$$

or

$$BW_0(K, \Omega) = \{u \in C_0(\Omega) \cap B_p^0(\Omega) : u = 1 \text{ on } K\}.$$

It is also useful to observe that if $\psi \in B_p^0(\Omega)$ is such that if $\varphi - \psi \in B_p^0(\Omega \setminus K)$ for some $\varphi \in BW_0(K, \Omega)$, then

$$\mathrm{cap}_{B_p}(K, \Omega) \le [\psi]^p_{B_p(\Omega)}.$$

3.2. Upper estimates for the relative Besov capacity

For every $x \in \mathbb{R}^n$ we obviously have $\mathrm{cap}_{B_p}(E, \Omega) = \mathrm{cap}_{B_p}(E + x, \Omega + x)$. Next we derive some upper estimates for the relative Besov capacity. Similar estimates have been obtained earlier by Bourdon in [6]. We follow his methods.

Theorem 3.4. *There exists a constant $C = C(n, p) > 0$ depending only on n and p such that*

$$\mathrm{cap}_{B_p}(B(x_0, r), B(x_0, R)) \le C \left(\ln \frac{R}{r}\right)^{1-p} \tag{3.4}$$

for every $0 < r < \frac{R}{2}$ and every $x_0 \in \mathbb{R}^n$.

Proof. We use the function $u : \mathbb{R}^n \to \mathbb{R}$,

$$u(x) = \begin{cases} 1 & \text{if } 0 \le |x - x_0| \le r \\ \left(\ln \frac{|x-x_0|}{R}\right)/\left(\ln \frac{r}{R}\right) & \text{if } r < |x - x_0| < R \\ 0 & \text{if } |x - x_0| \ge R. \end{cases}$$

Then $u \in B_p(\mathbb{R}^n)$ because it is Lipschitz with compact support. Since u is continuous on $\mathbb{R}^n$ and 0 outside $B(x_0, R)$, we have in fact from Lemma 2.11 that $u \in B_p^0(B(x_0, R))$. In fact $u \in BA(B(x_0, r), B(x_0, R))$ since $u = 1$ on $B(x_0, r)$. Let $v(x) = u(x) \ln \frac{R}{r}$. We will get an upper bound for $[v]_{B_p(B(x_0,R))}$. Let $k \ge 3$ be the smallest integer such that $2^{k-1}r \ge R$. For $i = 1, \dots, k$ we define $B_i = B(x_0, 2^i r) \setminus \overline{B}(x_0, 2^{i-1}r)$. We also define $B_0 = B(x_0, r)$ and $B_{k+1} = \mathbb{R}^n \setminus B(x_0, 2^k r)$. We have

$$[v]^p_{B_p(B(x_0,R))} = \sum_{0 \le i,j \le k+1} I_{i,j} = \sum_{0 \le i,j \le k+1} \int_{B_i} \int_{B_j} \frac{|v(x) - v(y)|^p}{|x - y|^{2n}}\, dx\, dy.$$

Obviously we have $I_{i,j} = I_{j,i}$. We majorize $I_{i,j}$ by distinguishing a few cases. For $j \le k$ and $0 \le i \le j - 2$ we have from the definition of v that $|v(x) - v(y)| \le j - i + 1$ whenever $x \in B_i$ and $y \in B_j$, hence

$$I_{i,j} \le C_0(j - i + 1)^p\, (2^j r)^{-2n}\, (2^i r)^n\, (2^j r)^n,$$

that is $I_{i,j} \le C_1(j - i)^p 2^{(i-j)n}$. For $0 \le i \le j \le k$ we notice, since v is $\frac{1}{2^{i-1}r}$-Lipschitz on $\bigcup_{j \ge i} B_j$ that

$$I_{i,j} \le (2^{i-1}r)^{-p} \int_{B_i} \int_{B_j} \frac{1}{|x - y|^{2n-p}}\, dx\, dy.$$

Moreover, we have

$$\int_{B_j} \frac{1}{|x - y|^{2n-p}}\, dx \le C_2(\operatorname{diam} B_j)^{p-n}$$

for every $y \in B(x_0, 2^i r)$, where C_2 depends only on p and n. Hence for $0 \le i \le j \le k$ we have

$$I_{i,j} \le C_3(2^{i-1}r)^{-p}(2^i r)^n (2^j r)^{p-n} \le C_4 2^{(j-i)(p-n)}.$$

In particular, for $j-1 \le i \le j \le k$, the integral $I_{i,j}$ is bounded by a constant that depends only on p and n. Now we have to bound $I_{i,j}$ when $j = k + 1$. Since v is constant on $B_k \cup B_{k+1}$, we have $I_{i,k+1} = 0$ for $i \in \{k, k + 1\}$. For $0 \le i \le k - 1$ we have

$$I_{i,k+1} \le (k - i + 1)^p \int_{B_i} \int_{B_{k+1}} \frac{1}{|x - y|^{2n}}\, dx\, dy.$$

But there exists $C_5 > 0$ such that

$$\int_{B_{k+1}} \frac{1}{|x-y|^{2n}}\, dx \leq C_5 (2^{k+1} r)^{-n}$$

for every $y \in \mathbb{R}^n$ with $|y - x_0| \leq 2^{k-1} r$. Hence

$$I_{i,k+1} \leq C_6 (k-i+1)^p 2^{(i-k-1)n}.$$

Finally we have

$$[v]^p_{B_p(B(x_0,R))} \leq C_7 k + C_8 \sum_{0 \leq i \leq j \leq k+1} (j-i)^p 2^{(i-j)n}.$$

The last sum is equal to

$$\sum_{l=1}^{k+1} (k+2-l) l^p 2^{-l\,n}.$$

But $k+2-l \leq k+1$ and there exists $a > 1$ such that $l^p 2^{-l\,n} \leq C_9 a^{-l}$ for $l \geq 1$. Hence

$$[v]^p_{B_p(B(x_0,R))} \leq C_{10} \ln \frac{R}{r}$$

and

$$[u]^p_{B_p(B(x_0,R))} \leq C_{10} \Big(\ln \frac{R}{r} \Big)^{1-p}.$$

The claim follows with $C = C_{10}$. ∎

Whenever $Q \subset \mathbb{R}^n$ is a cube with sides parallel to coordinate axes, we denote its edge length by $\ell(Q)$ and then set

$$\widehat{Q} = Q \times [\ell(Q), 2\ell(Q)] \subset \mathbb{R}^{n+1}_+,$$

so that $\widehat{Q}$ is the upper half of the $(n+1)$-dimensional box $Q \times [0, 2\ell(Q)]$. For definiteness, we assume that the cubes are closed. We denote $\mathcal{D}_r(\mathbb{R}^n)$, $r > 0$, the countable collection of all r-dyadic cubes in $\mathbb{R}^n$. Thus, $Q \in \mathcal{D}_r(\mathbb{R}^n)$ if and only if the corners of Q lie in $2^k r \mathbb{Z}^n$ and $\ell(Q) = 2^k r$ for some $k \in \mathbb{Z}$.

Similarly, if $Q_0 \subset \mathbb{R}^n$ is a r-dyadic cube, we denote by $\mathcal{D}_r(Q_0)$ the r-dyadic subcubes of Q_0. Finally, we set

$$\mathcal{W}_r = \{ \widehat{Q} : Q \in \mathcal{D}_r(\mathbb{R}^n) \}.$$

The members of $\mathcal{W}_r$ will be referred as to as *r-dyadic Whitney cubes of* $\mathbb{R}^{n+1}_+$. Two distinct r-dyadic Whitney cubes $\widehat{Q}, \widehat{Q}'$ are *adjacent* if there exists an integer k such that either:

(i) Q, Q' are dyadic cubes in $\mathcal{D}_r(\mathbb{R}^n)$ and both Q and Q' have side length $2^k r$ and a common face, or

(ii) one of the cubes Q, Q' has side length $2^k r$, the other has side length $2^{k+1}r$ and the one with the bigger side length includes the other one.

Given a function $f \in B_p(\mathbb{R}^n)$, we construct an extension $\widehat{f} : \mathcal{W}_r \to \mathbb{R}$. For $\widehat{Q} \in \mathcal{W}_r$ we let

$$\widehat{f}(\widehat{Q}) = \frac{1}{|Q|}\int_Q f(x)\,dx.$$

For two adjacent cubes $\widehat{Q}, \widehat{Q}' \in \mathcal{W}_r$ we have

$$\begin{aligned}
|\widehat{f}(\widehat{Q}) - \widehat{f}(\widehat{Q}')|^p &= \left|\frac{1}{|Q|}\int_Q f(x)\,dx - \frac{1}{|Q'|}\int_{Q'} f(y)\,dy\right|^p \\
&= \left|\frac{1}{|Q|}\frac{1}{|Q'|}\int_Q\int_{Q'} (f(x)-f(y))\,dx\,dy\right|^p \\
&\le \frac{1}{|Q|}\frac{1}{|Q'|}\int_Q\int_{Q'} |f(x)-f(y)|^p\,dx\,dy \\
&\le C(n)\int_Q\int_{Q'} \frac{|f(x)-f(y)|^p}{|x-y|^{2n}}\,dx\,dy.
\end{aligned}$$

For the following lemma see [7, Lemma 3.5].

Lemma 3.5. *There exists a constant $C = C(n)$ depending only on n such that we have*

$$C(n)^{-1}|\eta-\zeta|^{-2n} \le \sum_{\widehat{Q},\widehat{Q}' \in \mathcal{W}_r \text{ adjacent}} \frac{\chi_Q(\eta)\chi_{Q'}(\zeta)}{|Q||Q'|} \le C(n)|\eta-\zeta|^{-2n}$$

for a.e. $\eta, \zeta \in \mathbb{R}^n$.

We also have (see [7, Theorem 3.4]):

Lemma 3.6. *There exists a constant $C = C(n,p)$ such that*

$$\begin{aligned}
C^{-1}[f]^p_{B_p(\mathbb{R}^n)} &\le \sum_{\widehat{Q},\widehat{Q}' \in \mathcal{W}_r \text{ adjacent}} \frac{1}{|Q|}\frac{1}{|Q'|}\int_Q\int_{Q'} |f(x)-f(y)|^p\,dx\,dy \\
&\le C[f]^p_{B_p(\mathbb{R}^n)}
\end{aligned}$$

for every $f \in B_p(\mathbb{R}^n)$.

This implies (see [7, Lemma 3.5]):

Lemma 3.7. *There exists a constant $C = C(n,p)$ such that*

$$\sum_{\widehat{Q},\widehat{Q}' \in \mathcal{W}_r \text{ adjacent}} |\widehat{f}(\widehat{Q}) - \widehat{f}(\widehat{Q}')|^p \le C[f]^p_{B_p(\mathbb{R}^n)} \tag{3.5}$$

for every $f \in B_p(\mathbb{R}^n)$.

3.3. Hausdorff measure and relative Besov capacity

Now we examine the relationship between Hausdorff measures and the B_p-capacity. Let h be a real-valued, increasing function on $[0,\infty)$ with

$$\lim_{t\to 0} h(t) = h(0) = 0.$$

Such a function h is called a *measure function.* Let $0<\delta\le\infty$. For $E\subset\mathbb{R}^n$ we define

$$\Lambda_h^\delta(E) = \inf \sum_i h(r_i),$$

where the infimum is taken over all coverings of E by balls B_i with diameter r_i not exceeding δ. The set function Λ_h^∞ is called the *h-Hausdorff content.* Clearly Λ_h^δ is an outer measure for every $\delta\in(0,\infty]$.

Moreover, for every $E\subset\mathbb{R}^n$, there exists a Borel set $\widetilde{E}$ such that $E\subset\widetilde{E}\subset\mathbb{R}^n$ and $\Lambda_h^\delta(E)=\Lambda_h^\delta(\widetilde{E})$. Clearly $\Lambda_h^\delta(E)$ is a decreasing function of δ. This allows us to define the *h-Hausdorff measure* of $E\subset\mathbb{R}^n$ by

$$\Lambda_h(E) = \sup_{\delta>0} \Lambda_h^\delta(E) = \lim_{\delta\to 0} \Lambda_h^\delta(E).$$

The measure Λ_h is Borel regular; that is, it is an additive measure on Borel sets of $\mathbb{R}^n$ and for each $E\subset\mathbb{R}^n$ there is a Borel set G such that $E\subset G$ and $\Lambda_h(E)=\Lambda_h(G)$. (See [13, p. 170] and [25, Chapter 4].) If $h(t)=t^s$, we write Λ_s for Λ_{t^s}. It is immediate from the definition that $\Lambda_s(E)<\infty$ implies $\Lambda_u(E)=0$ for all $u>s$. The smallest $s\ge 0$ that satisfies $\Lambda_u(E)=0$ for all $u>s$ is called the *Hausdorff dimension of E.*

The set function Λ_h^∞ satisfies the following three properties:

(i) If K_i is a decreasing sequence of compact sets, then

$$\Lambda_h^\infty\Big(\bigcap_{i=1}^\infty K_i\Big) = \lim_{i\to\infty} \Lambda_h^\infty(K_i).$$

(ii) If E_i is an increasing sequence of arbitrary sets, then

$$\Lambda_h^\infty(\bigcup_{i=1}^\infty E_i) = \lim_{i\to\infty} \Lambda_h^\infty(E_i).$$

(iii) $\Lambda_h^\infty(E)=\sup\{\Lambda_h^\infty(K) : K\subset E \text{ compact}\}$ whenever $E\subset\mathbb{R}^n$ is a Borel set. (See [1, p. 138] and [8, Theorem II.2].)

If $h:[0,\infty)\to[0,\infty)$ is a measure function, we know that $\Lambda_h(E)=0$ if and only if $\Lambda_h^\infty(E)=0$. (See [1, Proposition 5.1.5].) If $h(t)=t^s$, $0<s<\infty$, we write Λ_s^∞ for $\Lambda_{t^s}^\infty$.

We notice that for every $0 < s < \infty$, there exists a constant $C = C(n,s) > 0$ such that

$$\Lambda_s^\infty(B) = C\,(\operatorname{diam} B)^s \tag{3.6}$$

for every ball $B \subset \mathbb{R}^n$.

Since $\Lambda_h(E) = 0$ for every $E \subset \mathbb{R}^n$ whenever $h : [0,\infty) \to [0,\infty)$ is a measure function such that $\liminf_{t\to 0} h(t)t^{-n} = 0$, it is enough to assume that $h : [0,\infty) \to [0,\infty)$ is an increasing homeomorphism such that $t \mapsto h(t)\,t^{-n}$, $0 < t < \infty$ is decreasing if we hope to get $\Lambda_h(E) > 0$ where $E \subset \mathbb{R}^n$. (See [1, Proposition 5.1.8].)

Let $Q_{0,r} \in \mathcal{D}_r(\mathbb{R}^n)$ be a cube with side length r.

Theorem 3.8. *Suppose $h : [0,\infty) \to [0,\infty)$ is an increasing homeomorphism such that $t \mapsto h(t)t^{-n}$, $0 < t < \infty$ is decreasing. There exists a positive constant $C_1' = C_1'(n,p)$ such that*

$$\frac{\Lambda_h^\infty(E\cap Q_{k,r})}{\left(\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \le C_1' k^{p-1} \operatorname{cap}_{B_p}(E\cap Q_{k,r}, \operatorname{int}(Q_{0,r})) \tag{3.7}$$

for every $E \subset \mathbb{R}^n$, every $k > 1, r > 0$, and for every $Q_{k,r} \in \mathcal{D}_r(Q_{0,r})$ with side length $2^{-k}r$ and with one corner at the center of $Q_{0,r}$.

Here and throughout the paper $\operatorname{int}(E)$ denotes the interior of a set E whenever $E \subset \mathbb{R}^n$.

Proof. Fix $r > 0$, $k > 1$, $Q_{0,r}$ a r-dyadic cube of side length r and $Q_{k,r} \in \mathcal{D}_r(Q_{0,r})$ r-dyadic subcube of $Q_{0,r}$ with side length $2^{-k}r$ and with one corner at the center of $Q_{0,r}$. Let $E \subset \mathbb{R}^n$. From the fact that there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E} \subset \mathbb{R}^n$ and

$$\operatorname{cap}_{B_p}(E\cap Q_{k,r}, \operatorname{int}(Q_{0,r})) = \operatorname{cap}_{B_p}(\widetilde{E}\cap Q_{k,r}, \operatorname{int}(Q_{0,r})),$$

we can assume that E is a Borel set. Moreover, since

$$\Lambda_h^\infty(E) = \sup\{\Lambda_h^\infty(K) : K \subset E \text{ compact}\}$$

whenever $E \subset \mathbb{R}^n$ is a Borel set and since $\operatorname{cap}_{B_p}(\cdot, \operatorname{int}(Q_{0,r}))$ is a Choquet capacity, we can assume that E is compact.

There is nothing to prove if we have either $\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t} = \infty$ or $\Lambda_h^\infty(E\cap Q_{k,r}) = 0$. So we can assume without loss of generality that $\alpha = \Lambda_h^\infty(E\cap Q_{k,r}) > 0$ and that $\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t} < \infty$.

For every $\zeta \in \partial Q_{0,r}$ there is an increasing sequence $(Q_{s,\zeta})_{s\le 0}$ of r-dyadic subcubes of $Q_{0,r}$ containing ζ such that $Q_{s,\zeta}$ has side length $2^s r$ for every integer $s \le 0$ and

$$\bigcap_{s\le 0} Q_{s,\zeta} = \{\zeta\}.$$

We denote by s^0_ζ the sequence $(\widehat{Q}_{s,\zeta})_{s\le 0}$.

For every $\eta \in Q_{k,r}$ there is a decreasing sequence $(Q_{s+k,\eta})_{s\ge 0}$ of r-dyadic subcubes of $Q_{k,r}$ containing η such that $Q_{s+k,\eta}$ has side length $2^{-s-k}r$ for every $s \ge 0$ and

$$\bigcap_{s\ge 0} Q_{s+k,\eta} = \{\eta\}.$$

We denote by s^1_η the sequence $(\widehat{Q}_{s+k,\eta})_{s\ge 0}$. Let $I = \{\widehat{Q}_{0,r}, \ldots, \widehat{Q}_{k,r}\}$ be a shortest sequence of pairwise adjacent cubes connecting $\widehat{Q}_{0,r}$ and $\widehat{Q}_{k,r}$.

For $(\zeta,\eta) \in \partial Q_{0,r} \times Q_{k,r}$ we define $\gamma_{\zeta,\eta} = (\widehat{Q}_{s,\zeta,\eta})_{s\in\mathbb{Z}}$, where

$$\widehat{Q}_{s,\zeta,\eta} = \begin{cases} \widehat{Q}_{s,\zeta} & \text{if } s \le 0 \\ \widehat{Q}_{s,r} & \text{if } 0 \le s \le k \\ \widehat{Q}_{s,\eta} & \text{if } s \ge k. \end{cases}$$

For $\widehat{Q}, \widehat{Q}' \in \mathcal{W}_r$ we define

$$\mathcal{C}(\widehat{Q},\widehat{Q}') = \{(\zeta,\eta) \in \partial Q_{0,r} \times Q_{k,r} : \widehat{Q} = \widehat{Q}_{s,\zeta,\eta}, \widehat{Q}' = \widehat{Q}_{s+1,\zeta,\eta} \text{ for some } s \in \mathbb{Z}\}.$$

We notice that $\mathcal{C}(\widehat{Q},\widehat{Q}') = \emptyset$ if $\widehat{Q}, \widehat{Q}'$ are not adjacent or if they are adjacent but with the same side length.

Since the $\Lambda^\infty_1(\partial Q_{0,r}) = C(n)r$ and $\alpha = \Lambda^\infty_h(E \cap Q_{k,r}) > 0$, from Frostman lemma (see [1, Theorem 5.1.12]) there exists a constant $C > 0$ and probability measures ν_0 on $\partial Q_{0,r}$ and ν_1 on $E \cap Q_{k,r}$ such that for every ball $B(x,t)$ of radius t of $\mathbb{R}^n$ we have

$$\nu_0(B(x,t)) \le C\frac{t}{r} \text{ and } \nu_1(B(x,t)) \le C\frac{h(t)}{\alpha}. \tag{3.8}$$

For $\widehat{Q}, \widehat{Q}' \in \mathcal{W}_r$ we define

$$m(\widehat{Q},\widehat{Q}') = \nu_0 \times \nu_1(\mathcal{C}(\widehat{Q},\widehat{Q}')).$$

We notice that $m(\widehat{Q},\widehat{Q}')m(\widehat{Q}',\widehat{Q}) = 0$ for every pair of cubes $\widehat{Q}, \widehat{Q}' \in \mathcal{W}_r$. Moreover, if $m(\widehat{Q},\widehat{Q}') \ne 0$, then this implies that $\widehat{Q}$ and $\widehat{Q}'$ are adjacent but with different side length.

For $f \in BW(E \cap Q_{k,r}, \operatorname{int}(Q_{0,r}))$ let $\widehat{f}$ be its extension. Then, since f is continuous, we have that

$$\frac{1}{|Q_v|}\int_{Q_v} f(x)\,dx \to f(y)$$

for every $y \in \mathbb{R}^n$ for every nested sequence Q_v of r-dyadic cubes containing y and converging to y. It follows that

$$1 \le f(\eta) - f(\zeta) \le \sum_{s\in\mathbb{Z}} (\widehat{f}(\widehat{Q}_{s+1,\zeta,\eta}) - \widehat{f}(\widehat{Q}_{s,\zeta,\eta}))$$

whenever $\eta \in E \cap Q_{k,r}$ and $\zeta \in \partial Q_{0,r}$.

We obtain with the definition of $m(\widehat{Q},\widehat{Q}')$ and by Hölder's inequality that

$$\begin{aligned}
1 &\le \int_{\partial Q_{0,r}} \int_{E\cap Q_{k,r}} \sum_{s\in\mathbb{Z}} (\widehat{f}(\widehat{Q}_{s+1,\zeta,\eta}) - \widehat{f}(\widehat{Q}_{s,\zeta,\eta}))\, d\nu_1(\eta)\, d\nu_0(\zeta) \\
&\le \int_{\partial Q_{0,r}} \int_{Q_{k,r}} \sum_{s\in\mathbb{Z}} |\widehat{f}(\widehat{Q}_{s+1,\zeta,\eta}) - \widehat{f}(\widehat{Q}_{s,\zeta,\eta})|\, d\nu_1(\eta)\, d\nu_0(\zeta) \\
&= \sum_{\widehat{Q},\widehat{Q}'\in\mathcal{W}_r \text{ adjacent}} |\widehat{f}(\widehat{Q}) - \widehat{f}(\widehat{Q}')| m(\widehat{Q},\widehat{Q}') \\
&\le \Bigg(\sum_{\widehat{Q},\widehat{Q}'\in\mathcal{W}_r \text{ adjacent}} |\widehat{f}(\widehat{Q}) - \widehat{f}(\widehat{Q}')|^p\Bigg)^{1/p} \Bigg(\sum_{\widehat{Q},\widehat{Q}'\in\mathcal{W}_r} m(\widehat{Q},\widehat{Q}')^{p'}\Bigg)^{1/p'} \\
&\le C(n,p)[f]_{B_p(\mathbb{R}^n)} \Bigg(\sum_{\widehat{Q},\widehat{Q}'\in\mathcal{W}_r} m(\widehat{Q},\widehat{Q}')^{p'}\Bigg)^{1/p'},
\end{aligned}$$

where we used (3.5) for the last inequality. For a nonnegative integer s we let

$$E_{0,s} = \{(\widehat{Q},\widehat{Q}') \in \mathcal{W}_r \times \mathcal{W}_r : \widehat{Q} = \widehat{Q}_{-s-1,\zeta},\ \widehat{Q}' = \widehat{Q}_{-s,\zeta} \text{ for some } \zeta \in \partial Q_{0,r}\}$$

and similarly

$$E_{1,s} = \{(\widehat{Q},\widehat{Q}') \in \mathcal{W}_r \times \mathcal{W}_r : \widehat{Q} = \widehat{Q}_{s+k,\eta},\ \widehat{Q}' = \widehat{Q}_{s+k+1,\eta} \text{ for some } \eta \in Q_{k,r}\}.$$

We notice that we can break $\sum = \sum_{\widehat{Q},\widehat{Q}'\in\mathcal{W}_r} m(\widehat{Q},\widehat{Q}')^{p'}$ into 3 parts, namely

$$\sum = \sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{0,s}} m(\widehat{Q},\widehat{Q}')^{p'} + \sum_{\widehat{Q},\widehat{Q}'\in I} m(\widehat{Q},\widehat{Q}')^{p'} + \sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{1,s}} m(\widehat{Q},\widehat{Q}')^{p'}.$$

We recall that $I = \{\widehat{Q}_{0,r}, \ldots, \widehat{Q}_{k,r}\}$ is a shortest sequence of pairwise adjacent cubes in $\mathcal{W}_r$ connecting $\widehat{Q}_{0,r}$ and $\widehat{Q}_{k,r}$. Thus, the sum in the middle is exactly k. We get upper bounds for the first and the third term in the sum. We notice that for every $s \geq 0$ we have

$$\sum_{(\widehat{Q},\widehat{Q}')\in E_{0,s}} m(\widehat{Q},\widehat{Q}') = 1$$

since $\nu_0 \times \nu_1$ is a probability measure. On the other hand, there exists a constant $C' = C'(p,n)$ depending only on p and n such that

$$m(\widehat{Q},\widehat{Q}') \leq C' \frac{h(2^{-s-k}r)}{\alpha} \text{ for every } (\widehat{Q},\widehat{Q}') \in E_{1,s}$$

for every integer $s \geq 0$ and

$$m(\widehat{Q},\widehat{Q}') \leq C' 2^{-s} \text{ for every } (\widehat{Q},\widehat{Q}') \in E_{0,s}$$

for every integer $s \geq 0$.

Therefore

$$\begin{aligned}\sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{1,s}} m(\widehat{Q},\widehat{Q}')^{p'} &= \sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{1,s}} m(\widehat{Q},\widehat{Q}')^{p'-1} m(\widehat{Q},\widehat{Q}') \\ &\leq C\alpha^{1-p'} \sum_{s\geq 0} h(2^{-s-k}r)^{p'-1} \left(\sum_{(\widehat{Q},\widehat{Q}')\in E_{1,s}} m(\widehat{Q},\widehat{Q}') \right).\end{aligned}$$

But there exists a constant $C_0 = C_0(n,p) > 1$ such that

$$\frac{1}{C_0} \int_0^{2^{-k}r} h(t)^{p'-1} \frac{dt}{t} \leq \sum_{s\geq 0} h(2^{-k-s}r)^{p'-1} \leq C_0 \int_0^{2^{-k}r} h(t)^{p'-1} \frac{dt}{t}$$

for every $r > 0$, every integer $k > 1$ and every increasing homeomorphism $h : [0,\infty) \to [0,\infty)$ such that $t \mapsto h(t)t^{-n}$, $0 < t < \infty$, is decreasing. Hence

$$\sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{1,s}} m(\widehat{Q},\widehat{Q}')^{p'} \leq C\, \alpha^{1-p'} \int_0^{2^{-k}r} h(t)^{p'-1} \frac{dt}{t}.$$

From a similar computation we get

$$\begin{aligned}\sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{0,s}} m(\widehat{Q},\widehat{Q}')^{p'} &= \sum_{s=0}^{\infty} \sum_{(\widehat{Q},\widehat{Q}')\in E_{0,s}} m(\widehat{Q},\widehat{Q}')^{p'-1} m(\widehat{Q},\widehat{Q}') \\ &\leq C \sum_{s\geq 0} 2^{-(p'-1)s} \left(\sum_{(\widehat{Q},\widehat{Q}')\in E_{0,s}} m(\widehat{Q},\widehat{Q}') \right) = C.\end{aligned}$$

So we get

$$\sum \leq C\left(\alpha^{1-p'}\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t} + k + 1\right).$$

It is easy to see that there exists a constant $C = C(n,p)$ such that

$$\frac{\Lambda_h^\infty(Q_{k,r})}{\left(\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \leq C.$$

for every $r > 0$, every integer $k > 1$ and every increasing homeomorphism $h : [0,\infty) \to [0,\infty)$ such that $t \mapsto h(t)t^{-n}$, $0 < t < \infty$, is decreasing. Hence

$$\sum \leq Ck\,\alpha^{1-p'}\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}.$$

Therefore we obtain

$$1 \leq C[f]_{B_p(\mathbb{R}^n)}\left(k\,\alpha^{1-p'}\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{1/p'}$$

for every integer $k > 1$ and for every $f \in BW(E \cap Q_{k,r}, \mathrm{int}(Q_{0,r}))$.

This implies that there exists $C_1' = C_1'(n,p) > 0$ such that

$$\frac{\Lambda_h^\infty(E \cap Q_{k,r})}{\left(\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}}k^{1-p} \leq C_1'\mathrm{cap}_{B_p}(E \cap Q_{k,r}, \mathrm{int}(Q_{0,r})).$$

This finishes the proof. ∎

Theorem 3.8 helps us formulate and prove the following theorem. We leave the details to the reader.

Theorem 3.9. *Suppose $h : [0,\infty) \to [0,\infty)$ is an increasing homeomorphism such that $t \mapsto h(t)t^{-n}$, $0 < t < \infty$ is decreasing. There exists a positive constant $C_1 = C_1(n,p)$ such that*

$$\frac{\Lambda_h^\infty(E \cap Q(x, 2^{-k}r))}{\left(\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \leq C_1 k^{p-1}\mathrm{cap}_{B_p}(E \cap Q(x, 2^{-k}r), \mathrm{int}(Q(x,r)))$$

for every $E \subset \mathbb{R}^n$, every integer $k > 1$, every $x \in \mathbb{R}^n$, and every $r > 0$.

From Theorem 3.9 it follows easily that there exists a constant $C = C(n,p)$ such that

$$\frac{\Lambda_1^\infty(E \cap B(a,3R))}{R} \leq C\mathrm{cap}_{B_p}(E \cap B(a,3R), B(a,6R)) \tag{3.9}$$

whenever $E \subset \mathbb{R}^n$, $R > 0$, and $a \in \mathbb{R}^n$.

As a corollary we have the following.

Corollary 3.10. *There exists a positive constant $C_2 = C_2(n,p)$ such that*

$$C_2 \left(\ln \frac{R}{r}\right)^{1-p} \le \operatorname{cap}_{B_p}(Q(x,r), \operatorname{int}(Q(x,R))) \tag{3.10}$$

for every $x \in \mathbb{R}^n$ and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

Proof. We apply Theorem 3.9 for $h(t) = t$. We notice that there exists a constant $C_2' = C_2'(n,p)$ such that

$$\frac{\Lambda_1^\infty(Q(x,2^{-k}r))}{\left(\int_0^{2^{-k}r} t^{p'-1}\frac{dt}{t}\right)^{p-1}} = C_2' \tag{3.11}$$

for every $x \in \mathbb{R}^n$, every integer $k \ge 2$ and every $r > 0$. The rest is routine. ■

Theorem 3.4 and Corollary 3.10 easily yield the following theorem, (cf. [6]).

Theorem 3.11. *There exists $C_0 = C_0(n,p) > 0$ such that*

$$\frac{1}{C_0}\left(\ln \frac{R}{r}\right)^{1-p} \le \operatorname{cap}_{B_p}(B(x,r), B(x,R)) \le C_0 \left(\ln \frac{R}{r}\right)^{1-p} \tag{3.12}$$

for every $x \in \mathbb{R}^n$ and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

A set $E \subset \mathbb{R}^n$ is said to be *of Besov B_p-capacity zero* if $\operatorname{cap}_{B_p}(E\cap\Omega, \Omega) = 0$ for all open and bounded $\Omega \subset \mathbb{R}^n$. In this case we write $\operatorname{cap}_{B_p}(E) = 0$.

The following lemma is obvious.

Lemma 3.12. *A countable union of sets of Besov B_p-capacity zero has Besov B_p-capacity zero.*

The next lemma shows that, if E is bounded, one needs to test only a single bounded open set Ω containing E in showing that E has zero Besov B_p-capacity.

Lemma 3.13. *Suppose that E is bounded and that there is a bounded neighborhood Ω of E with $\operatorname{cap}_{B_p}(E,\Omega) = 0$. Then $\operatorname{cap}_{B_p}(E) = 0$.*

Proof. Let Ω' be an open set. Since there exists a Borel set $\widetilde{E} \subset \Omega$ such that $E \subset \widetilde{E}$ and $\operatorname{cap}_{B_p}(E,\Omega) = \operatorname{cap}_{B_p}(\widetilde{E},\Omega) = 0$, we may assume that E is itself a Borel set.

Thus, by invoking Theorem 3.2, we may assume that $E \cap \Omega'$ is compact. Since $\operatorname{cap}_{B_p}(E\cap\Omega', \Omega) = 0$, there exists a sequence $\varphi_i \in BW(E\cap\Omega', \Omega)$ such that $[\varphi_i]_{B_p(\Omega)} < 2^{-i}$ for every integer $i \geq 1$. We notice that the sequence φ_i is bounded in $B_p^0(\Omega)$. Therefore, from the reflexivity of $B_p^0(\Omega)$, there exists $\psi \in B_p^0(\Omega)$ and a subsequence denoted again by φ_i such that $\varphi_i \to \psi$ weakly in $B_p^0(\Omega)$. From Mazur's lemma, there exists a sequence $\widetilde{\varphi}_i$ of convex combinations of φ_j,

$$\widetilde{\varphi}_i = \sum_{j=i}^{j_i} \lambda_{i,j}\varphi_j, \quad \lambda_{i,j} \geq 0, \text{ and } \sum_{j=i}^{j_i} \lambda_{i,j} = 1,$$

such that $\widetilde{\varphi}_i \to \psi$ in $B_p^0(\Omega)$. Without loss of generality we can assume that $\widetilde{\varphi}_i \to \psi$ pointwise a.e. in $\mathbb{R}^n$ as $i \to \infty$. The convexity of the Besov seminorm and the choice of the sequence φ_i imply, together with the closedness of $BW(E \cap \Omega', \Omega)$ under finite convex combinations, that $\widetilde{\varphi}_i$ is a sequence in $BW(E\cap\Omega', \Omega)$ and $[\widetilde{\varphi}_i]_{B_p(\Omega)} < 2^{-i}$ for every integer $i \geq 1$. Since $[\widetilde{\varphi}_i]_{B_p(\Omega)} < 2^{-i}$ for every integer $i \geq 1$ and $\widetilde{\varphi}_i \to \psi$ in $B_p^0(\Omega)$, it follows that $[\psi]_{B_p(\Omega)} = 0$. Therefore, from (2.1) and the fact that $\psi \in L^p(\mathbb{R}^n)$, it follows that in fact $\psi = 0$ a.e. in $\mathbb{R}^n$, which means that

$$||\widetilde{\varphi}_i||_{B_p(\Omega)} \to 0 \tag{3.13}$$

as $i \to \infty$. Let $\eta \in BW(E\cap\Omega', \Omega')$. Then $\eta\, \widetilde{\varphi}_i$ is a sequence in $BW(E\cap\Omega', \Omega')$, hence

$$\operatorname{cap}_{B_p}(E \cap \Omega', \Omega') \leq [\eta\, \widetilde{\varphi}_i]^p_{B_p(\Omega')} \tag{3.14}$$

for every integer $i \geq 1$. From Lemma 2.5, (3.13) and (3.14) we have

$$\begin{aligned} 0 &\leq \operatorname{cap}_{B_p}(E \cap \Omega', \Omega') \leq \lim_{i\to\infty} [\eta\, \widetilde{\varphi}_i]^p_{B_p(\Omega')} \leq \lim_{i\to\infty} ||\eta\, \widetilde{\varphi}_i||^p_{B_p(\Omega)} \\ &\leq C||\widetilde{\varphi}_i||^p_{B_p(\Omega)} \to 0, \end{aligned}$$

where C depends only on n, p, the Lipschitz constant of η and the diameter of supp η. Hence $\operatorname{cap}_{B_p}(E \cap \Omega', \Omega') = 0$. This finishes the proof. ■

Corollary 3.14. *Let $E \subset \mathbb{R}^n$ be such that $\operatorname{cap}_{B_p}(E) = 0$. Then $\Lambda_h(E) = 0$ for every measure function $h : [0, \infty) \to [0, \infty)$ such that*

$$\int_0^1 h(t)^{p'-1} \frac{dt}{t} < \infty. \tag{3.15}$$

In particular, the Hausdorff dimension of E is zero.

Note that for every $\varepsilon > 0$ we can take $h = h_\varepsilon : [0, \infty) \to [0, \infty)$ in Corollary 3.14, where $h_\varepsilon(t) = |\ln t|^{1-p-\varepsilon}$ for every $t \in (0, 1/2)$.

Proof. It is enough to assume, without loss of generality, that $h : [0, \infty) \to [0, \infty)$ is an increasing homeomorphism such that $t \mapsto h(t)t^{-n}$, $0 < t < \infty$ is decreasing. (See [1, Proposition 5.1.8].)

If $\mathrm{cap}_{B_p}(E) = 0$, then there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E}$ and $\mathrm{cap}_{B_p}(\widetilde{E}) = 0$, hence we can assume without loss of generality that E is itself Borel.

Since Λ_h is a Borel regular measure and $\Lambda_h(E) = 0$ if and only if $\Lambda_h^\infty(E) = 0$, (see [1, Proposition 5.1.5]) it is enough to assume that E is in fact compact.

For E compact the claim follows obviously from Theorem 3.9.

The second claim is a consequence of (i) because for every $s \in (0, n)$, the function $h_s : [0, \infty) \to [0, \infty)$ defined by $h_s(t) = t^s$ has the property (3.15). ■

As another corollary, we have the following:

Corollary 3.15. *Suppose $E \subset \mathbb{R}^n$ is such that $\mathrm{cap}_{B_p}(E) = 0$. Then $\mathbb{R}^n \setminus E$ is connected.*

Proof. We fix $s \in (1, n)$. If $\mathrm{cap}_{B_p}(E) = 0$, then in particular we have $\Lambda_{n-s}(E) = 0$ and this implies via [20, Theorem 2.27 and Corollary 2.39] that $\mathrm{Cap}_s(E) = 0$. Here Cap_s denotes the Sobolev s-capacity as in [20, p. 48]. An appeal to [20, Lemma 2.46] finishes the proof. ■

We close this section with another sufficient condition to get sets of Besov B_p-capacity zero.

Theorem 3.16. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\ln \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Then $\Lambda_h(E) < \infty$ implies $\mathrm{cap}_{B_p}(E) = 0$ for every set E in $\mathbb{R}^n$.*

Before we prove Theorem 3.16, we state and prove the following proposition.

Proposition 3.17. *Let E be a compact set in $\mathbb{R}^n$. There exists a constant $C = C(n, p)$ such that $\mathrm{cap}_{B_p}(E, \Omega) \le C\Lambda_h(E)$ for every bounded and open set Ω containing E.*

Proof. We can assume without loss of generality that $\Lambda_h(E) < \infty$. Let Ω be a bounded open set containing E.

We denote by δ the distance from E to the complement of Ω. Without loss of generality we can assume that $0 < \delta < 1$.

We fix $0 < \varepsilon < 1$ such that $0 < \varepsilon < \frac{\delta^2}{4}$. Then $r < \varepsilon$ implies

$$\ln\left(\frac{\delta}{2r}\right) \ge \frac{1}{2} \ln\left(\frac{1}{r}\right).$$

We cover E by open balls $B(x_i, r_i)$ such that $r_i < \frac{\varepsilon}{2}$. Since we may assume that the balls $B(x_i, r_i)$ intersect E, we have $B(x_i, \frac{\delta}{2}) \subset \Omega$. In fact, since E is compact, E is covered by only finitely many of the balls $B(x_i, r_i)$.

As in [20, p. 48] we obtain

$$\begin{aligned} \operatorname{cap}_{B_p}(E, \Omega) &\leq \sum_i \operatorname{cap}_{B_p}(B(x_i, r_i), \Omega) \leq \sum_i \operatorname{cap}_{B_p}(B(x_i, r_i), B(x_i, \frac{\delta}{2})) \\ &\leq C(n,p) \sum_i \left(\ln \frac{1}{r_i} \right)^{1-p}. \end{aligned}$$

In the last step we also used formula (3.12) for the Besov B_p-capacity of spherical condensers together with our choice of ε. Taking the infimum over all such coverings and letting $\varepsilon \to 0$, we conclude $\operatorname{cap}_{B_p}(E, \Omega) \leq C\Lambda_h(E)$. This finishes the proof of the proposition. ■

We prove now Theorem 3.16.

Proof. Since Λ_h is a Borel regular measure, we may assume that E is a Borel set and furthermore, in light of the Choquet capacitability theorem, we may assume that E is compact. We let $M = C\Lambda_h(E)$, where C is the constant from Proposition 3.17. Since $\Lambda_h(E) < \infty$, we have that $|E| = 0$, while Proposition 3.17 implies that $\operatorname{cap}_{B_p}(E, \Omega) \leq M$ for every bounded and open set Ω containing E. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set containing E. From Lemma 3.13 it is enough to show that $\operatorname{cap}_{B_p}(E, \Omega) = 0$. We choose a descending sequence of bounded open sets

$$\Omega = \Omega_1 \supset\supset \Omega_2 \supset\supset \ldots \supset\supset \cap_i \Omega_i = E$$

and a sequence $\varphi_i \in BW(E, \Omega_i)$ with $[\varphi_i]^p_{B_p(\Omega_i)} < M + 1$. Then φ_i is a bounded sequence in $B_p(\Omega)$. Because φ_i converges pointwise to a function ψ which is 0 in $\mathbb{R}^n \setminus E$ and 1 on E, we have from Mazur's lemma and the reflexivity of $B^0_p(\Omega)$ that $\psi \in B^0_p(\Omega)$. That is, there exists a subsequence denoted again by φ_i such that $\varphi_i \to \psi$ weakly in $B^0_p(\Omega)$ and a sequence $\widetilde{\varphi}_i$ of convex combinations of φ_j,

$$\widetilde{\varphi}_i = \sum_{j=i}^{j_i} \lambda_{i,j} \varphi_j, \quad \lambda_{i,j} \geq 0, \text{ and } \sum_{j=i}^{j_i} \lambda_{i,j} = 1,$$

such that $\widetilde{\varphi}_i \to \psi$ in $B^0_p(\Omega)$. Without loss of generality we can assume that $\widetilde{\varphi}_i \to \psi$ pointwise in $\mathbb{R}^n$ as $i \to \infty$. The convexity of the Besov seminorm and the choice of the sequence φ_i imply, together with the closedness of $BW(E, \Omega_i)$ under finite convex combinations, that $\widetilde{\varphi}_i \in BW(E, \Omega_i)$ for every integer $i \geq 1$. Since $|E| = 0$, $\psi = 0$ in $\mathbb{R}^n \setminus E$, and $\widetilde{\varphi}_i \to \psi$ in $B_p^0(\Omega)$, it follows that $||\psi||_{B_p(\Omega)} = 0$. This implies $||\widetilde{\varphi}_i||_{B_p(\Omega)} \to 0$ as $i \to \infty$, hence

$$0 \leq \mathrm{cap}_{B_p}(E,\Omega) \leq \lim_{i\to\infty} [\widetilde{\varphi}_i]^p_{B_p(\Omega)} = 0\,.$$

■

4. Besov capacity and quasicontinuous functions

In this section we study a global Besov capacity and quasicontinuous functions in Besov spaces.

4.1. Besov Capacity

Definition 4.1. For a set $E \subset \mathbb{R}^n$ define

$$\mathrm{Cap}_{B_p}(E) = \inf\{||u||^p_{L^p(\mathbb{R}^n)} + [u]^p_{B_p(\mathbb{R}^n)} : u \in S(E)\},$$

where u runs through the set

$$S(E) = \{u \in B_p(\mathbb{R}^n) : \; u = 1 \text{ in a neighborhood of } E\}.$$

Since $B_p(\mathbb{R}^n)$ is closed under truncations from below by 0 and from above by 1 and since these truncations do not increase the norm, we may restrict ourselves to those functions $u \in S(E)$ for which $0 \leq u \leq 1$. We get the same capacity if we consider the apparently larger set of admissible functions, namely

$$\widetilde{S}(E) = \{u \in B_p(\mathbb{R}^n) : \; u \geq 1 \text{ a.e. in a neighborhood of } E\}.$$

Moreover, we have the following lemma:

Lemma 4.2. *If K is compact, then*

$$\mathrm{Cap}_{B_p}(K) = \inf\{||u||^p_{L^p(\mathbb{R}^n)} + [u]^p_{B_p(\mathbb{R}^n)} : u \in S_0(K)\}$$

where $S_0(K) = S(K) \cap C_0^\infty(\mathbb{R}^n)$.

Proof. Let $u \in S(K)$. Since $B_p(\mathbb{R}^n) = B_p^0(\mathbb{R}^n)$, we may choose a sequence of functions $\varphi_j \in C_0^\infty(\mathbb{R}^n)$ converging to u in $B_p(\mathbb{R}^n)$. Let U be a bounded and open neighborhood of K such that $u = 1$ in U. Let $\psi \in C^\infty(\mathbb{R}^n)$, $0 \leq \psi \leq 1$ be such that $\psi = 1$ in $\mathbb{R}^n \setminus U$ and $\psi = 0$ in $\widetilde{U} \subset\subset U$, an open neighborhood of K. From Lemma 2.6 we see that the functions $\psi_j = 1-(1-\varphi_j)\psi$ converge to $1-(1-u)\psi$ in $B_p(\mathbb{R}^n)$. This establishes the assertion since $1-(1-u)\psi = u$. ■

We can apply almost verbatim the proof of Theorem 3.1 to conclude:

Theorem 4.3. *The set function* $E \mapsto \mathrm{Cap}_{B_p}(E), E \subset \mathbb{R}^n$ *is a Choquet capacity. In particular*

(i) *If* $E_1 \subset E_2$*, then* $\mathrm{Cap}_{B_p}(E_1) \le \mathrm{Cap}_{B_p}(E_2)$.

(ii) *If* $E = \bigcup_i E_i$*, then*

$$\mathrm{Cap}_{B_p}(E) \le \sum_i \mathrm{Cap}_{B_p}(E_i).$$

We have introduced two different capacities, and it is next shown that they have the same zero sets.

Let $\Omega, \widetilde{\Omega}$ be bounded and open subsets of $\mathbb{R}^n$ such that $\Omega \subset\subset \widetilde{\Omega}$. Let $\eta \in C_0^\infty(\widetilde{\Omega})$ be a cut-off function satisfying (2.15). Suppose K is a compact subset of Ω. Then, if $u \in S_0(K)$, we have that $u\eta$ is admissible for the condenser $(K, \widetilde{\Omega})$. Therefore

$$\mathrm{cap}_{B_p}(K, \widetilde{\Omega}) \le [u\eta]^p_{B_p(\widetilde{\Omega})} \le ||u\eta||^p_{B_p(\widetilde{\Omega})} \le C\,||u||^p_{B_p(\mathbb{R}^n)} \tag{4.1}$$

where C depends only on n, p, diam $\widetilde{\Omega}$ and $\mathrm{dist}(\Omega, \partial\widetilde{\Omega})$. (See Remark 2.7.) Since $||u||_{B_p(\mathbb{R}^n)} = ||u||_{L^p(\mathbb{R}^n)} + [u]_{B_p(\mathbb{R}^n)}$, we have

$$||u||^p_{B_p(\mathbb{R}^n)} \le 2^{p-1}(||u||^p_{L^p(\mathbb{R}^n)} + [u]^p_{B_p(\mathbb{R}^n)}). \tag{4.2}$$

From (4.1) and (4.2) we get, by taking the infimum over all $u \in S_0(K)$, that

$$\mathrm{cap}_{B_p}(K, \widetilde{\Omega}) \le 2^{p-1}\, C\, \mathrm{Cap}_{B_p}(K), \tag{4.3}$$

where C is the constant from (4.1).

Since both $\mathrm{cap}_{B_p}(\cdot, \widetilde{\Omega})$ and $\mathrm{Cap}_{B_p}(\cdot)$ are Choquet capacities, we obtain:

Theorem 4.4. *There exists* $C > 0$ *depending only on* n*,* p*,* diam $\widetilde{\Omega}$ *and* $\mathrm{dist}(\Omega, \partial\widetilde{\Omega})$ *such that*

$$\mathrm{cap}_{B_p}(E, \widetilde{\Omega}) \le C\, \mathrm{Cap}_{B_p}(E) \tag{4.4}$$

for every $E \subset \Omega$.

Corollary 4.5. *If* $\mathrm{Cap}_{B_p}(E) = 0$*, then* $\mathrm{cap}_{B_p}(E) = 0$.

We also have a converse result, namely:

Theorem 4.6. *If* $\mathrm{cap}_{B_p}(E) = 0$*, then* $\mathrm{Cap}_{B_p}(E) = 0$.

Proof. Without loss of generality we can assume that E is bounded. Since we have $\mathrm{cap}_{B_p}(E) = 0$, there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E}$ and $\mathrm{cap}_{B_p}(\widetilde{E}) = 0$. Since $\mathrm{Cap}_{B_p}(\cdot)$ is a Choquet capacity, we can in fact assume that E is compact. Then we have $\mathrm{cap}_{B_p}(E, \Omega) = 0$ for every Ω open and bounded, $\Omega \supset E$. We fix $\Omega \subset \mathbb{R}^n$ bounded and open such that $E \subset \Omega$. Like in the proof of Lemma 3.13, we construct a sequence of functions $\widetilde{\varphi}_i$ in $BW(E, \Omega)$ such that $||\widetilde{\varphi}_i||_{B_p(\Omega)} \to 0$ as $i \to \infty$. This implies in particular that the sequence $\widetilde{\varphi}_i$ is in $S_0(E)$ with

$$0 \le \mathrm{Cap}_{B_p}(E) \le \lim_{i\to\infty} \left(||\widetilde{\varphi}_i||^p_{L^p(\mathbb{R}^n)} + [\widetilde{\varphi}_i]^p_{B_p(\mathbb{R}^n)} \right) = 0,$$

hence $\mathrm{Cap}_{B_p}(E) = 0$. This proves the claim. ■

Remark 4.7. For $E \subset \mathbb{R}^n$ compact we see from the proof of Lemma 3.13 and Theorem 4.6 that it is enough to have $\mathrm{cap}_{B_p}(E, \Omega) = 0$ for one bounded open set $\Omega \subset \mathbb{R}^n$ with $E \subset \Omega$ in order to have $\mathrm{Cap}_{B_p}(E) = 0$.

It is desirable to know when a set is negligible for a Besov space. If there is an isometric isomorphism between two normed spaces X and Y we write $X = Y$. In particular, if E is relatively closed subset of Ω, then by

$$B^0_p(\Omega \setminus E) = B^0_p(\Omega)$$

we mean that each function $u \in B^0_p(\Omega)$ can be approximated in B_p-norm by functions from $C^\infty_0(\Omega \setminus E)$.

Theorem 4.8. *Suppose that E is a relatively closed subset of Ω. Then*

$$B^0_p(\Omega \setminus E) = B^0_p(\Omega)$$

if and only $\mathrm{Cap}_{B_p}(E) = 0$.

Proof. Suppose that $\mathrm{Cap}_{B_p}(E) = 0$. Let $\varphi \in C^\infty_0(\Omega)$ and choose a sequence u_j of functions in $B_p(\mathbb{R}^n)$ such that $0 \le u_j \le 1$, $u_j = 1$ in a neighborhood of E and $u_j \to 0$ in $B_p(\mathbb{R}^n)$. For every $j \ge 1$ we define $w_j = (1 - u_j)\varphi$. Then from Remark 2.8 and the properties of the functions φ and u_j, it follows that w_j is a bounded sequence of functions in $B_p(\mathbb{R}^n)$, compactly supported in $\Omega \setminus E$. Lemma 2.10 implies that w_j is a sequence in $B^0_p(\Omega \setminus E)$. Moreover, Lemma 2.6 implies, since $\varphi - w_j = u_j\varphi$ for every $j \ge 1$ and since $||u_j||_{B_p(\mathbb{R}^n)} \to 0$, that w_j converges to φ in $B_p(\mathbb{R}^n)$. Since w_j is a sequence in $B^0_p(\Omega \setminus E)$, it follows that $\varphi \in B^0_p(\Omega \setminus E)$. Hence

$$B^0_p(\Omega) \subset B^0_p(\Omega \setminus E)$$

and since the reverse inclusion is trivial, the sufficiency is established.

For the only if part, let $K \subset E$ be compact. It suffices to show that $\mathrm{Cap}_{B_p}(K) = 0$. Choose $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ in a neighborhood of K. Since $B_p^0(\Omega \setminus E) = B_p^0(\Omega)$, we may choose a sequence of functions $\varphi_j \in C_0^\infty(\Omega \setminus K)$ such that $\varphi_j \to \varphi$ in $B_p(\Omega)$. Consequently

$$\mathrm{Cap}_{B_p}(K) \leq \left(\lim_{j\to\infty} ||\varphi_j - \varphi||^p_{L^p(\mathbb{R}^n)} + [\varphi_j - \varphi]^p_{B_p(\mathbb{R}^n)} \right) = 0,$$

and the theorem follows. ■

4.2. Quasicontinuous functions

We show that for each $u \in B_p(\mathbb{R}^n)$ there is a function v such that $u = v$ a.e. and that v is B_p-quasicontinuous, i.e. v is continuous when restricted to a set whose complement has arbitrarily small Besov B_p-capacity. Moreover, this quasicontinuous representative is unique up to a set of Besov B_p-capacity zero.

Definition 4.9. A function $u : \mathbb{R}^n \to \mathbb{R}$ is B_p-*quasicontinuous* if for every $\varepsilon > 0$ there is an open set $G \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{B_p}(G) < \varepsilon$ and the restriction of u to $\mathbb{R}^n \setminus G$ is continuous.

A sequence of functions $\psi_j : \mathbb{R}^n \to \mathbb{R}$ converges B_p-*quasiuniformly* in $\mathbb{R}^n$ to a function ψ if for every $\varepsilon > 0$ there is an open set G such that $\mathrm{Cap}_{B_p}(G) < \varepsilon$ and $\psi_j \to \psi$ uniformly in $\mathbb{R}^n \setminus G$.

We say that a property holds B_p-*quasieverywhere*, or simply q.e., if it holds except on a set of Besov B_p-capacity zero.

Theorem 4.10. *Let $\varphi_j \in C(\mathbb{R}^n) \cap B_p(\mathbb{R}^n)$ be a Cauchy sequence in $B_p(\mathbb{R}^n)$. Then there is a subsequence φ_k which converges B_p-quasiuniformly in $\mathbb{R}^n$ to a function $u \in B_p(\mathbb{R}^n)$. In particular, u is B_p-quasicontinuous and $\varphi_k \to u$ B_p-quasieverywhere in $\mathbb{R}^n$.*

Proof. The proof is similar to the proof of [20, Theorem 4.3] and omitted. ■

Theorem 4.10 implies the following corollary.

Corollary 4.11. *Suppose that $u \in B_p(\mathbb{R}^n)$. Then there exists a Borel B_p-quasicontinuous function $v \in B_p(\mathbb{R}^n)$ such that $u = v$ a.e.*

Proof. Since $u \in B_p(\mathbb{R}^n)$, from Theorem 2.4 there exists a sequence of functions φ_j in $C_0^\infty(\mathbb{R}^n)$ converging to u in $B_p(\mathbb{R}^n)$. Passing to subsequences if necessary, we can assume that $\varphi_j \to u$ pointwise a.e. in $\mathbb{R}^n$ and that

$$2^{jp} \left(||\varphi_{j+1} - \varphi_j||^p_{L^p(\mathbb{R}^n)} + [\varphi_{j+1} - \varphi_j]^p_{B_p(\mathbb{R}^n)} \right) < 2^{-j}$$

for every $j = 1, 2, \ldots$ Defining $E_j = \{x \in \mathbb{R}^n : |\varphi_{j+1}(x) - \varphi_j(x)| > 2^{-j}\}$ and letting $E = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j$, the proof of the previous theorem yields the existence of a function $v \in B_p(\mathbb{R}^n)$, such that $\varphi_j \to v$ in $B_p(\mathbb{R}^n)$ and pointwise in $\mathbb{R}^n \setminus E$. Since E is a Borel set of Besov B_p-capacity zero and the functions φ_j are continuous, this finishes the proof. ∎

Theorem 4.12. *Let $u \in B_p(\mathbb{R}^n)$. Then $u \in B_p^0(\Omega)$ if and only if there exists a B_p-quasicontinuous function v in $\mathbb{R}^n$ such that $u = v$ a.e. in Ω and $v = 0$ q.e. in $\mathbb{R}^n \setminus \Omega$.*

Proof. Fix $u \in B_p^0(\Omega)$ and let $\varphi_j \in C_0^{\infty}(\Omega)$ be a sequence converging to u in $B_p(\Omega)$. By Theorem 4.10 there is a subsequence of φ_j which converges B_p-quasieverywhere in $\mathbb{R}^n$ to a B_p-quasicontinuous function v in $\mathbb{R}^n$ such that $u = v$ a.e. in Ω and $v = 0$ q.e. in $\mathbb{R}^n \setminus \Omega$. Hence v is the desired function.

To prove the converse, we assume first that Ω is bounded. Because the truncations of v converge to v in $B_p(\Omega)$, we can assume that v is bounded. Without loss of generality, since v is B_p-quasicontinuous and $v = 0$ q.e. outside Ω we can assume that in fact $v = 0$ everywhere in $\mathbb{R}^n \setminus \Omega$.

Choose open sets G_j such that v is continuous on $\mathbb{R}^n \setminus G_j$ and

$$\mathrm{Cap}_{B_p}(G_j) \longrightarrow 0.$$

By passing to a subsequence, we may pick a sequence φ_j in $B_p(\mathbb{R}^n)$ such that $0 \le \varphi_j \le 1$, $\varphi_j = 1$ everywhere in G_j, $\varphi_j \to 0$ a.e. in $\mathbb{R}^n$, and

$$||\varphi_j||_{L^p(\mathbb{R}^n)}^p + [\varphi_j]_{B_p(\mathbb{R}^n)}^p \to 0.$$

Then from Remark 2.8 we have that $w_j = (1 - \varphi_j)v$ is a bounded sequence in $B_p(\Omega)$. Moreover, for every $j \ge 1$, we have $\lim_{x \to y, x \in \Omega} w_j(x) = 0$ for all $y \in \partial\Omega$.

Thus, from Lemma 2.11, we have that w_j is a sequence in $B_p^0(\Omega)$. Clearly $w_j \to v$ in $L^p(\mathbb{R}^n)$ and pointwise a.e. in $\mathbb{R}^n$.

This, together with the boundedness of the sequence w_j in $B_p^0(\Omega)$, implies via Mazur's lemma that $v \in B_p^0(\Omega)$. The proof is complete in case Ω is bounded.

Assume that Ω is unbounded. We can assume again, without loss of generality, that v is bounded and that $v = 0$ everywhere in $\mathbb{R}^n \setminus \Omega$. For every $k \ge 2$ let $\varphi_k \in C_0^{\infty}(B(0, k^2))$ be such that $0 \le \varphi_k \le 1$, $\varphi_k = 1$ on $B(0, k)$ and $[\varphi_k]_{B_p(\mathbb{R}^n)} \le C(\ln k)^{1-p}$. (See (3.4).) Then

$$v_k = v\varphi_k \in B_p^0(\Omega \cap B(0, k^2)) \subset B_p^0(\Omega)$$

for every $k \ge 2$ and like in Theorem 2.4, we get

$$||v - v_k||_{B_p(\mathbb{R}^n)} \to 0,$$

which implies that $v \in B_p^0(\Omega)$. ∎

We denote by

$$Q^{B_p} = Q^{B_p}(\mathbb{R}^n)$$

the set of all functions $u \in B_p(\mathbb{R}^n)$ such that there exists a sequence $\varphi_j \in C(\mathbb{R}^n) \cap B_p(\mathbb{R}^n)$ converging to u both in $B_p(\mathbb{R}^n)$ and B_p-quasiuniformly. It follows immediately from Theorem 4.10 that the functions in Q^{B_p} are B_p-quasicontinuous and for each $v \in B_p(\mathbb{R}^n)$ there is $u \in Q^{B_p}$ such that $u = v$ a.e. We soon show that, conversely, each B_p-quasicontinuous function v of $B_p(\mathbb{R}^n)$ belongs to Q^{B_p}.

Theorem 4.13. *Let $u \in Q^{B_p}$. If $u \geq 1$ B_p-quasieverywhere on E, then*

$$\operatorname{Cap}_{B_p}(E) \leq ||u||^p_{L^p(\mathbb{R}^n)} + [u]^p_{B_p(\mathbb{R}^n)}.$$

Proof. The proof is similar to the proof of [20, Lemma 4.7] and omitted.■

This result has the following corollary.

Corollary 4.14. *Suppose that Ω is open and bounded and let $E \subset\subset \Omega$. Let $u \in Q^{B_p}$. Suppose that $u \geq 1$ quasieverywhere on E and that u has compact support in Ω. Then*

$$\operatorname{cap}_{B_p}(E, \Omega) \leq [u]^p_{B_p(\Omega)}.$$

We know that Cap_{B_p} is an outer capacity. We will show that it satisfies the following compatibility condition (see [21]):

Theorem 4.15. *Suppose that G is open and $|E| = 0$. Then*

$$\operatorname{Cap}_{B_p}(G) = \operatorname{Cap}_{B_p}(G \setminus E). \tag{4.5}$$

Proof. Obviously we have $\operatorname{Cap}_{B_p}(G \setminus E) \leq \operatorname{Cap}_{B_p}(G)$. Conversely, we can assume without loss of generality that $\operatorname{Cap}_{B_p}(G \setminus E) < \infty$. We fix $\varepsilon > 0$. There exists a function $u_\varepsilon \in B_p(\mathbb{R}^n)$ and an open neighborhood W of $G \setminus E$ such that $u_\varepsilon = 1$ on W and

$$||u_\varepsilon||^p_{L^p(\mathbb{R}^n)} + [u_\varepsilon]^p_{B_p(\mathbb{R}^n)} < \operatorname{Cap}_{B_p}(G \setminus E) + \varepsilon.$$

Since $|E| = 0$, we can assume without loss of generality that in fact $u_\varepsilon = 1$ on E. But then $u_\varepsilon = 1$ on $W \cup G$ which is an open neighborhood of G, hence

$$\operatorname{Cap}_{B_p}(G) \leq ||u_\varepsilon||^p_{L^p(\mathbb{R}^n)} + [u_\varepsilon]^p_{B_p(\mathbb{R}^n)} < \operatorname{Cap}_{B_p}(G \setminus E) + \varepsilon.$$

Letting $\varepsilon \to 0$ we get the desired conclusion. ■

We state now the uniqueness of a B_p-quasicontinuous representative.

Theorem 4.16. *Let f and g be B_p-quasicontinuous functions on $\mathbb{R}^n$ such that*

$$|\{x : f(x) \neq g(x)\}| = 0.$$

Then $f = g$ B_p-quasieverywhere on $\mathbb{R}^n$.

Proof. The proof is verbatim the proof from [21, p. 262]. ∎

Combining Theorem 4.13 and Theorem 4.16 we obtain the following corollary.

Corollary 4.17. *Suppose that $E \subset \mathbb{R}^n$. Then*

$$\mathrm{Cap}_{B_p}(E) = \inf\{||u||^p_{L^p(\mathbb{R}^n)} + [u]^p_{B_p(\mathbb{R}^n)}\},$$

where the infimum is taken over all B_p-quasicontinuous $u \in B_p(\mathbb{R}^n)$ such that $u = 1$ B_p-quasieverywhere on E.

Corollary 4.11 and Theorem 4.16 imply that each $u \in B_p(\mathbb{R}^n)$ has a "unique" quasicontinuous Borel version.

Corollary 4.18. *Suppose that $u \in B_p(\mathbb{R}^n)$. Then there exists a Borel B_p-quasicontinuous function v such that $u = v$ a.e. Moreover, if $\widetilde{v}$ is another Borel B_p-quasicontinuous function such that $u = \widetilde{v}$ a.e., then $v = \widetilde{v}$ B_p-quasieverywhere.*

We have a result similar to Corollary 4.18 for locally integrable functions with finite B_p-seminorm.

Corollary 4.19. *Suppose that $u \in L^1_{loc}(\mathbb{R}^n)$ such that $[u]_{B_p(\mathbb{R}^n)} < \infty$. Then there exists a B_p-quasicontinuous Borel function v such that $u = v$ a.e. Moreover, if $\widetilde{v}$ is another B_p-quasicontinuous Borel function such that $u = \widetilde{v}$ a.e., then $v = \widetilde{v}$ B_p-quasieverywhere.*

Proof. We prove the "uniqueness" first. Suppose $v, \widetilde{v}$ are two Borel B_p-quasicontinuous functions such that $v = u$ a.e. and $\widetilde{v} = u$ a.e. Let $w = v - \widetilde{v}$. We notice that w is B_p-quasicontinuous and belongs to $B_p(\mathbb{R}^n)$ because $w = 0$ a.e. in $\mathbb{R}^n$. Hence from Corollary 4.18 we have that $w = 0$ B_p-quasieverywhere. The "uniqueness" is proved.

We prove now the existence. For every integer $k \geq 1$ we choose a function $\eta_k \in C_0^\infty(B(0, 2^{k+1}))$ such that $\eta_k = 1$ on $B(0, 2^k)$ and $|\nabla \eta_k|_{L^\infty(\mathbb{R}^n)} \leq 2^{1-k}$. We have

$$\eta_{k+1}\eta_k = \eta_k \tag{4.6}$$

for every integer $k \geq 1$. For a fixed integer $k \geq 1$, we define $u_k = \eta_k u$. Then $u_k \in L^p(\mathbb{R}^n)$ because $u \in L^p_{loc}(\mathbb{R}^n)$ and $\eta_k \in C_0^\infty(B(0, 2^{k+1}))$. Moreover, from Lemma 2.9, it follows that $[\eta_k u - \eta_k u_{B(0,2^k)}]_{B_p(\mathbb{R}^n)} < \infty$. This, together with the fact that $\eta_k \in B_p(\mathbb{R}^n)$, imply that $u_k \in B_p(\mathbb{R}^n)$. Therefore, from Corollary 4.11 it follows that there exists $\widetilde{u}_k \in B_p(\mathbb{R}^n)$ a B_p-quasicontinuous Borel function such that $\widetilde{u}_k = u_k$ a.e. in $\mathbb{R}^n$. In particular, since $\eta_k = 1$ in $B(0,2^k)$, this implies that $\widetilde{u}_k = u$ a.e. in $B(0,2^k)$. So, for every integer $k \geq 1$ we have that $\widetilde{u}_{k+1}$ is a B_p-quasicontinous Borel representative of $\eta_{k+1}u$, hence $\eta_k\widetilde{u}_{k+1}$ is a B_p-quasicontinuous Borel representative of $\eta_k\eta_{k+1}u = u_k$, where the equality follows from the definition of u_k and (4.6). This implies that both $\eta_k\widetilde{u}_{k+1}$ and $\widetilde{u}_k$ are two B_p-quasicontinuous Borel representatives of $u_k \in B_p(\mathbb{R}^n)$, hence from Corollary 4.18 we can assume that $\widetilde{u}_k = \eta_k\widetilde{u}_{k+1}$ in $B(0,2^k)$. Since $\eta_k = 1$ on $B(0,2^k)$, this means in particular that we can assume that $\widetilde{u}_k(x) = \widetilde{u}_{k+1}(x)$ for every x in $B(0,2^k)$.

So, we constructed a sequence of B_p-quasicontinuous Borel functions $\widetilde{u}_k$ in $B_p(\mathbb{R}^n)$ satisfying the following properties:

$$\begin{aligned} \widetilde{u}_k(x) &= u(x) && \text{for a.e. } x \text{ in } B(0,2^k) \\ \widetilde{u}_l(x) &= \widetilde{u}_k(x) && \text{for every } x \text{ in } B(0,2^k) \text{ and } l \geq k \geq 1. \end{aligned}$$

We define $\widetilde{u} : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$\widetilde{u}(x) = \lim_{k\to\infty} \widetilde{u}_k(x).$$

Thus, $\widetilde{u}$ is a B_p-quasicontinuous Borel function and $u = \widetilde{u}$ a.e. This proves the existence of a B_p-quasicontinuous Borel representative of u. The claim follows. ∎

5. Strong A_∞-weights

In this section we apply results from previous sections to study strong A_∞-weights, as promised in the introduction. We prove the following theorems.

Theorem 5.1. *Let $s \in (n-1, n]$ and let u be in $L^1_{loc}(\mathbb{R}^n)$ such that its distributional gradient ∇u is in the Morrey space $\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)$. There exists $\varepsilon = \varepsilon(n,s) > 0$ such that if $||\nabla u||_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)} < \varepsilon$, then $w = e^{nu}$ is a strong A_∞-weight with data depending only on n and s.*

Theorem 5.2. *Let $p \in (n,\infty)$ and let u be in $L^1_{loc}(\mathbb{R}^n)$ such that $[u]_{B_p(\mathbb{R}^n)} < \infty$. There exists $\varepsilon = \varepsilon(n,p) > 0$ such that if $[u]_{B_p(\mathbb{R}^n)} < \varepsilon$, then $w = e^{nu}$ is a strong A_∞-weight with data depending only on n and p.*

A corollary to Theorem 5.1 is the following (see [11], [18] and (1.3)):

Corollary 5.3. *Let μ be a signed Radon measure on $\mathbb{R}^n$ of finite total mass $||\mu||_{\mathcal{M}} = |\mu|(\mathbb{R}^n)$ and let u be a distributional solution of the equation*

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = \mu$$

such that $\nabla u \in L^{n,\infty}(\mathbb{R}^n)$. There exists $\varepsilon(n) > 0$ such that if $||\mu||_{\mathcal{M}} < \varepsilon(n)$, then $w = e^{nu}$ is a strong A_∞-weight.

Corollary 5.3 was known (in a stronger form) for $n = 2$. (See [5].) Theorem 5.1 yields another consequence that will be proved later:

Theorem 5.4. *Let $s \in (n-1, n]$ and let u be in $L^1_{loc}(\mathbb{R}^n)$ such that its distributional gradient ∇u is in the Morrey space $\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)$. There exists $\varepsilon = \varepsilon(n,s) > 0$ such that if $||\nabla u||_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)} < \varepsilon$, then*

$$C^{-1}\delta_\mu(x_1,x_2) \le D_\mu(x_1,x_2) \le C\delta_\mu(x_1,x_2) \text{ for all } x_1, x_2 \text{ in } \mathbb{R}^n, \tag{5.1}$$

where $C = C(n,s) > 0$,

$$D_\mu(x,y) = \inf\left\{\int_\gamma e^{\widetilde{u}} ds : \gamma \text{ a rectifiable curve connecting } x, y\right\} \tag{5.2}$$

and $\widetilde{u}$ is an s-quasicontinuous Borel representative of u.

For the definition of an s-quasicontinuous function, see [20, p. 87]. For a discussion and definition of line integration, see [19, Chapter 7].

Theorem 5.2 has also a consequence that will be proved later:

Theorem 5.5. *Let $p \in (n,\infty)$ and let u be in $L^1_{loc}(\mathbb{R}^n)$ such that $[u]_{B_p(\mathbb{R}^n)} < \infty$. There exists $\varepsilon = \varepsilon(n,p) > 0$ such that if $[u]_{B_p(\mathbb{R}^n)} < \varepsilon$, then*

$$C^{-1}\delta_\mu(x_1,x_2) \le D_\mu(x_1,x_2) \le C\delta_\mu(x_1,x_2) \text{ for all } x_1, x_2 \text{ in } \mathbb{R}^n,$$

where $C = C(n,p) > 0$,

$$D_\mu(x,y) = \inf\left\{\int_\gamma e^{\widetilde{u}} ds : \gamma \text{ a rectifiable curve connecting } x, y\right\} \tag{5.3}$$

and $\widetilde{u}$ is a B_p-quasicontinuous Borel representative of u.

One should compare the metrics D_μ in Theorems 5.4 and 5.5 to those studied in [5] and [29].

Before we start the proof of Theorems 5.1 and 5.2, we mention the following auxiliary results:

Lemma 5.6. (See [4, Lemma 3.11].) *Let $x, y \in \mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ be a Borel set. Suppose that $B_1, ..., B_k$ are open balls in $\mathbb{R}^n$ such that $x \in B_1$, $y \in B_k$, and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$. Then there exists a constant $c_1 = c_1(n) > 0$ with the following property: if*

$$\Lambda_1^\infty(E) \le c_1|x-y|, \tag{5.4}$$

then

$$\sum_{i\in\mathcal{G}} \operatorname{diam} B_i > \frac{1}{10}|x-y|, \tag{5.5}$$

where

$$\mathcal{G} = \left\{ i = 1, \ldots, k : |E \cap B_i| \le \frac{1}{2}|B_i| \right\}. \tag{5.6}$$

Lemma 5.7. (See [24, Theorem 3.1].) *Suppose $s \in (n-1, n]$. There exists a constant $C = C(n,s) > 0$ such that*

$$\frac{\Lambda_1^\infty(E \cap B(a,3R))}{R} \le C\,\frac{\operatorname{cap}_s(E \cap B(a,3R), B(a,6R))}{(6R)^{n-s}} \tag{5.7}$$

for every $a \in \mathbb{R}^n$, every $R > 0$ and every Borel set $E \subset \mathbb{R}^n$.

Here cap_s denotes the variational s-capacity as in [20, p. 27].

We will prove now the following lemma.

Lemma 5.8. *Suppose $s \in (1, n]$. Let $B = B(x_0, R) \subset \mathbb{R}^n$. Let $\eta \in C_0^\infty(2B)$ such that $0 \le \eta \le 1$, $\eta = 1$ on B, and that $||\nabla\eta||_{L^\infty(2B)} < \frac{2}{R}$. Then*

$$v = \eta(u - u_B) \in H_0^{1,s}(2B)$$

whenever $u \in L^1_{loc}(\mathbb{R}^n)$ with $||\nabla u||_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)} < \infty$. Moreover, there exists a constant $C = C(n,s)$ such that

$$\int_{2B} |\nabla v|^s dx \le C(n,s) \int_{2B} |\nabla u|^s dx.$$

Proof. Let $u \in L^1_{loc}(\mathbb{R}^n)$ be such that $||\nabla u||_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)} < \infty$. Then from the Poincaré inequality it follows that

$$u - u_B \in H^{1,s}(2B).$$

This, together with the fact that $\eta \in C_0^\infty(2B)$ implies via [20, p. 21] that $v = \eta(u - u_B) \in H_0^{1,s}(2B)$.

Moreover,

$$\begin{aligned}\int_{2B} |\nabla v|^s dx &\leq \int_{2B} (|u-u_B||\nabla\eta| + \eta|\nabla u|)^s dx \\ &\leq 2^{s-1}\left(\int_{2B} (|u-u_B|^s|\nabla\eta|^s + |\nabla u|^s) dx\right) \\ &\leq 2^{s-1}\left(\left(\frac{2}{R}\right)^s \int_{2B} |u-u_B|^s dx + \int_{2B} |\nabla u|^s\right) \\ &\leq C(n,s)\int_{2B} |\nabla u|^s dx,\end{aligned}$$

where for the last inequality we used the Poincaré inequality. The claim follows. ■

We will prove Theorem 5.1 now.

Proof. Since $\nabla u \in \mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)$, we can assume without loss of generality due to [20, Theorem 4.4] that u is s-quasicontinuous and Borel. Since ∇u has small $\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)$-norm, it follows from (1.4) that u is in $\mathrm{BMO}(\mathbb{R}^n)$ with small BMO-seminorm. Therefore, from John-Nirenberg lemma, it follows that $w(x) = e^{nu(x)}$ is an A_∞-weight and doubling measure with data depending on n and s. That is, (see [15, Theorem IV.2.15]), there exists a constant $C = C(n,s)$ such that

$$\frac{1}{|B|}\int_B e^{n(u(x)-u_B)} dx < C \text{ and } \int_{2B} w(x)dx \leq C\int_B w(x)dx \tag{5.8}$$

for every ball $B \subset \mathbb{R}^n$. We write $d\mu(x) = w(x)dx$. We recall the definition of δ_μ from (1.1). We shall show that there exists a constant $C = C(n,s) \in (0,1]$ such that

$$d_\mu(x_1,x_2) := \inf \sum_{i=1}^k \mu(B_i)^{\frac{1}{n}} \geq C\mu(B_{x_1,x_2})^{\frac{1}{n}} = C\delta_\mu(x_1,x_2) \tag{5.9}$$

for all $x_1, x_2 \in \mathbb{R}^n$, where the infimum is taken over finite chains of open balls connecting x_1 and x_2 satisfying

$$x_1 \in B_1, x_2 \in B_k \text{ and } B_i \cap B_{i+1} \neq \emptyset \text{ for all } i = 1,\ldots,k-1. \tag{5.10}$$

Indeed, (5.9) implies both that d_μ is a distance and that is comparable to δ_μ as required in (1.2). Towards this end, fix $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, and let

$$a = \frac{x_1+x_2}{2},\ R = |x_1 - x_2|,\ B = B(a,R).$$

Now let $B_1, \dots, B_k$ be an arbitrary chain of balls connecting x_1 and x_2 as in (5.10). Let $\eta \in C_0^\infty(6B)$ be such that $\eta = 1$ on $3B$ and $||\nabla \eta||_{L^\infty(\mathbb{R}^n)} < \frac{2}{3R}$. Since u is s-quasicontinuous and Borel, it follows from Lemma 5.8 that $v(x) = \eta(x)\,|u(x) - u_{3B}|$ is an s-quasicontinuous Borel function in $H_0^{1,s}(6B)$ compactly supported in $6B$. Let $E = \{x \in 3B : |u(x) - u_{3B}| > 1\}$. We have that E is a Borel set since u is a Borel function. Since v is an s-quasicontinuous function in $H_0^{1,s}(6B)$ compactly supported in $6B$, we have from Lemma 5.8 that

$$\begin{aligned}(5.11)\quad \operatorname{cap}_s(E, 6B) &\le \int_{6B} |\nabla v(x)|^s dx \le C(n,s)\int_{6B} |\nabla u(x)|^s dx \\ &\le C(n,s)(6R)^{n-s}||\nabla u||^s_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)}.\end{aligned}$$

This implies that

$$(5.12)\qquad \frac{\operatorname{cap}_s(E,6B)}{(6R)^{n-s}} \le C(n,s)||\nabla u||^s_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)},$$

which together with (5.7) yields

$$(5.13)\qquad \frac{\Lambda_1^\infty(E)}{R} \le C(n,s)\,\frac{\operatorname{cap}_s(E,6B)}{(6R)^{n-s}} \le C(n,s)||\nabla u||^s_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)}.$$

We choose $\varepsilon = \varepsilon(n,s) > 0$ such that $C(n,s)\varepsilon^s < c_1$ where c_1 is the constant from (5.4). We assume first that $B_i \subset 3B$ for all $i = 1, \dots, k$. Let $\mathcal{G}$ be defined like in (5.6). We have

$$\begin{aligned}(5.14)\ \sum_{i=1}^k \mu(B_i)^{\frac{1}{n}} &\ge \sum_{i\in\mathcal{G}} \mu(B_i)^{\frac{1}{n}} \ge \sum_{i\in\mathcal{G}} \mu(B_i\setminus E)^{\frac{1}{n}} \\ &= \sum_{i\in\mathcal{G}}\left(\int_{B_i\setminus E} e^{nu(x)}dx\right)^{\frac{1}{n}} \ge \sum_{i\in\mathcal{G}}\left(\int_{B_i\setminus E} e^{n(u_{3B}-1)}dx\right)^{\frac{1}{n}} \\ &= e^{u_{3B}-1}\left(\sum_{i\in\mathcal{G}} |B_i\setminus E|^{\frac{1}{n}}\right) \ge e^{u_{3B}-1}\sum_{i\in\mathcal{G}}\left(\frac{1}{2}|B_i|\right)^{\frac{1}{n}} \\ &\ge C(n)e^{u_{3B}}|3B|^{\frac{1}{n}}.\end{aligned}$$

From (5.8) and (5.14) there exists $C = C(n,s)$ such that

$$\begin{aligned}(5.15)\quad \sum_{i=1}^k \mu(B_i)^{\frac{1}{n}} &\ge C\left(\int_{3B} e^{nu(x)}dx\right)^{\frac{1}{n}} \ge C\left(\int_{\frac{1}{2}B} e^{nu(x)}dx\right)^{\frac{1}{n}} \\ &= C\delta_\mu(x_1,x_2).\end{aligned}$$

Next, if the chain (B_i) does not lie entirely in $3B$, then there exists a smallest number k' with $1 \le k' \le k$ such that $B_{k'} \cap \partial 2B \ne \emptyset$. Let $x_0 \in B_{k'} \cap \partial 2B$. Then $B_1, \dots, B_{k'}$ is a chain of balls connecting x_1 and x_0. We note that the definition of k' and the fact that $x_1 \in B_1 \cap 2B$ implies

$$B_1 \cup \ldots \cup B_{k'-1} \subset 2B.$$

Let $u = \frac{x_1+x_0}{2}$. We let $B_{x_0,x_1} = B(u, |x_0 - x_1|)$.

It is easy to see that $3B \subset 3B_{x_0,x_1}$.

If $B_{k'} \subset 3B$, then the subchain $B_1, \dots, B_{k'}$ is contained in $3B \subset 3B_{x_0,x_1}$ and we can apply the preceding argument with x_0 in place of x_2 to conclude that (5.15) holds; in the opposite case, diam $B_{k'} \ge R$. The doubling condition for μ then implies $\mu(B) \le C\mu(B_{k'})$. Thus, (5.15) is true in all cases. This finishes the proof. ■

We prove now Theorem 5.4.

Proof. We see that D_μ is independent of the choice of the s-quasicontinuous Borel representative of u. Indeed, if $\widetilde{u}$ and $\widetilde{v}$ are two s-quasicontinuous Borel representatives of u, then from an argument similar to [20, Theorem 4.14] we have $\widetilde{u} = \widetilde{v}$ s-quasieverywhere, which implies via [20, Theorem 2.27], since $s \in (n-1, n]$, that

$$\int_\gamma e^{\widetilde{u}} ds = \int_\gamma e^{\widetilde{v}} ds$$

for every rectifiable curve γ in $\mathbb{R}^n$.

It is easy to see that D_μ is indeed symmetric, nonnegative and satisfies the triangle inequality. From (5.1) it would follow immediately that D_μ is a distance comparable to δ_μ. So fix x_1, x_2 in $\mathbb{R}^n$. We can assume without loss of generality that $x_1 \ne x_2$. Like before, let $a = \frac{x_1+x_2}{2}$, $R = |x_1 - x_2|$, $B = B(a, R)$. Like in the proof of Theorem 5.1, let $v = \eta|u - u_{3B}|$ and $E = \{x \in 3B : |u(x) - u_{3B}| > 1\}$. Like before, we have that E is a Borel set and v is an s-quasicontinuous function in $H_0^{1,s}(6B)$ compactly supported in $6B$.

Let γ be a rectifiable curve connecting x_1 and x_2 and let $|\gamma|$ be its image. We assume first that $|\gamma| \subset 3B$. We obviously have

$$\int_\gamma e^u ds \ge \int_{\gamma \cap (3B \setminus E)} e^u ds. \tag{5.16}$$

As in the proof of Theorem 5.1, we have

$$\frac{\Lambda_1^\infty(E)}{R} \le C(n,s) \frac{\operatorname{cap}_s(E, 6B)}{(6R)^{n-s}} \le C(n,s) ||\nabla u||^s_{\mathcal{L}^{s,n-s}(\mathbb{R}^n;\mathbb{R}^n)},$$

hence

$$\begin{aligned}\Lambda_1^\infty(|\gamma| \cap (3B \setminus E)) &\geq \Lambda_1^\infty(|\gamma| \cap 3B) - \Lambda_1^\infty(|\gamma| \cap E)\\ &\geq R - \Lambda_1^\infty(E) \geq (1-c_1)R\end{aligned}$$

if $\varepsilon = \varepsilon(n,s)$ is small enough, where c_1 is the constant from Lemma 5.6.

We get

$$\begin{aligned}\int_\gamma e^u ds &\geq \int_{\gamma\cap(3B\setminus E)} e^u ds \geq \int_{\gamma\cap(3B\setminus E)} e^{u_{3B}-1} ds\\ &\geq \Lambda_1^\infty(|\gamma| \cap (3B \setminus E))\, e^{u_{3B}-1} > C(n)\, R e^{u_{3B}-1}\\ &= C(n) \left(\int_{3B} e^{n(u_{3B}-1)} dz\right)^{\frac{1}{n}} \geq C(n,s) \left(\int_{3B} e^{nu} dz\right)^{\frac{1}{n}},\end{aligned} \tag{5.17}$$

where the last inequality follows from (5.8). Hence we get

$$\begin{aligned}\int_\gamma e^u ds &\geq C(n,s)\Big(\int_{3B} e^{nu(x)} dx\Big)^{\frac{1}{n}}\\ &\geq C(n,s)\Big(\int_{\frac{1}{2}B} e^{nu(x)} dx\Big)^{\frac{1}{n}}\\ &= C(n,s)\delta_\mu(x_1,x_2).\end{aligned} \tag{5.18}$$

Now we assume that $|\gamma| \setminus 3B \neq \emptyset$. We assume that γ is parametrized by its arc length parametrization. Let $t_0 = \inf\{t \in [0,\ell(\gamma)] : \gamma(t) \notin 2B.\}$. Then, since γ is a path with $\gamma(0)$, $\gamma(\ell(\gamma)) \in 2B$, it follows that $0 < t_0 < \ell(\gamma)$ and moreover, $\gamma([0,t_0]) \subset 2\overline{B}$. Let $x_0 = \gamma(t_0)$ and let $\tilde{\gamma}$ be the restriction of γ to $[0,t_0]$. Let $u = \frac{x_1+x_0}{2}$. We let $B_{x_0,x_1} = B(u,|x_0-x_1|)$. It is easy to see that $3B \subset 3B_{x_0,x_1}$, where we recall that $B = B(a,|x_1-x_2|)$ with $a = \frac{x_1+x_2}{2}$. We can apply (5.18) with x_0 in place of x_2 and $\tilde{\gamma}$ in place of γ to conclude that

$$\int_\gamma e^u ds \geq \int_{\tilde{\gamma}} e^u ds \geq C\Big(\int_{3B_{x_0,x_1}} e^{nu(x)} dx\Big)^{\frac{1}{n}} \geq C \left(\int_{3B} e^{nu(z)} dz\right)^{\frac{1}{n}}.$$

So we proved that there exists a constant $C = C(n,s)$ such that

$$\int_\gamma e^u ds \geq C \left(\int_{3B} e^{nu(x)} dx\right)^{\frac{1}{n}} \geq C\Big(\int_{\frac{1}{2}B} e^{nu(x)} dx\Big)^{\frac{1}{n}} = C\delta_\mu(x_1,x_2)$$

for every $x_1, x_2 \in \mathbb{R}^n$ and every rectifiable curve γ connecting x_1 and x_2.

To prove the converse inequality, we need to find a path from x_1 to x_2 whose length we can control. Let H denote the hyperplane through a that is orthogonal to the line segment that joins x_1 and x_2. For each $v \in B(a,R) \cap H$ let $\gamma_v : [0,R] \to \mathbb{R}^n$ be the path such that $\gamma_v(0) = x_1$, $\gamma_v(R) = x_2$, $\gamma_v(R/2) = v$ and $\dot{\gamma_v}$ is constant on $(0, R/2)$ and on $(R/2, R)$.

Obviously we have

$$D_\mu(x_1,x_2) \leq \frac{1}{m_{n-1}(B(a,R)\cap H)} \int_{B(a,R)\cap H} \int_{\gamma_v} e^u ds\, dv. \tag{5.19}$$

We can bound this last expression by

$$C(n)\left(I(x_1,R) + I(x_2,R)\right), \tag{5.20}$$

where

$$I(x_i,R) = \int_{B(x_i,2R)} e^{u(z)}|x_i - z|^{1-n} dz$$

for $i = 1,2$. (The iterated integral on the right side of (5.19) can be split into two pieces corresponding to s in $[0,R/2]$ and $[R/2,R]$, each piece being bounded by an integral in polar coordinates centered at x_1 or x_2. See [30, Proposition 3.12].) Since $w = e^{nu}$ is an A_∞-weight, there exists a constant $C_1 = C_1(n,s) > 0$ and $q = q(n,s) > 1$ such that

$$\left(\frac{1}{|B|}\int_B e^{nqu(z)} dz\right)^{\frac{1}{q}} \leq C_1 \frac{1}{|B|}\int_B e^{nu(z)} dz. \tag{5.21}$$

(See [30, Proposition 3.5] and [16, Lemma 2].) Let r be the conjugate exponent to nq. Using Hölder inequality and (5.8), we get

$$\begin{aligned} I(x_1,R) &\leq \left(\int_{B(x_1,2R)} e^{nqu(z)} dz\right)^{\frac{1}{nq}} \left(\int_{B(x_1,2R)} |x_1 - z|^{r-rn} dz\right)^{\frac{1}{r}} \\ &\leq C(n,p) C_1^{\frac{1}{q}} |B(x_1,2R)|^{\frac{1}{nq}-\frac{1}{n}} \left(\int_{B(x_1,2R)} e^{nu(z)} dz\right)^{\frac{1}{n}} R^{1-\frac{1}{q}} \\ &\leq C(n,s)\left(\int_{B(a,3R)} e^{nu(z)} dz\right)^{\frac{1}{n}} \\ &\leq C(n,s)\left(\int_{B(a,\frac{1}{2}R)} e^{nu(z)} dz\right)^{\frac{1}{n}} = C(n,s)\delta_\mu(x_1,x_2). \end{aligned} \tag{5.22}$$

Similarly we get

$$I(x_2,R) \leq C(n,s)\left(\int_{B(a,\frac{1}{2}R)} e^{nu(z)} dz\right)^{\frac{1}{n}} = C(n,s)\delta_\mu(x_1,x_2).$$

■

Now we prove Theorem 5.2.

Proof. Since $u \in L^1_{loc}(\mathbb{R}^n) \cap B_p(\mathbb{R}^n)$, with small B_p-seminorm, it follows that $u \in \mathrm{BMO}(\mathbb{R}^n)$ with small BMO-seminorm. Therefore, by John-Nirenberg lemma, there exists a constant $C = C(n,p)$ such that $w = e^{nu}$ is an A_∞-weight and a doubling measure satisfying (5.8) with C. We write $d\mu(x) = w(x)dx$. We recall the definition of δ_μ from (1.1). We shall show that there exists a constant $C = C(n,p) \in (0,1]$ such that

$$d_\mu(x_1, x_2) := \inf \sum_{i=1}^{k} \mu(B_i)^{\frac{1}{n}} \geq C\mu(B_{x_1,x_2})^{\frac{1}{n}} = C\delta_\mu(x_1, x_2)$$

for all $x_1, x_2 \in \mathbb{R}^n$, where the infimum is taken over finite chains of open balls connecting x_1 and x_2 satisfying (5.10). Indeed, (5.9) implies both that d_μ is a distance and that is comparable to δ_μ as required in (1.2). Towards this end, fix $x_1, x_2 \in \mathbb{R}^n$. We can assume without loss of generality that $x_1 \neq x_2$. Let

$$a = \frac{x_1 + x_2}{2},\ R = |x_1 - x_2|, B = B(a, R).$$

Now let $B_1, \ldots, B_k$ be an arbitrary chain of balls connecting x_1 and x_2 as in (5.10). Let $\eta \in C_0^\infty(6B)$ be such that $\eta = 1$ on $3B$ and $||\nabla\eta||_{L^\infty(\mathbb{R}^n)} < \frac{2}{3R}$. Since $u \in B_p(\mathbb{R}^n)$ is a B_p-quasicontinuous Borel function, it follows that the function v defined by $v = \eta|u - u_{3B}|$ is a B_p-quasicontinuous Borel function supported in $6B$. Hence $v \in B_p^0(6B)$.

Let $E = \{x \in 3B : |u(x) - u_{3B}| > 1\}$. We have that E is a Borel set since u is a Borel function. From Lemma 2.9, (3.9) and Corollary 4.14, we get

$$\begin{aligned} \frac{\Lambda_1^\infty(E)}{R} &\leq C(n,p)\,\mathrm{cap}_{B_p}(E, 6B) \leq C(n,p)[v]^p_{B_p(6B)} \\ &\leq C(n,p)\,[u]^p_{B_p(\mathbb{R}^n)}, \end{aligned} \tag{5.23}$$

where R is the radius of B. We choose $\varepsilon = \varepsilon(n,p) > 0$ such that $C(n,p)\,\varepsilon^p < c_1$ where c_1 is the constant from (5.4) and $C(n,p)$ is the constant from the last inequality in (5.23). We assume first that $B_i \subset 3B$ for all $i = 1, \ldots, k$. Let $\mathcal{G}$ be defined like in (5.6). The proof now continues like in Theorem 5.1, with the only difference that the constants who depended on n and s will now depend on n and p. ■

Now we prove Theorem 5.5.

Proof. We see that D_μ is independent of the choice of the B_p-quasicontinuous Borel representative of u. Indeed, if $\widetilde{u}$ and $\widetilde{v}$ are two B_p-quasicontinuous Borel representatives of u, then from Corollary 4.19 we have $\widetilde{u} = \widetilde{v}$

B_p-quasieverywhere, which implies via Corollary 3.14 that

$$\int_\gamma e^{\tilde{u}} ds = \int_\gamma e^{\tilde{v}} ds$$

for every rectifiable curve γ in $\mathbb{R}^n$.

Like in the proof of Theorem 5.2, we have

$$\frac{\Lambda_1^\infty(E)}{R} \le C(n,p) \operatorname{cap}_{B_p}(E, 6B) \le C(n,p)[u]^p_{B_p(\mathbb{R}^n)}.$$

The proof now continues like in Theorem 5.4, with the only difference that the constants who depended on n and s will now depend on n and p. ■

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