SCALING INVARIANT SOBOLEV-LORENTZ CAPACITY ON $\mathbf{R}^n$

ŞERBAN COSTEA

ABSTRACT. We develop a capacity theory based on the definition of Sobolev functions on $\mathbf{R}^n$ with respect to the Lorentz norm. Basic properties of capacity, including monotonicity, finite subadditivity and convergence results are included. We also provide sharp estimates for the capacity of balls. Sobolev-Lorentz capacity and Hausdorff measures are related.

1. INTRODUCTION

We recall that for $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the *Morrey space* $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ is defined to be the linear space of measurable functions $u \in L^1_{loc}(\mathbf{R}^n)$ such that

$$||u||_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} = \sup_{x\in\mathbf{R}^n} \sup_{r>0} \left(r^{-\lambda} \int_{B(x,r)} |u(y)|^p dy \right)^{1/p} < \infty.$$

In other words, the fractional maximal function

$$M_{n-\lambda}u(x) = \sup_{r>0} \left(r^{n-\lambda} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|^p dy \right)^{1/p}$$

is bounded in $\mathbf{R}^n$. In particular, $\mathcal{L}^{n,0}(\mathbf{R}^n) = L^n(\mathbf{R}^n)$. We refer to [Gia83, p. 65] for more information about Morrey spaces and their use in the theory of partial differential equations. One notices that the weak Lebesgue space $L^{n,\infty}(\mathbf{R}^n)$ is contained in $\mathcal{L}^{p,n-p}(\mathbf{R}^n)$ for every $p \in [1, n)$. Similarly we can define the Morrey space $\mathcal{L}^{p,\lambda}(\mathbf{R}^n;\mathbf{R}^m)$ for vector-valued measurable functions. Capacities related to Morrey spaces were studied by Adams and Xiao in [AX04].

We have already noticed that the Lorentz spaces embed continuously into the Morrey spaces; that is to say, $L^{n,q}(\mathbf{R}^n) \hookrightarrow L^{n,\infty}(\mathbf{R}^n) \hookrightarrow \mathcal{L}^{p,n-p}(\mathbf{R}^n)$ whenever $1 \leq p < n < q \leq \infty$. Lorentz spaces have been studied extensively by Bennett and Sharpley in [BS88]. Sobolev-Lorentz spaces have recently been studied by Kauhanen, Koskela, and Malý in [KKM99] and by Malý, Swanson, and Ziemer in [MSZ05].

Our results concerning the Sobolev-Lorentz capacity generalize some of the results concerning s-capacity on $\mathbf{R}^n$ for $s \in (1, n]$. See [HKM93, Chapter 2] for the s-capacity on $\mathbf{R}^n$ and [KM96], [KM00] for capacity on general metric spaces.

We provide sharp estimates for the Sobolev-Lorentz n, q relative capacity of pairs $(\overline{B}(0, r), B(0, 1))$ for $1 \leq q \leq \infty$ and small r. The Sobolev-Lorentz capacity and Hausdorff measures are also related; we obtain results that are Sobolev-Lorentz

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analogues of those obtained by Reshetnyak in [Res69], Martio in [Mar79], Maz'ja in [Maz85] and others.

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2. Preliminaries

Our notation in this paper is standard and generally as in [HKM93]. Here Ω will denote a nonempty open subset of $\mathbf{R}^n$, while $dx = dm_n(x)$ will denote the Lebesgue n-measure in $\mathbf{R}^n$, where $n \geq 2$ is integer. For two sets $A, B \subset \mathbf{R}^n$, we define $\mathrm{dist}(A, B)$, the distance between A and B, by

$$\mathrm{dist}(A, B) = \inf_{a \in A, b \in B} |a - b|.$$

For $n \geq 2$ integer $\Omega_n = |B(0,1)|$ denotes the measure of the n-dimensional unit ball, that is $\Omega_n = |B(0,1)|$. Thus, $\omega_{n-1} = n\Omega_n$, where ω_{n-1} denotes the spherical measure of the $n-1$-dimensional sphere.

For a measurable $u : \Omega \to \mathbf{R}$, supp u is the smallest closed set such that u vanishes outside supp u. We also define

$$\begin{aligned} C_0(\Omega) &= \{\varphi \in C(\Omega) : \mathrm{supp}\ \varphi \subset\subset \Omega\} \\ Lip(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\}. \end{aligned}$$

For a function $\varphi \in Lip(\Omega) \cap C_0(\Omega)$ we write

$$\nabla \varphi = (\partial_1 \varphi, \partial_2 \varphi, \dots, \partial_n \varphi)$$

for the gradient of φ. This notation makes sense, since from Rademacher's theorem ([Fed69, Theorem 3.1.6]) every Lipschitz function on $\mathbf{R}^n$ is a.e. differentiable.

Throughout this section we will assume that $m \geq 1$ is a positive integer. Let $f : \Omega \to \mathbf{R}^m$ be a measurable function. We define $\lambda_{[f]}$, the *distribution function* of f as follows (see [BS88, Definition II.1.1] and [SW75, p. 57]):

$$\lambda_{[f]}(t) = |\{x \in \Omega : |f(x)| > t\}|, \qquad t \geq 0.$$

We define f^*, the *nonincreasing rearrangement* of f by

$$f^*(t) = \inf\{v : \lambda_{[f]}(v) \leq t\}, \quad t \geq 0.$$

(See [BS88, Definition II.1.5] and [SW75, p. 189].) We notice that f and f^* have the same distribution function. Moreover, for every positive α we have $(|f|^\alpha)^* = (|f|^*)^\alpha$ and if $|g| \leq |f|$ a.e. on Ω, then $g^* \leq f^*$. (See [BS88, Proposition II.1.7].) We also define f^{**}, the *maximal function* of f^* by

$$f^{**}(t) = m_{f^*}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0.$$

(See [BS88, Definition II.3.1] and [SW75, p. 203].)

Throughout this paper, we will denote by p' the Hölder conjugate of $p \in [1, \infty]$, that is

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

The *Lorentz space* $L^{p,q}(\Omega; \mathbf{R}^m)$, $1 < p < \infty$, $1 \leq q \leq \infty$, is defined as follows:

$$L^{p,q}(\Omega; \mathbf{R}^m) = \{f : \Omega \to \mathbf{R}^m : f \text{ is measurable and } ||f||_{L^{p,q}(\Omega; \mathbf{R}^m)} < \infty\},$$

where

$$||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} = ||\,|f|\,||_{p,q} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \,\frac{dt}{t}\right)^{\frac{1}{q}} & 1 \le q < \infty \\ \sup_{t>0} t\lambda_{[f]}(t)^{\frac{1}{p}} = \sup_{s>0} s^{\frac{1}{p}} f^*(s) & q = \infty. \end{cases}$$

(See [BS88, Definition IV.4.1] and [SW75, p. 191].) If $1 \le q \le p$, then $||\cdot||_{L^{p,q}(\Omega;\mathbf{R}^m)}$ already represents a norm, but for $p < q \le \infty$ it represents a quasinorm, equivalent to the norm $||\cdot||_{L^{(p,q)}(\Omega;\mathbf{R}^m)}$, where

$$||f||_{L^{(p,q)}(\Omega;\mathbf{R}^m)} = ||\,|f|\,||_{(p,q)} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \,\frac{dt}{t}\right)^{\frac{1}{q}} & 1 \le q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & q = \infty. \end{cases}$$

(See [BS88, Definition IV.4.4].) Namely, from [BS88, Lemma IV.4.5] we have that

$$||\,|f|\,||_{L^{p,q}(\Omega)} \le ||\,|f|\,||_{L^{(p,q)}(\Omega)} \le \frac{p}{p-1}||\,|f|\,||_{L^{p,q}(\Omega)}$$

for every $1 \le q \le \infty$ and every measurable function $f : \Omega \to \mathbf{R}^m$.

It is known that $(L^{p,q}(\Omega;\mathbf{R}^m), ||\cdot||_{L^{p,q}(\Omega;\mathbf{R}^m)})$ is a Banach space for $1 \le q \le p$, while $(L^{p,q}(\Omega;\mathbf{R}^m), ||\cdot||_{L^{(p,q)}(\Omega;\mathbf{R}^m)})$ is a Banach space for $1 < p < \infty$, $1 \le q \le \infty$. These spaces are reflexive if $1 < q < \infty$. (See [BS88, Theorem IV.4.7, Corollaries I.4.3 and IV.4.8], the definition of $L^{p,q}(\Omega;\mathbf{R}^m)$ and the discussion after Definition 2.1.)

Definition 2.1. (See [BS88, Definition I.3.1].) Let $1 < p < \infty$ and $1 \le q \le \infty$. Let $X = L^{p,q}(\Omega;\mathbf{R}^m)$. A function f in X is said to have *absolutely continuous norm* in X if and only if $||f\chi_{E_k}||_X \to 0$ for every sequence E_k satisfying $E_k \to \emptyset$ a.e.

Let X_a be the subspace of X consisting of functions of absolutely continuous norm and let X_b be the closure in X of the set of simple functions. It is known that $X_a = X_b$. (See [BS88, Theorem I.3.13].) Moreover, we have $X_a = X_b = X$ whenever $1 \le q < \infty$. (See [BS88, Theorem IV.4.7 and Corollary IV.4.8] and the definition of $L^{p,q}(\Omega;\mathbf{R}^m)$.)

We prove now that $X_a \ne X$ for $X = L^{p,\infty}(\Omega;\mathbf{R}^m)$. Without loss of generality we can assume that $m = 1$ and that $\Omega = B(0,2) \setminus \{0\}$. We define $u : \Omega \to \mathbf{R}$,

$$u(x) = \begin{cases} |x|^{-\frac{n}{p}} & \text{if } 0 < |x| < 1 \\ 0 & \text{if } 1 \le |x| \le 2. \end{cases} \tag{1}$$

It is easy to see that $u \in L^{p,\infty}(\Omega)$ and moreover,

$$||u\chi_{B(0,\alpha)}||_{L^{p,\infty}(\Omega)} = ||u||_{L^{p,\infty}(\Omega)} = \Omega_n^{1/p}$$

for every $\alpha > 0$. This shows that u does not have absolutely continuous weak L^p-norm and therefore $L^{p,\infty}(\Omega)$ does not have absolutely continuous norm. Since $L^{p,\infty}(\Omega)$ can be identified with $(L^{p',1}(\Omega))^*$ (see [BS88, Corollary IV.4.8]), it follows from [BS88, Corollaries I.4.3, I.4.4, IV.4.8 and Theorem IV.4.7] that neither $L^{p,1}(\Omega)$, nor $L^{p,\infty}(\Omega)$ are reflexive whenever $1 < p < \infty$.

Remark 2.2. It is also known (see [BS88, Proposition IV.4.2]) that for every $p \in (1,\infty)$ and $1 \le r < s \le \infty$ there exists a constant $C(p,r,s)$ such that

$$||\,|f|\,||_{L^{p,s}(\Omega)} \le C(p,r,s)||\,|f|\,||_{L^{p,r}(\Omega)} \tag{2}$$

for all measurable functions $f \in L^{p,r}(\Omega;\mathbf{R}^m)$ and all integers $m \ge 1$. In particular, we have the embedding $L^{p,r}(\Omega;\mathbf{R}^m) \hookrightarrow L^{p,s}(\Omega;\mathbf{R}^m)$.

We have the following generalized Hölder inequality for Lorentz spaces.

Theorem 2.3. *Let* $\Omega \subset \mathbf{R}^n$. *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *If* $f \in L^{p,q}(\Omega)$ *and* $g \in L^{p',q'}(\Omega)$, *then*

$$\int_\Omega |f(x)g(x)|\, dx \leq ||f||_{L^{p,q}(\Omega)}\, ||g||_{L^{p',q'}(\Omega)}.$$

Proof. We have to analyze two situations, depending on whether $q \in (1,\infty)$ or not.

Suppose first that $1 < q < \infty$. Then $1 < q' < \infty$ and by Hölder's inequality, we have

$$\int_0^\infty f^*(s)g^*(s)ds = \int_0^\infty f^*(s)s^{\frac{1}{p}-\frac{1}{q}}g^*(s)s^{\frac{1}{p'}-\frac{1}{q'}} \leq ||f||_{L^{p,q}(\Omega)}\, ||g||_{L^{p',q'}(\Omega)}.$$

By using this and [BS88, Theorem II.2.2], we get the desired conclusion for $1 < q < \infty$.

We assume now without loss of generality that $q = 1$. The case $q = \infty$ is similar. If $q = 1$ then $q' = \infty$ and we have

$$\begin{aligned}\int_0^\infty f^*(s)g^*(s)ds &= \int_0^\infty f^*(s)s^{\frac{1}{p}-1}g^*(s)s^{\frac{1}{p'}}ds \leq \sup_{s>0} g^*(s)s^{\frac{1}{p'}} \int_0^\infty f^*(s)s^{\frac{1}{p}-1}ds \\ &= ||g||_{L^{p',\infty}(\Omega)}\, ||f||_{L^{p,1}(\Omega)}.\end{aligned}$$

By using this and [BS88, Theorem II.2.2], we get the desired conclusion for $q = \infty$ as well. This finishes the proof.

□

As an application of Theorem 2.3 we have the following result.

Corollary 2.4. *Let* $1 < p < q \leq \infty$ *and* $\varepsilon \in (0, p-1)$ *be fixed. Suppose* $\Omega \subset \mathbf{R}^n$ *has finite measure. Then*

$$||f||_{L^{p-\varepsilon}(\Omega;\mathbf{R}^m)} \leq C(p,q,\varepsilon)\, |\Omega|^{\frac{\varepsilon}{p(p-\varepsilon)}}\, ||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} \tag{3}$$

for every integer $m \geq 1$, *where*

$$C(p,q,\varepsilon) = \begin{cases} \left(\frac{p(q-p+\varepsilon)}{q}\right)^{\frac{1}{p-\varepsilon}-\frac{1}{q}} \varepsilon^{\frac{1}{q}-\frac{1}{p-\varepsilon}}, & p < q < \infty \\ p^{\frac{1}{p-\varepsilon}}\, \varepsilon^{-\frac{1}{p-\varepsilon}}, & q = \infty. \end{cases}$$

Proof. From the definition of the Lorentz norms and quasinorms for vector-valued functions, it follows that it is enough to assume that $m = 1$ and that $f \geq 0$. We have to consider two cases, depending on whether $q < \infty$ or $q = \infty$.

Suppose first that $q < \infty$. Then from [BS88, Proposition II.1.7, Definition IV.4.1] and Theorem 2.3 we have

$$||f^{p-\varepsilon}||_{L^1(\Omega)} \leq ||f^{p-\varepsilon}||_{L^{\frac{p}{p-\varepsilon},\frac{q}{p-\varepsilon}}(\Omega)}\, ||\chi_\Omega||_{L^{\frac{p}{\varepsilon},\frac{q}{q-p+\varepsilon}}(\Omega)}. \tag{4}$$

By taking the $p-\varepsilon$th root, we get the desired conclusion for $q < \infty$.

Assume now that $q = \infty$. Then from [BS88, Proposition II.1.7, Definition IV.4.1] and Theorem 2.3 we have

$$||f^{p-\varepsilon}||_{L^1(\Omega)} \leq ||f^{p-\varepsilon}||_{L^{\frac{p}{p-\varepsilon},\infty}(\Omega)}\, ||\chi_\Omega||_{L^{\frac{p}{\varepsilon},1}(\Omega)}. \tag{5}$$

By taking the $p-\varepsilon$th root, we get the desired conclusion for $q = \infty$. This finishes the proof.

□

We have a few interesting results concerning Lorentz spaces.

Theorem 2.5. *Suppose* $1 < p < q \leq \infty$. *Let* $\Omega \subset \mathbf{R}^n$ *and let* $f_1, f_2 \in L^{p,q}(\Omega)$. *We let* $f_3 = \max(|f_1|, |f_2|)$. *Then* $f_3 \in L^{p,q}(\Omega)$ *and*

$$||f_3||^p_{L^{p,q}(\Omega)} \leq ||f_1||^p_{L^{p,q}(\Omega)} + ||f_2||^p_{L^{p,q}(\Omega)}.$$

Proof. Without loss of generality we can assume that both f_1 and f_2 are nonnegative. We have to consider two cases, depending on whether $p < q < \infty$ or $q = \infty$.

Suppose $p < q < \infty$. We have ([KKM99, Proposition 2.1])

$$||f_i||^p_{L^{p,q}(\Omega)} = \left(p \int_0^\infty s^{q-1} \lambda_{[f_i]}(s)^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}},$$

where $\lambda_{[f_i]}$ is the distribution function of f_i for $i = 1, 2, 3$. From the definition of f_3 we obviously have $\lambda_{[f_3]}(s) \leq \lambda_{[f_1]}(s) + \lambda_{[f_2]}(s)$ for every $s \geq 0$, which implies that

$$\begin{aligned} ||f_3||^p_{L^{p,q}(\Omega)} &\leq \left(p \int_0^\infty s^{q-1} (\lambda_{[f_1]}(s) + \lambda_{[f_2]}(s))^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}} \\ &\leq \left(p \int_0^\infty s^{q-1} \lambda_{[f_1]}(s)^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}} + \left(p \int_0^\infty s^{q-1} \lambda_{[f_2]}(s)^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}} \\ &= ||f_1||^p_{L^{p,q}(\Omega)} + ||f_2||^p_{L^{p,q}(\Omega)}. \end{aligned}$$

Suppose now $q = \infty$. From the definition of f_3 we obviously have as before $\lambda_{[f_3]}(s) \leq \lambda_{[f_1]}(s) + \lambda_{[f_2]}(s)$ for every $s \geq 0$. Therefore

$$s^p\, \lambda_{[f_3]}(s) \leq s^p\, \lambda_{[f_1]}(s) + s^p\, \lambda_{[f_3]}(s)$$

for every $s \geq 0$, which implies

$$s^p\, \lambda_{[f_3]}(s) \leq ||f_1||^p_{L^{p,\infty}(\Omega)} + ||f_2||^p_{L^{p,\infty}(\Omega)} \tag{6}$$

for every $s \geq 0$. By taking the supremum over all $s \geq 0$ in (6), we get the desired conclusion. □

Theorem 2.6. *Suppose* $1 < p < q \leq \infty$ *and* $\varepsilon \in (0, 1)$. *Let* $\Omega \subset \mathbf{R}^n$ *and let* $f_1, f_2 \in L^{p,q}(\Omega)$. *We denote* $f_3 = f_1 + f_2$. *Then* $f_3 \in L^{p,q}(\Omega)$ *and*

$$||f_3||^p_{L^{p,q}(\Omega)} \leq (1-\varepsilon)^{-p} ||f_1||^p_{L^{p,q}(\Omega)} + \varepsilon^{-p} ||f_2||^p_{L^{p,q}(\Omega)}.$$

Proof. Without loss of generality we can assume that both f_1 and f_2 are nonnegative. We have to consider two cases, depending on whether $p < q < \infty$ or $q = \infty$.

Suppose $p < q < \infty$. We have ([KKM99, Proposition 2.1])

$$||f_i||^p_{L^{p,q}(\Omega)} = \left(p \int_0^\infty s^{q-1} \lambda_{[f_i]}(s)^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}},$$

where $\lambda_{[f_i]}$ is the distribution function of f_i for $i = 1, 2, 3$. From the definition of f_3 we obviously have $\lambda_{[f_3]}(s) \leq \lambda_{[f_1]}((1-\varepsilon)s) + \lambda_{[f_2]}(\varepsilon s)$ for every $s \geq 0$, which implies that

$$\begin{aligned} ||f_3||^p_{L^{p,q}(\Omega)} &\leq \left(p \int_0^\infty s^{q-1} (\lambda_{[f_1]}((1-\varepsilon)s) + \lambda_{[f_2]}(\varepsilon s))^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}} \\ &\leq \left(p \int_0^\infty s^{q-1} \lambda_{[f_1]}((1-\varepsilon)s)^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}} + \left(p \int_0^\infty s^{q-1} \lambda_{[f_2]}(\varepsilon s)^{\frac{q}{p}}\, ds\right)^{\frac{p}{q}} \\ &= (1-\varepsilon)^{-p} ||f_1||^p_{L^{p,q}(\Omega)} + \varepsilon^{-p} ||f_2||^p_{L^{p,q}(\Omega)}. \end{aligned}$$

Suppose now $q = \infty$. From the definition of f_3 we obviously have as before $\lambda_{[f_3]}(s) \le \lambda_{[f_1]}((1-\varepsilon)s) + \lambda_{[f_2]}(\varepsilon s)$ for every $s \ge 0$. Therefore

$$s^p\, \lambda_{[f_3]}(s) \le s^p\, \lambda_{[f_1]}((1-\varepsilon)s) + s^p\, \lambda_{[f_3]}(\varepsilon s)$$

for every $s \ge 0$, which implies

$$s^p\, \lambda_{[f_3]}(s) \le (1-\varepsilon)^{-p} ||f_1||^p_{L^{p,\infty}(\Omega)} + \varepsilon^{-p} ||f_2||^p_{L^{p,\infty}(\Omega)} \tag{7}$$

for every $s \ge 0$. By taking the supremum over all $s \ge 0$ in (7), we get the desired conclusion. □

Theorem 2.6 has an interesting corollary.

Corollary 2.7. *Let $\Omega \subset \mathbf{R}^n$ be open. Suppose $1 < p < \infty$ and $1 \le q \le \infty$. Let f_k be a sequence of functions in $L^{p,q}(\Omega; \mathbf{R}^m)$ converging to f with respect to the p,q-quasinorm and pointwise a.e. in Ω. Then*

$$\lim_{k\to\infty} ||f_k||_{L^{p,q}(\Omega;\mathbf{R}^m)} = ||f||_{L^{p,q}(\Omega;\mathbf{R}^m)}.$$

Proof. We can assume without loss of generality that $m = 1$. Since $||\cdot||_{L^{p,q}(\Omega)}$ is already a norm for $1 \le q \le p$, the claim is trivial in this case. Hence we can assume without loss of generality that $p < q \le \infty$. The proof for the case $q = \infty$ was presented to me by Jan Malý.

Since $f^* \le \liminf f_k^*$ (see [BS88, Proposition II.1.7]), it follows easily that

$$\liminf_{k\to\infty} ||f_k||_{L^{p,q}(\Omega)} \ge ||f||_{L^{p,q}(\Omega)}.$$

We would be done if we show that

$$\limsup_{k\to\infty} ||f_k||_{L^{p,q}(\Omega)} \le ||f||_{L^{p,q}(\Omega)}. \tag{8}$$

In order to do that we fix $\varepsilon \in (0,1)$. From Theorem 2.6 we have

$$||f_k||^p_{L^{p,q}(\Omega)} \le (1-\varepsilon)^{-p} ||f||^p_{L^{p,q}(\Omega)} + \varepsilon^{-p} ||f_k - f||^p_{L^{p,q}(\Omega)}$$

for every $k = 1, 2, \ldots$ Taking lim sup on both sides and using the fact that f_k converges to f with respect to the $L^{p,q}$-quasinorm, we get

$$\limsup_{k\to\infty} ||f_k||^p_{L^{p,q}(\Omega)} \le (1-\varepsilon)^{-p} ||f||^p_{L^{p,q}(\Omega)}. \tag{9}$$

Letting $\varepsilon \to 0$ in (9) yields (8). This finishes the proof.

□

We use the notation

$$u^+ = \max(u, 0) \text{ and } u^- = \min(u, 0).$$

If $u \in C_0(\Omega) \cap Lip(\Omega)$, then obviously $u^+ \in C_0(\Omega) \cap Lip(\Omega)$ and from [HKM93, Lemmas 1.11 and 1.19] we have

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0 \\ 0 & \text{if } u \le 0. \end{cases} \tag{10}$$

Theorem 2.8. *Suppose $1 \le q < p < \infty$. Let $\Omega \subset \mathbf{R}^n$ and let $f_1, f_2 \in C_0(\Omega) \cap Lip(\Omega)$. We denote $f_3 = (|f_1|^q + |f_2|^q)^{1/q}$. Then $f_3 \in C_0(\Omega) \cap Lip(\Omega)$ and*

(i) $|\nabla f_3|^q \le |\nabla f_1|^q + |\nabla f_2|^q$ *a.e. in Ω.*

(ii) $||\nabla f_3||^q_{L^{p,q}(\Omega;\mathbf{R}^n)} \le ||\nabla f_1||^q_{L^{p,q}(\Omega;\mathbf{R}^n)} + ||\nabla f_2||^q_{L^{p,q}(\Omega;\mathbf{R}^n)}$.

Proof. Without loss of generality we can assume that both f_1 and f_2 are nonnegative. We have $|f_3(x) - f_3(y)|^q \le |f_1(x) - f_1(y)|^q + |f_2(x) - f_2(y)|^q$ for every $x, y \in \mathbf{R}^n$, hence it follows easily that $f_3 \in C_0(\Omega) \cap Lip(\Omega)$.

(i) We can assume without loss of generality that $q > 1$. We would be done immediately if $f_i \in C_0^1(\Omega)$ for $i = 1, 2, 3$ by using the previous inequality. Otherwise, since $f_i \in C_0(\Omega) \cap Lip(\Omega)$ for $i = 1, 2, 3$, it follows immediately from [HKM93, Lemma 1.11 and Theorem 1.18] that

$$\nabla(f_i^q) = q f_i^{q-1} \nabla f_i \text{ a.e. in } \mathbf{R}^n \text{ for } i = 1, 2, 3. \tag{11}$$

The definition of f_3 together with (11) implies

$$f_3^{q-1} \nabla f_3 = f_1^{q-1} \nabla f_1 + f_2^{q-1} \nabla f_2 \text{ a.e. in } \mathbf{R}^n. \tag{12}$$

By using the definition of f_3 one more time together with the Cauchy-Schwarz inequality, (12), and [HKM93, Lemma 1.19], we get the desired conclusion.

(ii) For $i = 1, 2, 3$ we denote $g_i = |\nabla f_i|^q$. Then, since $1 \le q < p$, we see via [BS88, Proposition II.1.7 and Definition IV.4.1] that

$$g_i \in L^{\frac{p}{q},1}(\Omega) \text{ and } ||g_i||_{L^{\frac{p}{q},1}(\Omega)} = ||\nabla f_i||^q_{L^{p,q}(\Omega;\mathbf{R}^n)} \text{ for } i = 1, 2, 3. \tag{13}$$

The claim follows by using (13) together with (i), the definition of the functions g_i and the fact that $|| \cdot ||_{L^{\frac{p}{q},1}(\Omega)}$ is a norm when $1 \le q < p$. This finishes the proof. □

Let μ be the right-invariant Haar probability measure defined on $SO(n)$, the compact topological group of orthonormal $n \times n$ matrices with entries from $\mathbf{R}$. (For the existence of left-invariant and right-invariant Haar measures on locally compact topological groups see [Hal50, Theorem B.58] and the discussion afterwards. For the uniqueness of such measures see [Hal50, Theorem C.60].)

The following definition was suggested by Eero Saksman.

Definition 2.9. For every measurable function $f : \mathbf{R}^n \to \mathbf{R}$ we define Tf as follows:

$$(Tf)(x) = \int_{SO(n)} f(Hx) d\mu(H).$$

Since μ is right-invariant, it follows that $(Tf)(x) = (Tf)(Hx)$ whenever $x \in \mathbf{R}^n, H \in SO(n)$ and f is a measurable function. This implies that $T(Tf) = Tf$ for every measurable function f. We notice that $|(Tf)(x) - (Tf)(y)| \le \int_{SO(n)} |f(Hx) - f(Hy)| d\mu(H)$. This implies easily that $T(C(\mathbf{R}^n)) \subset C(\mathbf{R}^n)$ and that $T(Lip(\mathbf{R}^n)) \subset Lip(\mathbf{R}^n)$. Moreover, for every $f \in C^1(\mathbf{R}^n)$ we have, via Lebesgue dominated convergence theorem

$$\begin{aligned} 0 &\le \lim_{h \to 0} \frac{1}{|h|} \left| Tf(x+h) - Tf(x) - \int_{SO(n)} \nabla f(Hx) \cdot Hh \, d\mu(H) \right| \\ &\le \int_{SO(n)} \frac{1}{|h|} |f(Hx + Hh) - f(Hx) - \nabla f(Hx) \cdot Hh| \, d\mu(H) = 0 \end{aligned}$$

whenever $x \in \mathbf{R}^n$. This implies immediately that $T(C^k(\mathbf{R}^n)) \subset C^k(\mathbf{R}^n)$ for every $k \ge 1$ with

$$\nabla(Tf)(x) = \int_{SO(n)} \nabla f(Hx) \cdot H \, d\mu(H) \tag{14}$$

pointwise in $\mathbf{R}^n$ whenever $f \in C^1(\mathbf{R}^n)$. From (14) it follows easily that

$$|\nabla(Tf)(x)| \le \int_{SO(n)} |\nabla f|(Hx)\, d\mu(H) = (T|\nabla f|)(x) \tag{15}$$

pointwise in $\mathbf{R}^n$ whenever $f \in C^1(\mathbf{R}^n)$.

Proposition 2.10. *Suppose* $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then*
(i) $||Tf||_{L^{(p,q)}(\mathbf{R}^n)} \le ||f||_{L^{(p,q)}(\mathbf{R}^n)}$ *for every* $f \in C_0(\mathbf{R}^n)$.
(ii) *If* $1 \le q \le p$, *then* $||Tf||_{L^{p,q}(\mathbf{R}^n)} \le ||f||_{L^{p,q}(\mathbf{R}^n)}$ *for every* $f \in C_0(\mathbf{R}^n)$.

Proof. We fix $p \in (1,\infty)$ and $q \in [1,\infty]$. Let $f \in C_0(\mathbf{R}^n)$. It is easy to see (see Definition 2.9 and the discussion afterwards) that $Tf \in C_0(\mathbf{R}^n)$.

Let $g \in L^{p',q'}(\mathbf{R}^n)$. Without loss of generality we can assume that g is supported in supp Tf. Then it follows from Theorem 2.3 that $g \in L^1(\mathbf{R}^n)$. Moreover, we have

$$\begin{aligned}\left|\int_{\mathbf{R}^n} (Tf)(x)g(x)\, dx\right| &\le \int_{\mathbf{R}^n} |(Tf)(x)g(x)|\, dx \\ &\le \int_{\mathbf{R}^n} \left(\int_{SO(n)} |f(Hx)| d\mu(H)\right) |g(x)|\, dx \\ &= \int_{SO(n)} \left(\int_{\mathbf{R}^n} |f(Hx)g(x)|\, dx\right) d\mu(H),\end{aligned} \tag{16}$$

where we used Fubini's theorem for the equality in the sequence. It is easy to see that $|f \circ H|^* = |f|^*$ for every $H \in SO(n)$. Since μ is a probability measure, we obtain, via (16) and [BS88, Theorem II.2.2]:

$$\int_{\mathbf{R}^n} |(Tf)(x)g(x)|\, dx \le \int_0^\infty f^*(s)g^*(s)\, ds. \tag{17}$$

From (17) it follows, via [BS88, Theorem II.2.7] that

$$\int_0^\infty (Tf)^*(s)g^*(s)\, ds \le \int_0^\infty f^*(s)g^*(s)\, ds. \tag{18}$$

By using (18) together with [BS88, Proposition II.4.2, Theorem IV.4.3, Theorem IV.4.6 and Theorem IV.4.7], we get the desired conclusion. □

Proposition 2.11. *Suppose* $1 < p < \infty$ *and* $1 \le q \le \infty$. *Let* $w : [\Omega_n r^n, \Omega_n] \to [0,\infty)$ *be defined by* $w(t) = (t/\Omega_n)^{1/n}$. *Suppose* $f : [r,1] \to [0,\infty)$ *is continuous and let* $g : [\Omega_n r^n, \Omega_n] \to [0,\infty)$ *be defined by* $g(t) = f(w(t))$. *Then*

$$||g||_{L^{p,q}([\Omega_n r^n,\Omega_n])} \ge n\Omega_n ||f||_{L^1([r,1])} ||(t/\Omega_n)^{-1/n'}||^{-1}_{L^{p',q'}([\Omega_n r^n,\Omega_n])}. \tag{19}$$

Proof. From the change of variable formula we get

$$\int_r^1 f(t)dt = \int_{\Omega_n r^n}^{\Omega_n} g(t)w'(t)dt = \frac{1}{n\Omega_n}\int_{\Omega_n r^n}^{\Omega_n} g(t)\,(t/\Omega_n)^{-1/n'}\, dt. \tag{20}$$

The claim follows by using (20) and [BS88, Theorem II.2.2], via Hölder's inequality. □

Lemma 2.12. *Suppose* $q \in [1,\infty]$ *and let* q' *be the Hölder conjugate of* q. *Then there exists* $C = C(n,q)$ *such that*

$$||(t/\Omega_n)^{-1/n'}||_{L^{n',q'}([\Omega_n r^n,\Omega_n])} \le C\left(1 + \ln\frac{1}{r}\right)^{\frac{1}{q'}}$$

for every $r \in (0,1)$. When $q = 1$, the right-hand side is interpreted as a constant.

Proof. Let $h : [\Omega_n r^n, \Omega_n] \to [0, \infty)$ be defined by $h(t) = (t/\Omega_n)^{-1/n'}$ and let $\lambda_{[h]}$ be the distribution function of h. Then

$$\lambda_{[h]}(s) = \begin{cases} 0 & \text{if } s > r^{1-n} \\ \Omega_n \left(s^{-n'} - r^n\right) & \text{if } 1 \le s \le r^{1-n} \\ \Omega_n \left(1 - r^n\right) & \text{if } 0 \le s \le 1. \end{cases} \tag{21}$$

We have to consider two cases, depending on whether $q = 1$ or $1 < q \le \infty$.

Suppose first that $q = 1$. Then $q' = \infty$. From (21) and [SW75, p. 191] we have

$$||h||^{n'}_{L^{n',\infty}([\Omega_n r^n, \Omega_n])} = \sup_{s>0} \lambda_{[h]}(s) s^{n'} = \Omega_n (1 - r^n),$$

hence the claim holds when $q = 1$.

Suppose now that $q > 1$. Then $q' < \infty$. We have ([KKM99, Proposition 2.1])

$$||h||^{q'}_{L^{n',q'}([\Omega_n r^n, \Omega_n])} = n' \int_0^\infty s^{q'-1} \lambda_{[h]}(s)^{\frac{q'}{n'}} ds.$$

We denote $J(n,q) = ||h||^{q'}_{L^{n',q'}([\Omega_n r^n, \Omega_n])}$. Then from (21) we have

$$\begin{aligned} J(n,q) &= n' \left(\int_0^1 |\Omega_n (1 - r^n)|^{\frac{q'}{n'}} s^{q'-1} \, ds + \int_1^{r^{1-n}} |\Omega_n (s^{-n'} - r^n)|^{\frac{q'}{n'}} s^{q'-1} \, ds \right) \\ &\le n \Omega_n^{\frac{q'}{n'}} \left(\int_0^1 s^{q'-1} ds + \int_1^{r^{1-n}} s^{-1} ds \right) \le n \Omega_n^{\frac{q'}{n'}} \left(\frac{1}{q'} + (n-1) \ln \frac{1}{r} \right). \end{aligned}$$

This yields the desired conclusion for $q > 1$. The Lemma is proved. □

3. Sobolev-Lorentz n, q relative capacity

Suppose $1 \le q \le \infty$. Let $\Omega \subset \mathbf{R}^n$ be an open set. Let $K \subset \Omega$ be compact. The Sobolev-Lorentz n, q-capacity of the pair (K, Ω) is denoted

$$\mathrm{cap}_{n,q}(K, \Omega) = \inf \{ ||\nabla u||^n_{L^{n,q}(\Omega; \mathbf{R}^n)} : u \in W(K, \Omega) \},$$

where

$$W(K, \Omega) = \{ u \in C_0^\infty(\Omega) : u \ge 1 \text{ in a neighborhood of } K \}.$$

We call $W(K, \Omega)$ the *set of admissible functions for the condenser* (K, Ω).

Lemma 3.1. *If $K \subset \Omega$ is compact, then we can get the same capacity if we restrict ourselves to a bigger set, namely*

$$W_0(K, \Omega) = \{ u \in C_0(\Omega) \cap Lip(\Omega) : u \ge 1 \textit{ on } K \}.$$

Proof. Let $u \in W_0(K, \Omega)$. We can assume without loss of generality that $u \ge 1$ in a neighborhood $U \subset\subset \Omega$ of K and that Ω is bounded. Let $\eta \in C_0^\infty(B(0,1))$ be a mollifier. For every integer $j \ge 1$ let $\eta_j(x) = j^n \eta(jx)$ and let $u_j = \eta_j * u$ be the convolution defined by

$$u_j(x) = (\eta_j * u)(x) = \int_{\mathbf{R}^n} \eta_j(x - y) u(y) dy.$$

For the basic properties of a mollifier see [Zie89, Theorems 1.6.1 and 2.1.3]. Let $\widetilde{U}$ be a neighborhood of K such that $\widetilde{U} \subset\subset U$ and let j_0 be a positive integer such that

$$1/j_0 < \min\{\text{dist}(\text{supp } u, \partial\Omega), \text{dist}(\widetilde{U}, \partial U)\}.$$

It is easy to see that $u_j, j \geq j_0$ is a sequence in $W(K,\Omega)$ and since $u \in C_0(\Omega) \cap Lip(\Omega)$, we have from [HKM93, Lemma 1.11] that

$$\lim_{j\to\infty} (||u_j - u||_{L^{n+1}(\Omega)} + ||\nabla u_j - \nabla u||_{L^{n+1}(\Omega;\mathbf{R}^n)}) = 0.$$

This together with (2) and Theorem 2.3 yields

$$\lim_{j\to\infty} (||u_j - u||_{L^{n,q}(\Omega)} + ||\nabla u_j - \nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)}) = 0. \tag{22}$$

An appeal to Corollary 2.7 applied for $p = n$ establishes the assertion, since $W(K,\Omega) \subset W_0(K,\Omega)$. □

Since truncation decreases the n,q-quasinorm whenever $1 \leq q \leq \infty$, it follows from Lemma 3.1 that we can choose only functions $u \in W_0(K,\Omega)$ that satisfy $0 \leq u \leq 1$ when computing the n,q relative capacity.

3.1. **Basic properties of the n,q relative capacity.** Usually, a capacity is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the n,q relative capacity. We follow [HKM93].

Theorem 3.2. *Suppose $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ be open. The set function $K \mapsto \text{cap}_{n,q}(K,\Omega)$, $K \subset \Omega$, K compact, enjoys the following properties:*

(i) *If $K_1 \subset K_2$, then $\text{cap}_{n,q}(K_1,\Omega) \leq \text{cap}_{n,q}(K_2,\Omega)$.*

(ii) *If $\Omega_1 \subset \Omega_2$ are open and K is a compact subset of Ω_1, then*

$$\text{cap}_{n,q}(K,\Omega_2) \leq \text{cap}_{n,q}(K,\Omega_1).$$

(iii) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^{\infty} K_i$, then*

$$\text{cap}_{n,q}(K,\Omega) = \lim_{i\to\infty} \text{cap}_{n,q}(K_i,\Omega).$$

(iv) *If Ω_i is an increasing sequence of open sets with $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and K is a compact subset of Ω_1, then*

$$\text{cap}_{n,q}(K,\Omega) = \lim_{i\to\infty} \text{cap}_{n,q}(K,\Omega_i).$$

(v) *Suppose $n \leq q \leq \infty$. If $K = \bigcup_{i=1}^{k} K_i \subset \Omega$ then*

$$\text{cap}_{n,q}(K,\Omega) \leq \sum_{i=1}^{k} \text{cap}_{n,q}(K_i,\Omega),$$

where $k \geq 1$ is a positive integer.

(vi) *Suppose $1 \leq q < n$. If $K = \bigcup_{i=1}^{k} K_i \subset \Omega$ then*

$$\text{cap}_{n,q}^{q/n}(K,\Omega) \leq \sum_{i=1}^{k} \text{cap}_{n,q}^{q/n}(K_i,\Omega),$$

where $k \geq 1$ is a positive integer.

Proof. Properties (i) and (ii) are immediate consequences of the definition.

(iii) Let $b =: \lim_{i\to\infty} \mathrm{cap}_{n,q}(K_i, \Omega)$. We fix a small $\varepsilon > 0$ and we pick a function $u \in W(K, \Omega)$ such that

$$||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} < \mathrm{cap}_{n,q}(K, \Omega) + \varepsilon.$$

When i is large, the sets K_i lie in the compact set $\{u \geq 1 - \varepsilon\}$. Therefore

$$\lim_{i\to\infty} \mathrm{cap}_{n,q}(K_i, \Omega) \leq \mathrm{cap}_{n,q}(\{u \geq 1-\varepsilon\}, \Omega) \leq \frac{1}{(1-\varepsilon)^{2n}} ||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)}.$$

Letting $\varepsilon \to 0$ yields $b \leq \mathrm{cap}_{n,q}(K, \Omega)$, hence (iii) follows because obviously $b \geq \mathrm{cap}_{n,q}(K, \Omega)$.

(iv) Let $b =: \lim_{i\to\infty} \mathrm{cap}_{n,q}(K, \Omega_i)$. We fix a small $\varepsilon > 0$ and we pick a function $u \in W(K, \Omega)$ such that

$$||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} < \mathrm{cap}_{n,q}(K, \Omega) + \varepsilon.$$

When i is large, the support of u lies in Ω_i. Therefore

$$\lim_{i\to\infty} \mathrm{cap}_{n,q}(K, \Omega_i) \leq ||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} < \mathrm{cap}_{n,q}(K, \Omega) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $b \leq \mathrm{cap}_{n,q}(K, \Omega)$, hence (iv) follows because we obviously have $b \geq \mathrm{cap}_{n,q}(K, \Omega)$.

It is enough to prove (v) and (vi) for $k = 2$ because then the general finite case follows by induction.

(v) When $q = n$ we are in the case of the n-capacity and then the claim holds. (See for example [HKM93, Theorem 2.2 (iii)].) So we can assume without loss of generality that $n < q \leq \infty$.

Let $u_i \in W_0(K_i, \Omega)$, $i = 1, 2$, such that

$$||\nabla u_i||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} < \mathrm{cap}_{n,q}(K_i, \Omega) + \varepsilon.$$

We define $u = \max(u_1, u_2)$. Since $u = (u_1 - u_2)^+ + u_2$, it follows from the discussion after Corollary 2.7 and (10) that $u \in W_0(K_1 \cup K_2, \Omega)$ with $|\nabla u| \leq \max(|\nabla u_1|, |\nabla u_2|)$. Using this and Theorem 2.5, we get

$$\begin{aligned} \mathrm{cap}_{n,q}(K_1 \cup K_2, \Omega) &\leq ||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} \leq ||\nabla u_1||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} + ||\nabla u_2||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} \\ &\leq \mathrm{cap}_{n,q}(K_1, \Omega) + \mathrm{cap}_{n,q}(K_2, \Omega) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we complete the proof in the case of two sets, and hence the general finite case.

(vi) The idea of the proof for the case $1 \leq q < n$ was suggested to me by Eero Saksman. I credit him for Theorem 2.8 as well.

Let $u_i \in W_0(K_i, \Omega)$, $i = 1, 2$, such that

$$0 \leq u_1 \leq 1 \text{ and } ||\nabla u_i||^q_{L^{n,q}(\Omega;\mathbf{R}^n)} < \mathrm{cap}^{q/n}_{n,q}(K_i, \Omega) + \varepsilon.$$

We define $u = (u_1^q + u_2^q)^{1/q}$. Then Theorem 2.8 implies that $u \in W_0(K_1 \cup K_2, \Omega)$ with

$$\begin{aligned} \mathrm{cap}^{q/n}_{n,q}(K_1 \cup K_2, \Omega) &\leq ||\nabla u||^q_{L^{n,q}(\Omega;\mathbf{R}^n)} \leq ||\nabla u_1||^q_{L^{n,q}(\Omega;\mathbf{R}^n)} + ||\nabla u_2||^q_{L^{n,q}(\Omega;\mathbf{R}^n)} \\ &\leq \mathrm{cap}^{q/n}_{n,q}(K_1, \Omega) + \mathrm{cap}^{q/n}_{n,q}(K_2, \Omega) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we complete the proof in the case of two sets, and hence the general finite case. The theorem is proved. $\square$

Remark 3.3. The definition of the n,q-capacity easily implies

$$\mathrm{cap}_{n,q}(K,\Omega) = \mathrm{cap}_{n,q}(\partial K,\Omega)$$

whenever K is a compact set in Ω.

3.1.1. **The scaling invariance of the n,q relative capacity.** Suppose $1 \leq q \leq \infty$. Obviously, $\mathrm{cap}_{n,q}(K,\Omega) = \mathrm{cap}_{n,q}(K+x,\Omega+x)$ whenever $\Omega \subset \mathbf{R}^n$ is open, $K \subset \Omega$ is compact and $x \in \mathbf{R}^n$. Indeed, the n,q-quasinorm is invariant under translations.

Lemma 3.4. *Suppose $1 \leq q \leq \infty$. Let Ω be open and $K \subset \Omega$ be compact. Then*

$$\mathrm{cap}_{n,q}(K,\Omega) = \mathrm{cap}_{n,q}(\alpha K,\alpha\Omega), \tag{23}$$

where $\alpha > 0$ and $\alpha A = \{\alpha a : a \in A\}$.

Proof. We have to analyze two cases, depending on whether $1 \leq q < \infty$ or $q = \infty$.

We assume first that $1 \leq q < \infty$. Let $u \in C_0^\infty(\Omega)$. We define $u_{(\alpha)} : \alpha\Omega \to \mathbf{R}$ by $u_{(\alpha)}(x) = u(\frac{x}{\alpha})$. Then $u \in W(K,\Omega)$ if and only if $u_{(\alpha)} \in W(\alpha K,\alpha\Omega)$. We notice that $\nabla u_{(\alpha)}(x) = \frac{1}{\alpha}\nabla u(\frac{x}{\alpha})$. We have

$$\begin{aligned} |\{x \in \alpha\Omega : |\nabla u_{(\alpha)}(x)| \geq t\}| &= |\{x \in \alpha\Omega : \frac{1}{\alpha}|\nabla u(\frac{x}{\alpha})| \geq t\}| \\ = \quad |\{x \in \alpha\Omega : |\nabla u(\tfrac{x}{\alpha})| \geq \alpha t\}| &= \alpha^n |\{\frac{x}{\alpha} \in \Omega : |\nabla u(\frac{x}{\alpha})| \geq \alpha t\}|. \end{aligned}$$

So $\lambda_{[|\nabla u_{(\alpha)}|]}(t) = \alpha^n \lambda_{[|\nabla u|]}(\alpha t)$ for every $t \geq 0$. Therefore

$$\begin{aligned} |\nabla u_{(\alpha)}|^*(t) &= \inf\{v \geq 0 : \lambda_{[|\nabla u_{(\alpha)}|]}(v) \leq t\} = \inf\{v \geq 0 : \alpha^n \lambda_{[|\nabla u|]}(\alpha v) \leq t\} \\ &= \frac{1}{\alpha}\inf\{\alpha v \geq 0 : \lambda_{[|\nabla u|]}(\alpha v) \leq \frac{t}{\alpha^n}\} = \frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n}). \end{aligned}$$

Hence we just proved that $|\nabla u_{(\alpha)}|^*(t) = \frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})$ for every $t \geq 0$. Therefore

$$||\nabla u_{(\alpha)}||^q_{L^{n,q}(\alpha\Omega;\mathbf{R}^n)} = \int_0^\infty t^{\frac{q}{n}} \left(|\nabla u_{(\alpha)}|^*(t)\right)^q \frac{dt}{t} = \int_0^\infty t^{\frac{q}{n}} \left(\frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})\right)^q \frac{dt}{t}.$$

By making the substitution $\frac{t}{\alpha^n} = s$, we have

$$\int_0^\infty t^{\frac{q}{n}} \left(\frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})\right)^q \frac{dt}{t} = \int_0^\infty (s\alpha^n)^{\frac{q}{n}} \left(\frac{1}{\alpha}|\nabla u|^*(s)\right)^q \frac{ds}{s} = ||\nabla u||^q_{L^{n,q}(\Omega;\mathbf{R}^n)}.$$

Thus we get $||\nabla u_{(\alpha)}||_{L^{n,q}(\alpha\Omega;\mathbf{R}^n)} = ||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)}$. This proves the claim when $1 \leq q < \infty$.

Now assume that $q = \infty$. We let $u \in C_0^\infty(\Omega)$ and we define $u_{(\alpha)}$ as before. Then as before, we have $u \in W(K,\Omega)$ if and only if $u_{(\alpha)} \in W(\alpha K,\alpha\Omega)$ and $|\nabla u_{(\alpha)}|^*(t) = \frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})$ for every $t \geq 0$. This implies

$$\begin{aligned} ||\nabla u_{(\alpha)}||^n_{L^{n,\infty}(\alpha\Omega;\mathbf{R}^n)} &= \sup_{t \geq 0} t\,(|\nabla u_{(\alpha)}|^*(t))^n = \sup_{t \geq 0} \frac{t}{\alpha^n}\,(|\nabla u|^*(\frac{t}{\alpha^n}))^n \\ &= \sup_{s \geq 0} s\,(|\nabla u|^*(s))^n = ||\nabla u||^n_{L^{n,\infty}(\Omega;\mathbf{R}^n)}. \end{aligned} \tag{24}$$

This finishes the proof.

$\square$

Lemma 3.5. *Suppose* $1 \leq q \leq \infty$. *Let* $\Omega \subset \mathbf{R^n}$ *be open and* $K \subset \Omega$ *compact. Then*

$$\text{cap}_{n,q}(K, \Omega) = \text{cap}_{n,q}(H^{-1}K, H^{-1}\Omega) \tag{25}$$

whenever $H \in SO(n)$.

Proof. Let $u \in C_0^\infty(\Omega)$. We define $u_H : H^{-1}\Omega \to \mathbf{R}$ by $u_H(x) = u(Hx)$. Then $u \in W(K, \Omega)$ if and only if $u_H \in W(H^{-1}K, H^{-1}\Omega)$. We notice that $\nabla u_H(x) = \nabla u(Hx) \cdot H$. Since $H \in SO(n)$, this implies immediately that $|\nabla u_H(x)| = |\nabla u(Hx)|$ for every $x \in \mathbf{R}^n$ and that $|\nabla u_H|^*(t) = |\nabla u|^*(t)$ for every $t \geq 0$. The desired conclusion follows easily from this, the definition of the Lorentz quasinorm for vector-valued functions and the definition of the n, q relative capacity. □

3.2. **Estimates for the n, q relative capacity.** Next we get some estimates for the n, q relative capacity of the spherical condenser $(\overline{B}(0, r), B(0, 1))$.

3.2.1. **Lower estimates for the n, q relative capacity.** The lower estimates for the relative capacity are always harder to get than the upper estimates. However, we start with the lower ones.

Let $r \in (0, 1)$. We define $\widetilde{W}(\overline{B}(0, r), B(0, 1)) = T(W(\overline{B}(0, r), B(0, 1)))$.

Lemma 3.6. *Let* $1 \leq q \leq \infty$ *be fixed. Then*

$$\begin{aligned} \text{cap}_{n,q}(\overline{B}(0, r), B(0, 1)) &\leq \inf\{||\nabla u||^n_{L^{n,q}(B(0,1);\mathbf{R}^n)} : u \in \widetilde{W}(\overline{B}(0, r), B(0, 1))\} \\ &\leq \left(\frac{n}{n-1}\right)^n \text{cap}_{n,q}(\overline{B}(0, r), B(0, 1)) \end{aligned}$$

for every $r \in (0, 1)$. *Moreover, the first inequality in the sequence becomes equality when* $1 \leq q \leq n$.

Proof. Let $r \in (0, 1)$ and $q \in [1, \infty]$ be fixed. From the discussion after Definition 2.9 it follows that $\widetilde{W}(\overline{B}(0, r), B(0, 1))$ is a subset of $W(\overline{B}(0, r), B(0, 1))$. By using this together with (15), Proposition 2.10, [BS88, Lemma IV.4.5] and the definition of the Sobolev-Lorentz capacity, we get the desired conclusion. □

Theorem 3.7. *Let* $1 \leq q \leq \infty$ *be fixed and let* q' *be the Hölder conjugate of* q. *There exists a constant* $C_1(n, q) > 0$ *such that*

$$C_1(n, q) \left(1 + \ln \frac{1}{r}\right)^{-\frac{n}{q'}} \leq \text{cap}_{n,q}(\overline{B}(0, r), B(0, 1))$$

for every $0 < r < 1$. *When* $q = 1$, *the left-hand side is interpreted as a constant.*

Proof. Let $q \in [1, \infty]$ be fixed and let $r \in (0, 1)$. When $q = n$ we are in the case of the n-capacity and then the result is a consequence of [HKM93, 2.13]. Therefore, we can assume without loss of generality that $q \neq n$. From Lemma 3.6, we see that it is enough to consider only functions in $\widetilde{W}(\overline{B}(0, r), B(0, 1))$ in order to get the desired lower bounds. So let $u \in \widetilde{W}(\overline{B}(0, r), B(0, 1))$. We can assume without loss of generality that $0 \leq u \leq 1$. We have $u \circ H = u$ for every $H \in SO(n)$, hence there exists a function $f \in C^\infty([0, 1])$ such that $u(x) = f(|x|)$ for every $x \in B(0, 1)$. Hence $|\nabla u(x)| = |f'(|x|)|$ for every $x \in B(0, 1)$. Moreover, $f'(t) = 0$ for every $t \in [0, r]$. If we define $g : [0, \Omega_n] \to [0, \infty)$ by $g(t) = |f'|((t/\Omega_n)^{1/n})$, we notice that g is a continuous function compactly supported in $(\Omega_n r^n, \Omega_n)$. Moreover, since $|\nabla u(x)| = g(\Omega_n |x|^n)$ for every $x \in B(0, 1)$, it follows that $|\nabla u|$ and g have the same distribution function. From this and the fact that g is supported in $(\Omega_n r^n, \Omega_n)$ we obtain

$$||\nabla u||_{L^{n,q}(B(0,1);\mathbf{R}^n)} = ||g||_{L^{n,q}([\Omega_n r^n, \Omega_n])}. \tag{26}$$

But $u \in \widetilde{W}(\overline{B}(0,r), B(0,1))$ with $u = 1$ on $\overline{B}(0,r)$. Hence for each $y \in \partial B(0,1)$ we have

$$1 \le \int_r^1 |\frac{d}{ds}u(sy)|ds \le \int_r^1 |\nabla u(sy)|ds = \int_r^1 |f'(s)|ds \tag{27}$$

From (26), (27), Proposition 2.11 and Lemma 2.12 we obtain

$$||\nabla u||_{L^{n,q}(B(0,1);\mathbf{R}^n)} \ge \widetilde{C}_1(n,q)\left(1 + \ln\frac{1}{r}\right)^{-\frac{1}{q'}}$$

for every $u \in \widetilde{W}(\overline{B}(0,r), B(0,1))$ such that $u = 1$ on $\overline{B}(0,r)$. By using this and Lemma 3.6, we get the desired conclusion. □

Corollary 3.8. *There exists a constant $C = C(n) > 0$ such that*

$$\text{cap}_{n,1}(\{x\}, \Omega) = C(n)$$

whenever $x \in \mathbf{R}^n$ and Ω is an open subset of $\mathbf{R}^n$ containing x.

Proof. Since the $n,1$ relative capacity is invariant under translations, we can assume without loss of generality that $x = 0$. (See the discussion before Lemma 3.4.) The claim follows from Theorem 3.2 (ii)-(iv), Lemma 3.4 and Theorem 3.7. It was easy to see the positivity of the aforementioned capacity for bounded open sets Ω containing x. The fact that this capacity is independent of both the open set and the point was observed by Ilkka Holopainen. I thank him for this fact. □

We can obtain the lower bound from Theorem 3.7 when $n < q \le \infty$ and $0 < r < e^{-\frac{1}{n-1}}$ via a different method. Before we do that we need the following result:

Proposition 3.9. *Let $\Omega \subset \mathbf{R}^n$ be bounded, let $n < q \le \infty$, and let $\varepsilon \in (0, n-1)$ be fixed. Then for every $K \subset \Omega$ compact we have*

$$\text{cap}_{n-\varepsilon}^{1/(n-\varepsilon)}(K,\Omega) \le C(n,q,\varepsilon)\, |\Omega|^{\frac{\varepsilon}{n(n-\varepsilon)}}\, \text{cap}_{n,q}^{1/n}(K,\Omega). \tag{28}$$

Proof. Let K be compact in Ω. Let $u \in W(K,\Omega)$. Then from Corollary 2.4 applied for $p = n$ and the definition of the $||\cdot||_{L^{n-\varepsilon}(\Omega;\mathbf{R}^n)}$-norm and $||\cdot||_{L^{(n,q)}(\Omega;\mathbf{R}^n)}$-quasinorm we have

$$||\nabla u||_{L^{n-\varepsilon}(\Omega;\mathbf{R}^n)} \le C(n,q,\varepsilon)\, |\Omega|^{\frac{\varepsilon}{n(n-\varepsilon)}}\, ||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)}.$$

Taking the infimum on both sides over such functions u, we get the claim for $K \subset \Omega$ compact. This finishes the proof. □

We now present the different method to obtain the lower bound from Theorem 3.7 when $n < q \le \infty$ and $0 < r < e^{-\frac{1}{n-1}}$.

Proof. (of Theorem 3.7) We have to consider two cases, depending on whether $n < q < \infty$ or $q = \infty$.

First we consider the case $n < q < \infty$. From (28) applied for $p = n$ and $n < q < \infty$, there exists a constant

$$C(n,\varepsilon,q) = \Omega_n^{\frac{\varepsilon}{n(n-\varepsilon)}}\,\varepsilon^{-\frac{1}{n-\varepsilon}+\frac{1}{q}}\left(\frac{n(q-n+\varepsilon)}{q}\right)^{\frac{1}{n-\varepsilon}-\frac{1}{q}}$$

such that

$$\mathrm{cap}_{n-\varepsilon}^{1/(n-\varepsilon)}(\overline{B}(0,r),B(0,1)) \le C(n,\varepsilon,q)\,\mathrm{cap}_{n,q}^{1/n}(\overline{B}(0,r),B(0,1))$$

for every $\varepsilon \in (0, n-1)$ and every $r \in (0,1)$. From [HKM93, 2.13] we have

$$\mathrm{cap}_{n-\varepsilon}(\overline{B}(0,r),B(0,1)) = \omega_{n-1}\left(\frac{\varepsilon}{n-\varepsilon-1}\right)^{n-\varepsilon-1}(r^{-\frac{\varepsilon}{n-\varepsilon-1}}-1)^{1-n+\varepsilon}.$$

Therefore,

$$\mathrm{cap}_{n,q}^{1/n}(\overline{B}(0,r),B(0,1)) \ge C_1(n,\varepsilon,q)\,\varepsilon^{1-\frac{1}{q}}\,r^{\frac{\varepsilon}{n-\varepsilon}} \tag{29}$$

for every $0 < \varepsilon < n-1$, where

$$C_1(n,\varepsilon,q) = \omega_{n-1}^{\frac{1}{n-\varepsilon}}\frac{\Omega_n^{-\frac{\varepsilon}{n(n-\varepsilon)}}}{(n-\varepsilon-1)^{\frac{n-\varepsilon-1}{n-\varepsilon}}}\left(\frac{n(q-n+\varepsilon)}{q}\right)^{\frac{1}{q}-\frac{1}{n-\varepsilon}}.$$

We define

$$C_1(n,q) = \inf_{0<\varepsilon<n-1} C_1(n,\varepsilon,q).$$

We notice that $C_1(n,q) > 0$. This together with (29) implies

$$\mathrm{cap}_{n,q}^{1/n}(\overline{B}(0,r),B(0,1)) \ge C_1(n,q)\,\varepsilon^{1-\frac{1}{q}}\,r^{\frac{\varepsilon}{n-\varepsilon}}. \tag{30}$$

For $r \in (0, e^{-\frac{1}{n-1}})$, we let $\varepsilon = \frac{1}{\ln\frac{1}{r}}$. Then $0 < \varepsilon < n-1$ and from (30) it follows that

$$\mathrm{cap}_{n,q}(\overline{B}(0,r),B(0,r)) \ge \frac{C_1(n,q)^n}{e^n}\left(\ln\frac{1}{r}\right)^{\frac{n}{q}-n} \tag{31}$$

for every $r \in (0, e^{-\frac{1}{n-1}})$. This yields the desired lower bound for the relative capacity whenever $n < q < \infty$ and $r \in (0, e^{-\frac{1}{n-1}})$.

Now we assume $q = \infty$. From (28) we have

$$\mathrm{cap}_{n-\varepsilon}^{1/(n-\varepsilon)}(\overline{B}(0,r),B(0,1)) \le \Omega_n^{\frac{\varepsilon}{n(n-\varepsilon)}}\,\varepsilon^{-\frac{1}{n-\varepsilon}}\,n^{\frac{1}{n-\varepsilon}}\,\mathrm{cap}_{n,\infty}^{1/n}(\overline{B}(0,r),B(0,1))$$

for every $\varepsilon \in (0, n-1)$. This together with [HKM93, 2.13] gives

$$\mathrm{cap}_{n,\infty}^{1/n}(\overline{B}(0,r),B(0,1)) \ge C_1(n,\varepsilon)\,\varepsilon\,r^{\frac{\varepsilon}{n-\varepsilon}} \tag{32}$$

for every $0 < \varepsilon < n-1$, where

$$C_1(n,\varepsilon) = \omega_{n-1}^{\frac{1}{n-\varepsilon}}\,\Omega_n^{-\frac{\varepsilon}{n(n-\varepsilon)}}\,(n-\varepsilon-1)^{-\frac{n-\varepsilon-1}{n-\varepsilon}}\,n^{-\frac{1}{n-\varepsilon}}.$$

We define

$$C_1(n) = \inf_{0<\varepsilon<n-1} C_1(n,\varepsilon).$$

We notice that $C_1(n) > 0$. This together with (32) implies

$$\mathrm{cap}_{n,\infty}^{1/n}(\overline{B}(0,r),B(0,1)) \ge C_1(n)\,\varepsilon\,r^{\frac{\varepsilon}{n-\varepsilon}}. \tag{33}$$

For $r \in (0, e^{-\frac{1}{n-1}})$ we let $\varepsilon = \frac{1}{\ln \frac{1}{r}}$. Then $0 < \varepsilon < n-1$ and from (33) it follows that

$$\mathrm{cap}_{n,\infty}(\overline{B}(0,r), B(0,1)) \geq \frac{C_1(n)^n}{e^n} \left(\ln \frac{1}{r}\right)^{-n} \tag{34}$$

for every $r \in (0, e^{-\frac{1}{n-1}})$. We let $C_1(n,q) = C_1(n)$ when $q = \infty$. This yields the desired lower bound for the relative capacity when $q = \infty$ and $r \in (0, e^{-\frac{1}{n-1}})$.

□

3.2.2. **Upper estimates for the n, q relative capacity.** Next we get some upper estimates for the Sobolev-Lorentz n, q relative capacity.

Theorem 3.10. *Let $1 \leq q \leq \infty$ be fixed and let q' be the Hölder conjugate of q. There exists a constant $C_2(n,q) > 0$ such that*

$$\mathrm{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \leq C_2(n,q) \left(\ln \frac{1}{r}\right)^{-\frac{n}{q'}}$$

for every $0 < r < e^{-\frac{1}{n-1}}$. When $q = 1$, the right-hand side is interpreted as a constant.

Proof. We let $r \in (0,1)$ be fixed. We use the function $u : B(0,1) \to \mathbf{R}$ defined by

$$u(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq r \\ \frac{\ln |x|}{\ln r} & \text{if } r < |x| < 1. \end{cases}$$

Then

$$|\nabla u(x)| = \begin{cases} 0 & \text{if } 0 \leq |x| < r \\ \frac{1}{\ln \frac{1}{r}} \frac{1}{|x|} & \text{if } r < |x| < 1. \end{cases}$$

We notice that $u \notin W_0(\overline{B}(0,r), B(0,1))$. However,

$$\mathrm{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \leq ||\nabla u||^n_{L^{n,q}(B(0,1);\mathbf{R}^n)} \tag{35}$$

because

$$||\nabla u||_{L^{n,q}(B(0,1);\mathbf{R}^n)} = \lim_{\delta \to 0} ||\nabla u_\delta||_{L^{n,q}(B(0,1);\mathbf{R}^n)},$$

where u_δ, $0 < \delta < \frac{1-r}{r}$ is a sequence in $W_0(\overline{B}(0,r), B(0,1))$ defined by

$$u_\delta(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq r \\ \frac{\ln (1+\delta)|x|}{\ln r(1+\delta)} & \text{if } r < |x| < \frac{1}{1+\delta} \\ 0 & \text{if } \frac{1}{1+\delta} \leq |x| \leq 1. \end{cases}$$

We want to get an upper estimate for $||\nabla u||_{L^{n,q}(B(0,1);\mathbf{R}^n)}$ whenever $1 \leq q \leq \infty$. We define $v : B(0,1) \to \mathbf{R}$ by $v(x) = -\ln r \, |\nabla u(x)|$. We compute $\lambda_{[v]}$. We recall that $\Omega_n = |B(0,1)|$. We have

$$\lambda_{[v]}(t) = |\{x \in B(0,1) \setminus B(0,r) : \frac{1}{|x|} > t\}| = |\{x \in B(0,1) \setminus B(0,r) : |x| < \frac{1}{t}\}|.$$

Hence

$$\lambda_{[v]}(t) = \begin{cases} 0 & \text{if } t > \frac{1}{r} \\ \Omega_n \left(\frac{1}{t^n} - r^n\right) & \text{if } 1 \leq t \leq \frac{1}{r} \\ \Omega_n (1 - r^n) & \text{if } 0 \leq t \leq 1. \end{cases}$$

We notice that

$$v^*(t) = \begin{cases} \left(\frac{1}{t/\Omega_n + r^n}\right)^{\frac{1}{n}} & \text{if } 0 \le t < \Omega_n\,(1-r^n) \\ 0 & \text{if } t \ge \Omega_n\,(1-r^n). \end{cases}$$

We compute $||v||_{L^{n,q}(B(0,1))}$. We have to consider two cases, depending on whether $1 \le q < \infty$ or $q = \infty$.

We assume first that $1 \le q < \infty$. Let

$$J =: ||v||^q_{L^{n,q}(B(0,1))} = \int_0^{\Omega_n(1-r^n)} t^{\frac{q}{n}} (v^*(t))^q \frac{dt}{t}.$$

By making the substitution $t = s\,\Omega_n r^n$, we get

$$\begin{aligned} J &= \int_0^{\Omega_n(1-r^n)} t^{\frac{q}{n}} \left(\frac{1}{t/\Omega_n + r^n}\right)^{\frac{q}{n}} \frac{dt}{t} = \Omega_n^{\frac{q}{n}} \int_0^{\frac{1-r^n}{r^n}} s^{\frac{q}{n}} \left(\frac{1}{s+1}\right)^{\frac{q}{n}} \frac{ds}{s} \\ &= \Omega_n^{\frac{q}{n}} \left(\int_0^1 s^{\frac{q}{n}-1} \left(\frac{1}{s+1}\right)^{\frac{q}{n}} ds + \int_1^{\frac{1-r^n}{r^n}} \left(\frac{s}{s+1}\right)^{\frac{q}{n}} \frac{ds}{s} \right) \\ &\le \Omega_n^{\frac{q}{n}} \left(\frac{n}{q} + \ln \frac{1-r^n}{r^n} \right) \le \Omega_n^{\frac{q}{n}} \left(\frac{n}{q} + n \ln \frac{1}{r} \right) \le \widetilde{C}_2(n,q) \ln \frac{1}{r} \end{aligned}$$

if $0 < r < e^{-\frac{1}{n-1}}$. From the above inequality, together with (35) and the fact that $v = -\ln r\,|\nabla u|$, it follows that

$$\text{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \le C_2(n,q) \left(\ln \frac{1}{r}\right)^{\frac{n}{q}-n} \tag{36}$$

whenever $1 \le q < \infty$ and $0 < r < e^{-\frac{1}{n-1}}$. Hence the claim holds for $1 \le q < \infty$.

Now assume $q = \infty$. We have

$$||v||^n_{L^{n,\infty}(B(0,1))} = \sup_{t \ge 0} t\,(v^*(t))^n = \sup_{0 \le t \le \Omega_n\,(1-r^n)} \frac{t}{t/\Omega_n + r^n} = \Omega_n\,(1-r^n).$$

Therefore

$$||\nabla u||^n_{L^{n,\infty}(B(0,1);\mathbf{R}^n)} = \left(\ln \frac{1}{r}\right)^{-n} ||v||^n_{L^{n,\infty}(B(0,1))} = \Omega_n\,(1-r^n) \left(\ln \frac{1}{r}\right)^{-n}.$$

From this and (35) we get

$$\text{cap}_{n,\infty}(\overline{B}(0,r), B(0,1)) \le \Omega_n \left(\ln \frac{1}{r}\right)^{-n} \tag{37}$$

for every $r \in (0,1)$, hence the claim holds also for $q = \infty$. This finishes the proof of the theorem.

□

By combining Theorems 3.7 and 3.10, we get the following:

Theorem 3.11. *Let $1 \le q \le \infty$ be fixed and let q' be its Hölder conjugate. Then there exists a constant $C(n,q) > 0$ such that*

$$C(n,q)^{-1} \left(\ln \frac{1}{r}\right)^{-\frac{n}{q'}} \le \text{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \le C(n,q) \left(\ln \frac{1}{r}\right)^{-\frac{n}{q'}}$$

for every $0 < r < e^{-\frac{1}{n-1}}$.

4. Hausdorff measure and the Sobolev-Lorentz n,q-capacity

In this section we examine the relationship between Hausdorff measures and the Sobolev-Lorentz n,q-capacity.

Definition 4.1. Let $1 \leq q < \infty$. Let K be a compact set in $\mathbf{R}^n$. We say that K is of *n,q-capacity zero* if

$$\mathrm{cap}_{n,q}(K,\Omega) = 0$$

whenever Ω is an open neighborhood of K. In this case we write $\mathrm{cap}_{n,q}(K) = 0$.

From Corollary 3.8 and Theorem 3.2 (i) it follows immediately that a compact set $K \subset \mathbf{R}^n$ is of $n,1$ capacity zero if and only if $K = \emptyset$.

Before proceeding, we recall the following version of the Poincaré inequality.

Theorem 4.2. Poincaré inequality for Sobolev-Lorentz spaces. *Let $\Omega \subset \mathbf{R}^n$ be bounded. Let $1 \leq q \leq \infty$ be fixed. Then there exists a constant $C(n,q)$ such that*

$$||u||_{L^{n,q}(\Omega)} \leq C(n,q)\,|\Omega|^{\frac{1}{n}}\,||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)} \tag{38}$$

for every $u \in C_0^\infty(\Omega)$.

Proof. For every $u \in C_0^\infty(\Omega)$ we have (see [GT83, Lemma 7.14]):

$$|u(x)| \leq \frac{1}{\omega_{n-1}}(I_1|\nabla u|)(x) \tag{39}$$

for every $x \in \mathbf{R}^n$. We recall that for every measurable function f in $\mathbf{R}^n$, $I_1 f$ is its Riesz potential of order 1. (See [BS88, Definition IV.4.17] and [Hei01, p. 20].) An application of Hardy-Littlewood-Sobolev theorem of fractional integration ([BS88, Theorem IV.4.18]) together with Theorem 2.3, [BS88, Proposition II.1.7] and (39) yields the desired conclusion. □

Theorem 4.3. *Suppose $1 < q < \infty$. Let E be a compact set in $\mathbf{R}^n$. If there exists a constant $M > 0$ such that*

$$\mathrm{cap}_{n,q}(E,\Omega) \leq M < \infty$$

for all open sets Ω containing E, then $\mathrm{cap}_{n,q}(E) = 0$.

Proof. When $q = n$ we are in the case of the n-capacity and then the claim holds. (See for example [HKM93, Lemma 2.34].) So we can assume without loss of generality that $q \neq n$. We let Ω be a fixed open neighborhood of E. We can assume without loss of generality that Ω is bounded. We choose a descending sequence of open sets

$$\Omega = \Omega_1 \supset\supset \Omega_2 \supset\supset \cdots \supset\supset \cap_i \Omega_i = E$$

and we choose $\varphi_i \in W(E,\Omega_i)$, $0 \leq \varphi_i \leq 1$ with $\varphi_i = 1$ on E and

$$||\nabla \varphi_i||^n_{L^{n,q}(\Omega_i;\mathbf{R}^n)} < M + 1.$$

From the Poincaré inequality for Sobolev-Lorentz spaces (38) we have that $(\varphi_i, \nabla\varphi_i)$ is bounded in the space $L^{n,q}(\Omega) \times L^{n,q}(\Omega;\mathbf{R}^n)$. We notice that φ_i converges pointwise to a function ψ which is 1 on E and 0 on $\mathbf{R}^n \setminus E$. Hence, from Mazur's lemma ([Yos80, p. 120]), [BS88, Lemma IV.4.5], and the reflexivity of $L^{n,q}(\Omega) \times L^{n,q}(\Omega;\mathbf{R}^n)$ it follows that there exists a subsequence denoted again by φ_i such that $(\varphi_i, \nabla\varphi_i)$

converges weakly to $(\psi, 0)$ in $L^{n,q}(\Omega) \times L^{n,q}(\Omega; \mathbf{R}^n)$ and a sequence $\widetilde{\varphi}_i$ of convex combinations of φ_i,

$$\widetilde{\varphi}_i = \sum_{j=i}^{j_i} \lambda_{i,j} \varphi_j, \quad \lambda_{i,j} \geq 0, \quad \sum_{j=i}^{j_i} \lambda_{i,j} = 1,$$

such that $(\widetilde{\varphi}_i, \nabla \widetilde{\varphi}_i)$ converges to $(\psi, 0)$ in $L^{n,q}(\Omega) \times L^{n,q}(\Omega; \mathbf{R}^n)$. The closedness of $W(E, \Omega_i)$ under finite convex combinations implies that $\widetilde{\varphi}_i \in W(E, \Omega_i)$ for every integer $i \geq 1$. Therefore

$$0 \leq \operatorname{cap}_{n,q}(E, \Omega) \leq \limsup_{i \to \infty} ||\nabla \widetilde{\varphi}_i||^n_{L^{n,q}(\Omega_i; \mathbf{R}^n)} = 0.$$

□

Theorem 4.4. *Suppose that $1 < q \leq \infty$ and that E is a compact set in $\mathbf{R}^n$. For $1 < q \leq \infty$ we let $h_{n,q} : [0, \infty) \to \mathbf{R}$ be defined by*

$$h_{n,q}(t) = \begin{cases} 0 & \text{if } t = 0 \\ \left(\ln \frac{1}{t}\right)^{-\frac{n}{q'}} & \text{if } 0 < t < \frac{1}{2} \\ 2\,(\ln 2)^{-\frac{n}{q'}}\, t & \text{if } t \geq \frac{1}{2}. \end{cases}$$

(i) *If $1 < q < n$, then $\Lambda_{h^{q/n}_{n,q}}(E) < \infty$ implies* $\operatorname{cap}_{n,q}(E) = 0$.

(ii) *If $n \leq q < \infty$, then $\Lambda_{h_{n,q}}(E) < \infty$ implies* $\operatorname{cap}_{n,q}(E) = 0$.

(iii) *If $q = \infty$, then $\Lambda_{h_{n,q}}(E) = 0$ implies* $\operatorname{cap}_{n,\infty}(E, \Omega) = 0$ *whenever Ω is an open neighborhood of E.*

Proof. We have to analyze three cases, depending on whether $1 < q < n$ or $n \leq q < \infty$ or $q = \infty$. It is enough to prove that $\operatorname{cap}_{n,q}(E, \Omega) = 0$ whenever Ω is a bounded open neighborhood of E. So let Ω be a bounded open set containing E. We denote by δ the distance from E to the complement of Ω. Without loss of generality we can assume that $0 < \delta < e^{-\frac{1}{2(n-1)}}$. Fix $0 < \varepsilon < 1$ such that $\varepsilon < \frac{1}{4}\delta^2$; then $r < \varepsilon$ implies $\ln(\frac{\delta}{2r}) \geq \frac{1}{2} \ln(\frac{1}{r})$. We cover E by open balls $B(x_i, r_i)$ such that $r_i < \frac{1}{2}\varepsilon$. Since we may assume that the balls $B(x_i, r_i)$ intersect E, we have $B(x_i, \frac{\delta}{2}) \subset \Omega$. In fact, since E is compact, E is covered by finitely many of the balls $B(x_i, r_i)$.

We assume first that $1 < q < n$. Using Theorem 3.2 (ii) and (v) we obtain

$$\begin{aligned} \operatorname{cap}^{q/n}_{n,q}(E, \Omega) &\leq \sum_i \operatorname{cap}^{q/n}_{n,q}(\overline{B}(x_i, r_i), \Omega) \leq \sum_i \operatorname{cap}^{q/n}_{n,q}(\overline{B}(x_i, r_i), B(x_i, \frac{\delta}{2})) \\ &= \sum_i \operatorname{cap}^{q/n}_{n,q}(\overline{B}(0, r_i), B(0, \frac{\delta}{2})) \leq C(n,q) \sum_i \left(\ln \frac{1}{r_i}\right)^{1-q}, \end{aligned}$$

where in the last step we also used (36) together with our choice of ε. Taking the infimum over all such coverings and letting $\varepsilon \to 0$, we conclude

$$\operatorname{cap}^{q/n}_{n,q}(E, \Omega) \leq C(n,q) \Lambda_{h^{q/n}_{n,q}}(E) < \infty.$$

Since Ω was an arbitrary bounded open set containing E, the desired conclusion follows from Theorems 3.2 (ii) and 4.3 when $1 < q < n$.

We assume now that $n \leq q < \infty$. When $q = n$ we are in the case of the n-capacity and then the claim holds. (See for example [HKM93, Theorem 2.27].) So we can assume without loss of generality that $n < q < \infty$. Using the finite subadditivity and the monotonicity property of the n,q-capacity we obtain

$$\begin{aligned}
\mathrm{cap}_{n,q}(E,\Omega) &\leq \sum_i \mathrm{cap}_{n,q}(B(x_i,r_i),\Omega) \leq \sum_i \mathrm{cap}_{n,q}(B(x_i,r_i),B(x_i,\frac{\delta}{2})) \\
&= \sum_i \mathrm{cap}_{n,q}(B(0,r_i),B(0,\frac{\delta}{2})) \leq C(n,q)\sum_i \left(\ln\frac{1}{r_i}\right)^{\frac{n}{q}-n},
\end{aligned}$$

where in the last step we also used (36) together with our choice of ε. Taking the infimum over all such coverings, we conclude

$$\mathrm{cap}_{n,q}(E,\Omega) \leq C(n,q)\Lambda_{h_{n,q}}(E) < \infty.$$

Since Ω was an arbitrary bounded open set containing E, the desired conclusion follows from Theorems 3.2 (ii) and 4.3 when $n < q < \infty$.

We assume now that $q = \infty$. Using the finite subadditivity and the monotonicity property of the n,∞-capacity we obtain

$$\begin{aligned}
\mathrm{cap}_{n,\infty}(E,\Omega) &\leq \sum_i \mathrm{cap}_{n,\infty}(B(x_i,r_i),\Omega) \leq \sum_i \mathrm{cap}_{n,\infty}(B(x_i,r_i),B(x_i,\frac{\delta}{2})) \\
&= \sum_i \mathrm{cap}_{n,\infty}(B(0,r_i),B(0,\frac{\delta}{2})) \leq C(n)\sum_i \left(\ln\frac{1}{r_i}\right)^{-n},
\end{aligned}$$

where in the last step we also used (37) together with our choice of ε. Taking the infimum over all such coverings, we conclude

$$\mathrm{cap}_{n,\infty}(E,\Omega) \leq C(n)\Lambda_{h_{n,\infty}}(E) = 0.$$

□

Remark 4.5. It is known that if $\mathrm{cap}_n(E) = 0$, then $\Lambda_h(E) = 0$ whenever E is a compact set in $\mathbf{R}^n$ and h is an increasing function on $[0,\infty)$ such that $h(0) = 0$, and

$$\int_0^1 h(r)^{1/(n-1)}\frac{dr}{r} < \infty.$$

(See [AH96, p. 20 and Theorem 5.1.13] and [HKM93, Corollary 2.40].) This corresponds to the case $q = n$. It is not known if we have similar results for $q \neq n$. A possible result would be the following:

Conjecture 4.6. *Let E be a compact set in $\mathbf{R}^n$ and let $1 < q \leq \infty$ be such that $q \neq n$. Then, if there exists a bounded open neighborhood Ω of E such that $\mathrm{cap}_{n,q}(E,\Omega) = 0$, we have $\Lambda_h(E) = 0$ whenever h is an increasing function on $[0,\infty)$ such that $h(0) = 0$, and*

$$\int_0^1 h(r)^{\frac{q'}{n}}\frac{dr}{r} < \infty.$$

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Ş. Costea: Serban.Costea@helsinki.fi, Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), University of Helsinki, FI-00014, Helsinki, Finland