CONDUCTOR INEQUALITIES AND CRITERIA FOR SOBOLEV-LORENTZ TWO-WEIGHT INEQUALITIES

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In memory of S. L. Sobolev

Abstract. We present integral conductor inequalities connecting the Lorentz p,q-(quasi)norm of a gradient of a function to a one-dimensional integral of the p,q-capacitance of the conductor between two level surfaces of the same function. These inequalities generalize an inequality obtained by the second author in the case of the Sobolev norm. Such conductor inequalities lead to necessary and sufficient conditions for Sobolev-Lorentz type inequalities involving two arbitrary measures.

1. Introduction

During the last decades Sobolev-Lorentz function spaces, which include classical Sobolev spaces, attracted attention not only as an interesting mathematical object, but also as a tool for a finer tuning of properties of solutions to partial differential equations. (See, for example, [Alb], [AFT1], [AFT2], [BBGGPV], [Cia], [CP], [Cos], [DHM], [HL], [KKM], [ST], *et al.*)

In the present paper we generalize the inequality

$$\int_0^\infty \mathrm{cap}_p(\overline{M_{at}}, M_t)\,d(t^p) \le c(a,p)\int_\Omega |\nabla f|^p\,dx \tag{1}$$

to the case of Sobolev-Lorentz spaces. Here $f \in Lip_0(\Omega)$, i.e. f is an arbitrary Lipschitz function compactly supported in the open set $\Omega \subset \mathbf{R}^n$, while M_t is the set $\{x \in \Omega : |f(x)| > t\}$ with $t > 0$. The inequality (1) was obtained in [M1]. (See also [M3, Chapter 2].) It has various extensions and applications to the theory of Sobolev-type spaces on domains in $\mathbf{R}^n$, Riemannian manifolds, metric and topological spaces, to linear and nonlinear partial differential equations, Dirichlet forms, and Markov processes etc. (See, for example, [Ad], [AH], [AP], [AX1], [AX2], [Ai], [CS], [DKX], [Dah], [Fi], [FU1], [FU2], [Gr], [Haj], [Han], [HMV], [Ka], [Ko1], [Ko2], [Mal], [M1], [M2], [M4], [M5], [MN], [MP], [Ne], [Ra], [Ta], [V1], [V2], [Vo], *et al*).

In the sequel, we prove the inequalities

$$\int_0^\infty \mathrm{cap}(\overline{M_{at}}, M_t)\,d(t^p) \le c(a,p,q)||\nabla f||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} \text{ when } 1 \le q \le p \tag{2}$$

and

$$\int_0^\infty \mathrm{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p}\,d(t^q) \le c(a,p,q)||\nabla f||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} \text{ when } p < q < \infty \tag{3}$$

for all $f \in Lip_0(\Omega)$.

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The proof of (2) and (3) is based on the superadditivity of the p, q-capacitance, also justified in this paper.

From (2) and (3) we derive necessary and sufficient conditions for certain two-weight inequalities involving Sobolev-Lorentz norms, generalizing results obtained in [M4] and [M5]. Specifically, let μ and ν be two locally finite nonnegative measures on Ω and let p, q, r, s be real numbers such that $1 < s \leq \max(p, q) \leq r < \infty$ and $q \geq 1$. We characterize the inequality

$$||f||_{L^{r,\max(p,q)}(\Omega,\mu)} \leq A\left(||\nabla f||_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} + ||f||_{L^{s,\max(p,q)}(\Omega,\nu)}\right) \tag{4}$$

restricted to functions $f \in Lip_0(\Omega)$ by requiring the condition

$$\mu(g)^{1/r} \leq K(\text{cap}_{p,q}(\overline{g}, G)^{1/p} + \nu(G)^{1/s}) \tag{5}$$

to be valid for all open bounded sets g and G subject to $\overline{g} \subset G$, $\overline{G} \subset \Omega$. When $n = 1$ inequality (4) becomes

$$||f||_{L^{r,\max(p,q)}(\Omega,\mu)} \leq A\left(||f'||_{L^{p,q}(\Omega,m_1)} + ||f||_{L^{s,\max(p,q)}(\Omega,\nu)}\right). \tag{6}$$

The requirement that (6) be valid for all functions $f \in Lip_0(\Omega)$ when $n = 1$ is shown to be equivalent to the condition

$$\mu(\sigma_d(x))^{1/r} \leq K(\tau^{(1-p)/p} + \nu(\sigma_{d+\tau}(x))^{1/s}) \tag{7}$$

whenever x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. Here and throughout the paper $\sigma_d(x)$ denotes the open interval $(x - d, x + d)$ for every $d > 0$.

2. **Preliminaries**

Denote by Ω a nonempty open subset of $\mathbf{R}^n$ and by m_n the Lebesgue n-measure in $\mathbf{R}^n$, where $n \geq 1$ is integer. For a Lebesgue measurable function $u : \Omega \to \mathbf{R}$ denote by supp u the smallest closed set such that u vanishes outside supp u. We also introduce

$$\begin{aligned} Lip(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\} \\ Lip_0(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz and with compact support in } \Omega\}. \end{aligned}$$

If $\varphi \in Lip(\Omega)$, we write $\nabla\varphi$ for the gradient of φ. This notation makes sense, since by Rademacher's theorem ([Fed, Theorem 3.1.6]) every Lipschitz function on Ω is m_n-a.e. differentiable.

Throughout this section, we assume that (Ω, μ) is a measure space. Let $f : \Omega \to \mathbf{R}^n$ be a μ-measurable function. We define $\mu_{[f]}$, the *distribution function* of f as follows (see [BS, Definition II.1.1]):

$$\mu_{[f]}(t) = \mu(\{x \in \Omega : |f(x)| > t\}), \qquad t \geq 0.$$

We define f^*, the *nonincreasing rearrangement* of f by

$$f^*(t) = \inf\{v : \mu_{[f]}(v) \leq t\}, \quad t \geq 0.$$

(See [BS, Definition II.1.5].) We note that f and f^* have the same distribution function. Moreover, for every positive α we have

$$(|f|^\alpha)^* = (|f|^*)^\alpha$$

and if $|g| \leq |f|$ a.e. on Ω, then $g^* \leq f^*$. (See [BS, Proposition II.1.7].) We also define f^{**}, the *maximal function* of f^* by

$$f^{**}(t) = m_{f^*}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad t > 0.$$

(See [BS, Definition II.3.1].)

Throughout the paper, we denote by p' the Hölder conjugate of $p \in [1, \infty]$.

The *Lorentz space* $L^{p,q}(\Omega, \mu; \mathbf{R}^n)$, $1 < p < \infty$, $1 \leq q \leq \infty$, is defined as follows:

$$L^{p,q}(\Omega, \mu; \mathbf{R}^n) = \{f : \Omega \to \mathbf{R}^n : f \text{ is } \mu\text{-measurable}, ||f||_{L^{p,q}(\Omega,\mu;\mathbf{R}^n)} < \infty\},$$

where

$$||f||_{L^{p,q}(\Omega,\mu;\mathbf{R}^n)} = ||\,|f|\,||_{p,q} = \begin{cases} \left(\displaystyle\int_0^\infty (t^{1/p} f^*(t))^q \, \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \displaystyle\sup_{t>0} t\mu_{[f]}(t)^{1/p} = \sup_{s>0} s^{1/p} f^*(s), & q = \infty. \end{cases}$$

(See [BS, Definition IV.4.1] and [SW, p. 191].) We omit $\mathbf{R}^n$ in the notation of function spaces for the scalar case, i.e. for $n = 1$.

If $1 \leq q \leq p$, then $|| \cdot ||_{L^{p,q}(\Omega,\mu;\mathbf{R}^n)}$ represents a norm, but for $p < q \leq \infty$ it represents a quasinorm, equivalent to the norm $|| \cdot ||_{L^{(p,q)}(\Omega,\mu;\mathbf{R}^n)}$, where

$$||f||_{L^{(p,q)}(\Omega,\mu;\mathbf{R}^n)} = ||\,|f|\,||_{(p,q)} = \begin{cases} \left(\displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \, \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^{**}(t), & q = \infty. \end{cases}$$

(See [BS, Definition IV.4.4].) Namely, from [BS, Lemma IV.4.5] we have that

$$||\,|f|\,||_{L^{p,q}(\Omega,\mu)} \leq ||\,|f|\,||_{L^{(p,q)}(\Omega,\mu)} \leq p'||\,|f|\,||_{L^{p,q}(\Omega,\mu)}$$

for every $q \in [1, \infty]$ and every μ-measurable function $f : \Omega \to \mathbf{R}^n$.

It is known that $(L^{p,q}(\Omega, \mu; \mathbf{R}^n), || \cdot ||_{L^{p,q}(\Omega,\mu;\mathbf{R}^n)})$ is a Banach space for $1 \leq q \leq p$, while $(L^{p,q}(\Omega, \mu; \mathbf{R}^n), || \cdot ||_{L^{(p,q)}(\Omega,\mu;\mathbf{R}^n)})$ is a Banach space for $1 < p < \infty$, $1 \leq q \leq \infty$.

Remark 2.1. It is also known (see [BS, Proposition IV.4.2]) that for every $p \in (1, \infty)$ and $1 \leq r < s \leq \infty$ there exists a constant $C(p, r, s)$ such that

$$||\,|f|\,||_{L^{p,s}(\Omega,\mu)} \leq C(p, r, s)||\,|f|\,||_{L^{p,r}(\Omega,\mu)} \tag{8}$$

for all measurable functions $f \in L^{p,r}(\Omega, \mu; \mathbf{R}^n)$ and all integers $n \geq 1$. In particular, the embedding $L^{p,r}(\Omega, \mu; \mathbf{R}^n) \hookrightarrow L^{p,s}(\Omega, \mu; \mathbf{R}^n)$ holds.

2.1. **The subadditivity and superadditivity of the Lorentz quasinorms.** In the second part of this paper, we will prove a few results by relying on the superadditivity of the Lorentz p, q-quasinorm. Therefore we recall the known results and present new results concerning the superadditivity and the subadditivity of the Lorentz p, q-quasinorm.

The superadditivity of the Lorentz p, q-norm in the case $1 \leq q \leq p$ was stated in [CHK, Lemma 2.5].

Proposition 2.2. *(See* [CHK, Lemma 2.5]*.) Let (Ω, μ) be a measure space. Suppose that $1 \leq q \leq p$. Let $\{E_i\}_{i\geq 1}$ be a collection of pairwise disjoint measurable subsets of Ω with $E_0 = \cup_{i\geq 1} E_i$ and let $f \in L^{p,q}(\Omega, \mu)$. Then*

$$\sum_{i\geq 1} ||\chi_{E_i} f||^p_{L^{p,q}(\Omega,\mu)} \leq ||\chi_{E_0} f||^p_{L^{p,q}(\Omega,\mu)}.$$

We obtain a similar result concerning the superadditivity in the case $1 < p < q < \infty$.

Proposition 2.3. *Let (Ω, μ) be a measure space. Suppose that $1 < p < q < \infty$. Let $\{E_i\}_{i \geq 1}$ be a collection of pairwise disjoint measurable subsets of Ω with $E_0 = \cup_{i \geq 1} E_i$ and let $f \in L^{p,q}(\Omega, \mu)$. Then*

$$\sum_{i \geq 1} ||\chi_{E_i} f||^q_{L^{p,q}(\Omega,\mu)} \leq ||\chi_{E_0} f||^q_{L^{p,q}(\Omega,\mu)}.$$

Proof. For every $i = 0, 1, 2, \dots$ we let $f_i = \chi_{E_i} f$, where χ_{E_i} is the characteristic function of E_i. We can assume without loss of generality that all the functions f_i are nonnegative. We have (see [KKM, Proposition 2.1])

$$||f_i||^q_{L^{p,q}(\Omega,\mu)} = p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds,$$

where $\mu_{[f_i]}$ is the distribution function of $f_i, i = 0, 1, 2, \dots$. From the definition of f_0 we have

$$\mu_{[f_0]}(s) = \sum_{i \geq 1} \mu_{[f_i]}(s) \text{ for every } s > 0, \tag{9}$$

which implies, since $1 < p < q < \infty$, that

$$\mu_{[f_0]}(s)^{q/p} \geq \sum_{i \geq 1} \mu_{[f_i]}(s)^{q/p} \text{ for every } s > 0.$$

This yields

$$\begin{aligned} ||f_0||^q_{L^{p,q}(\Omega,\mu)} &= p \int_0^\infty s^{q-1} \mu_{[f_0]}(s)^{q/p} ds \geq p \int_0^\infty s^{q-1} (\sum_{i \geq 1} \mu_{[f_i]}(s)^{q/p}) ds \\ &= \sum_{i \geq 1} p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds = \sum_{i \geq 1} ||f_i||^q_{L^{p,q}(\Omega,\mu)}. \end{aligned}$$

This finishes the proof of the superadditivity in the case $1 < p < q < \infty$.

□

We have a similar result for the subadditivity of the Lorentz p, q-quasinorm. When $1 < p < q \leq \infty$ we obtain a result that generalizes [Cos, Theorem 2.5].

Proposition 2.4. *Let (Ω, μ) be a measure space. Suppose that $1 < p < q \leq \infty$. Let $\{E_i\}_{i \geq 1}$ be a collection of pairwise disjoint measurable subsets of Ω with $E_0 = \cup_{i \geq 1} E_i$ and let $f \in L^{p,q}(\Omega, \mu)$. Then*

$$\sum_{i \geq 1} ||\chi_{E_i} f||^p_{L^{p,q}(\Omega,\mu)} \geq ||\chi_{E_0} f||^p_{L^{p,q}(\Omega,\mu)}.$$

Proof. Without loss of generality we can assume that all the functions $f_i = \chi_{E_i} f$ are nonnegative. We have to consider two cases, depending on whether $p < q < \infty$ or $q = \infty$.

Suppose that $p < q < \infty$. We have (see [KKM, Proposition 2.1])

$$||f_i||^p_{L^{p,q}(\Omega,\mu)} = \left(p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds \right)^{p/q},$$

where $\mu_{[f_i]}$ is the distribution function of f_i for $i = 0, 1, 2, \ldots$. From (9) we obtain

$$\begin{aligned} ||f_0||^p_{L^{p,q}(\Omega,\mu)} &= \left(p\int_0^\infty s^{q-1}\mu_{[f_0]}(s)^{q/p}ds\right)^{p/q} \leq \sum_{i\geq 1}\left(p\int_0^\infty s^{q-1}\mu_{[f_i]}(s)^{q/p}ds\right)^{p/q} \\ &= \sum_{i\geq 1}||f_i||^p_{L^{p,q}(\Omega,\mu)}. \end{aligned}$$

Now, suppose that $q = \infty$. From (9) we obtain

$$s^p\,\mu_{[f_0]}(s) = \sum_{i\geq 1}(s^p\,\mu_{[f_i]}(s)) \text{ for every } s > 0,$$

which implies

$$s^p\,\mu_{[f_0]}(s) \leq \sum_{i\geq 1}||f_i||^p_{L^{p,\infty}(\Omega,\mu)} \text{ for every } s > 0. \tag{10}$$

By taking the supremum over all $s > 0$ in (10), we get the desired conclusion. This finishes the proof.

□

3. **Sobolev-Lorentz** p, q-**capacitance**

Suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ be an open set, $n \geq 1$. Let $K \subset \Omega$ be compact. The Sobolev-Lorentz p, q-capacitance of the conductor (K, Ω) is denoted by

$$\mathrm{cap}_{p,q}(K,\Omega) = \inf\{||\nabla u||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} : u \in W(K,\Omega)\},$$

where

$$W(K,\Omega) = \{u \in Lip_0(\Omega) : u \geq 1 \text{ in a neighborhood of } K\}.$$

We call $W(K,\Omega)$ the *set of admissible functions for the conductor* (K,Ω).

Since $W(K,\Omega)$ is closed under truncations from below by 0 and from above by 1 and since these truncations do not increase the p, q-quasinorm of the gradients whenever $1 < p < \infty$ and $1 \leq q \leq \infty$, it follows that we can choose only functions $u \in W(K,\Omega)$ that satisfy $0 \leq u \leq 1$ when computing the p, q-capacitance of the conductor (K,Ω).

Lemma 3.1. *If Ω is bounded, then we get the same p, q-capacitance for the conductor (K,Ω) if we restrict ourselves to a bigger set, namely*

$$W_1(K,\Omega) = \{u \in Lip(\Omega) \cap C(\overline{\Omega}) : u \geq 1 \text{ on } K \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

Proof. Let $u \in W_1(K,\Omega)$. We can assume without loss of generality that $0 \leq u \leq 1$. Moreover, we can also assume that $u = 1$ in an open neighborhood U of K. Let $\widetilde{U}$ be an open neighborhood of K such that $\widetilde{U} \subset\subset U$. We choose a cutoff Lipschitz function η, $0 \leq \eta \leq 1$ such that $\eta = 1$ on $\Omega \setminus U$ and $\eta = 0$ on $\widetilde{U}$. We note that $1 - \eta(1 - u) = u$. We also note that there exists a sequence of functions $\varphi_j \in Lip_0(\Omega)$ such that

$$\lim_{j\to\infty}(||\varphi_j - u||_{L^{p+1}(\Omega,m_n)} + ||\nabla\varphi_j - \nabla u||_{L^{p+1}(\Omega,m_n;\mathbf{R}^n)}) = 0.$$

Without loss of generality the sequence φ_j can be chosen such that $\varphi_j \to u$ and $\nabla\varphi_j \to \nabla u$ pointwise a.e. in Ω. Then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $W(K,\Omega)$ and

$$\lim_{j\to\infty}(||\psi_j - u||_{L^{p+1}(\Omega,m_n)} + ||\nabla\psi_j - \nabla u||_{L^{p+1}(\Omega,m_n;\mathbf{R}^n)}) = 0.$$

This, Hölder's inequality for Lorentz spaces, and the behaviour of the Lorentz p,q-quasinorm in q yield

$$\lim_{j\to\infty}(||\psi_j - u||_{L^{p,q}(\Omega,m_n)} + ||\nabla\psi_j - \nabla u||_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}) = 0.$$

The desired conclusion follows.

$\square$

3.1. **Basic properties of the p,q-capacitance.** Usually, a capacitance is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the p,q-capacitance. We follow [Cos] for (i)-(vi). In addition we will prove some superadditivity properties of the p,q-capacitance.

Theorem 3.2. *Suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ be open. The set function $K \mapsto \mathrm{cap}_{p,q}(K,\Omega)$, $K \subset \Omega$, K compact, enjoys the following properties:*

(i) *If $K_1 \subset K_2$, then $\mathrm{cap}_{p,q}(K_1,\Omega) \leq \mathrm{cap}_{p,q}(K_2,\Omega)$.*

(ii) *If $\Omega_1 \subset \Omega_2$ are open and K is a compact subset of Ω_1, then*

$$\mathrm{cap}_{p,q}(K,\Omega_2) \leq \mathrm{cap}_{p,q}(K,\Omega_1).$$

(iii) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^{\infty} K_i$, then*

$$\mathrm{cap}_{p,q}(K,\Omega) = \lim_{i\to\infty} \mathrm{cap}_{p,q}(K_i,\Omega).$$

(iv) *If Ω_i is an increasing sequence of open sets with $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and K is a compact subset of Ω_1, then*

$$\mathrm{cap}_{p,q}(K,\Omega) = \lim_{i\to\infty} \mathrm{cap}_{p,q}(K,\Omega_i).$$

(v) *Suppose that $p \leq q \leq \infty$. If $K = \bigcup_{i=1}^{k} K_i \subset \Omega$ then*

$$\mathrm{cap}_{p,q}(K,\Omega) \leq \sum_{i=1}^{k} \mathrm{cap}_{p,q}(K_i,\Omega),$$

where $k \geq 1$ is a positive integer.

(vi) *Suppose that $1 \leq q < p$. If $K = \bigcup_{i=1}^{k} K_i \subset \Omega$ then*

$$\mathrm{cap}_{p,q}(K,\Omega)^{q/p} \leq \sum_{i=1}^{k} \mathrm{cap}_{p,q}(K_i,\Omega)^{q/p},$$

where $k \geq 1$ is a positive integer.

(vii) *Suppose that $1 \leq q \leq p$. Suppose that $\Omega_1, \ldots, \Omega_k$ are k pairwise disjoint open sets and K_i are compact subsets of Ω_i for $i = 1, \ldots, k$. Then*

$$\mathrm{cap}_{p,q}(\cup_{i=1}^{k} K_i, \cup_{i=1}^{k} \Omega_i) \geq \sum_{i=1}^{k} \mathrm{cap}_{p,q}(K_i,\Omega_i).$$

(viii) *Suppose that $p < q < \infty$. Suppose that $\Omega_1, \ldots, \Omega_k$ are k pairwise disjoint open sets and K_i are compact subsets of Ω_i for $i = 1, \ldots, k$. Then*

$$\mathrm{cap}_{p,q}(\cup_{i=1}^{k} K_i, \cup_{i=1}^{k} \Omega_i)^{q/p} \geq \sum_{i=1}^{k} \mathrm{cap}_{p,q}(K_i,\Omega_i)^{q/p}.$$

(ix) *Suppose that $1 \leq q < \infty$. If Ω_1 and Ω_2 are two disjoint open sets and $K \subset \Omega_1$, then*

$$\mathrm{cap}_{p,q}(K,\Omega_1 \cup \Omega_2) = \mathrm{cap}_{p,q}(K,\Omega_1).$$

Proof. Properties (i)-(vi) are proved by duplicating the proof of [Cos, Theorem 3.2], so we will prove only (vii)-(ix).

In order to prove (vii) and (viii), it is enough to assume that $k = 2$. A finite induction on k would prove each of these claims. So we assume that $k = 2$. Let $u \in Lip_0(\Omega_1 \cup \Omega_2)$ and let $u_i = \chi_{\Omega_i} u, i = 1, 2$. We let v_i be the restriction of u to Ω_i for $i = 1, 2$. Then $v_i \in Lip_0(\Omega_i)$ for $i = 1, 2$. We note that u_i can be regarded as the extension of v_i by 0 to $\Omega_1 \cup \Omega_2$ for $i = 1, 2$. We see that $u \in W(K_1 \cup K_2, \Omega_1 \cup \Omega_2)$ if and only if $v_i \in W(K_i, \Omega_i)$ for $i = 1, 2$.

First, suppose that $1 \le q \le p$. Since Ω_1 and Ω_2 are disjoint and $u = u_1 + u_2$ with the functions u_i supported in Ω_i for $i = 1, 2$, we obtain with the help of Proposition 2.2

$$\begin{aligned} ||\nabla u||^p_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)} &\ge ||\nabla u_1||^p_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)} + ||\nabla u_2||^p_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)} \\ &= ||\nabla v_1||^p_{L^{p,q}(\Omega_1, m_n; \mathbf{R}^n)} + ||\nabla v_2||^p_{L^{p,q}(\Omega_2, m_n; \mathbf{R}^n)}. \end{aligned}$$

This proves (vii).

Now, suppose that $p < q < \infty$. Since Ω_1 and Ω_2 are disjoint and $u = u_1 + u_2$ with the functions u_i supported in Ω_i for $i = 1, 2$, we obtain with the help of Proposition 2.3

$$\begin{aligned} ||\nabla u||^q_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)} &\ge ||\nabla u_1||^q_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)} + ||\nabla u_2||^q_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)} \\ &= ||\nabla v_1||^q_{L^{p,q}(\Omega_1, m_n; \mathbf{R}^n)} + ||\nabla v_2||^q_{L^{p,q}(\Omega_2, m_n; \mathbf{R}^n)}. \end{aligned}$$

This proves (viii).

We see that (ix) follows from (vii) and (ii) when $1 \le q \le p$. (We use (vii) with $k = 2$ by taking $K_1 = K$ and $K_2 = \emptyset$.) When $p < q < \infty$, (ix) follows from (viii) and (ii). (We use (viii) with $k = 2$ by taking $K_1 = K$ and $K_2 = \emptyset$.) This finishes the proof of the theorem.

□

Remark 3.3. The definition of the p, q-capacitance implies

$$\operatorname{cap}_{p,q}(K, \Omega) = \operatorname{cap}_{p,q}(\partial K, \Omega)$$

whenever K is a compact set in Ω. Moreover, if $n = 1$ and Ω is an open interval of $\mathbf{R}$, then

$$\operatorname{cap}_{p,q}(K, \Omega) = \operatorname{cap}_{p,q}(H, \Omega),$$

where H is the smallest compact interval containing K.

4. **Conductor inequalities**

Lemma 4.1. *Suppose that $\Omega \subset \mathbf{R}^n$ is open. Let $f \in Lip_0(\Omega)$ and let $a > 1$ be a constant. For $t > 0$ we denote $M_t = \{x \in \Omega : |f(x)| > t\}$. Then the function $t \mapsto \operatorname{cap}_{p,q}(\overline{M_{at}}, M_t)$ is upper semicontinuous.*

Proof. Let $t_0 > 0$ and $\varepsilon > 0$. Let $u \in W(\overline{M_{at_0}}, M_{t_0})$ be chosen such that

$$||\nabla u||^p_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)} < \operatorname{cap}_{p,q}(\overline{M_{at_0}}, M_{t_0}) + \varepsilon.$$

Let g be an open neighborhood of $\overline{M_{at_0}}$ such that $u \ge 1$ on g. Since g contains the compact set $\overline{M_{at_0}}$, there exists $\delta_1 > 0$ small such that $g \supset \overline{M_{a(t_0 - \delta_1)}}$. Let G be an open set such that supp $u \subset G \subset\subset M_{t_0}$. There exists a small $\delta_2 > 0$ such that $\overline{G} \subset M_{t_0+\delta_2}$. Thus we have $\overline{M_{a(t_0-\delta)}} \subset g$ and $\overline{G} \subset M_{t_0+\delta}$ for every $\delta \in (0, \min\{\delta_1, \delta_2\})$. From the choice of g and G we have that $u \in W(K, \Omega)$ whenever $K \subset g$ and $\overline{G} \subset \Omega$. This and the choice of u imply that

$$\operatorname{cap}_{p,q}(\overline{M_{a(t_0-\delta)}}, M_{t_0+\delta}) \le \operatorname{cap}_{p,q}(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$$

for every $\delta \in (0, \min\{\delta_1, \delta_2\})$. Using the monotonicity of $\operatorname{cap}_{p,q}$, we deduce that

$$\operatorname{cap}_{p,q}(\overline{M_{at}}, M_t) \le \operatorname{cap}_{p,q}(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$$

for every t sufficiently close to t_0. The result follows. $\square$

Theorem 4.2. *Let Φ denote an increasing convex (not necessarily strictly convex) function given on $[0, \infty)$, $\Phi(0) = 0$. Suppose that $a > 1$ is a constant.*

(i) *If $1 \le q \le p$, then*

$$\Phi^{-1}\left(\int_0^\infty \Phi(t^p \operatorname{cap}_{p,q}(\overline{M_{at}}, M_t))\frac{dt}{t}\right) \le c(a,p,q)||\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}$$

for every $\varphi \in Lip_0(\Omega)$.

(ii) *If $p < q < \infty$, then*

$$\Phi^{-1}\left(\int_0^\infty \Phi(t^q \operatorname{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p})\frac{dt}{t}\right) \le c(a,p,q)||\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}$$

for every $\varphi \in Lip_0(\Omega)$.

Proof. The proof follows [M4]. When $p = q$ we are in the case of the p-capacitance and for that case the result was proved in [M4, Theorem 1]. So we can assume without loss of generality that $p \ne q$. Let $\varphi \in Lip_0(\Omega)$. We set

$$\Lambda_t(\varphi) = \frac{1}{(a-1)t}\min\{(|\varphi| - t)_+, (a-1)t\}.$$

From Lemma 3.1 we note that

$$\Lambda_t(\varphi) \in W_1(\overline{M_{at}}, M_t) \text{ and } |\nabla\Lambda_t(\varphi)| = \frac{1}{(a-1)t}\chi_{M_t\setminus M_{at}}|\nabla\varphi| \ m_n\text{-a.e.} \tag{11}$$

The proof splits now, depending on whether $1 \le q < p$ or $p < q < \infty$.

We assume first that $1 \le q < p$. From (11) we have

$$t^p \operatorname{cap}_{p,q}(\overline{M_{at}}, M_t) \le \frac{1}{(a-1)^p}||\chi_{M_t\setminus M_{at}}\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}.$$

Hence

$$\int_0^\infty \Phi(t^p \operatorname{cap}_{p,q}(\overline{M_{at}}, M_t))\frac{dt}{t} \le \int_0^\infty \Phi(\frac{1}{(a-1)^p}||\chi_{M_t\setminus M_{at}}\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)})\frac{dt}{t}.$$

Let γ denote a locally integrable function on $(0, \infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. Then

$$\int_0^\infty (\gamma(t) - \gamma(at))\frac{dt}{t} = (\gamma(0) - \gamma(\infty))\log a. \tag{12}$$

We set

$$\gamma(t) = \Phi(\frac{1}{(a-1)^p}||\chi_{M_t}\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}).$$

Using the monotonicity and convexity of Φ together with Proposition 2.2 and the definition of γ, we see that

$$\Phi(\frac{1}{(a-1)^p}||\chi_{M_t\setminus M_{at}}\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}) \leq \gamma(t)-\gamma(at) \text{ for every } t>0.$$

Since

$$\gamma(0)=\Phi(\frac{1}{(a-1)^p}||\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}) \text{ and } \gamma(\infty)=0,$$

we get

$$\int_0^\infty \Phi(t^p \mathrm{cap}_{p,q}(\overline{M_{at}},M_t))\frac{dt}{t} \leq \log a\cdot\Phi(\frac{1}{(a-1)^p}||\nabla\varphi||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}).$$

This finishes the proof of the case $1\leq q<p$.

Now, we assume that $p<q<\infty$. From (11) we have

$$t^q \mathrm{cap}_{p,q}(\overline{M_{at}},M_t)^{q/p} \leq \frac{1}{(a-1)^q}||\chi_{M_t\setminus M_{at}}\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}.$$

Hence

$$\int_0^\infty \Phi(t^q \mathrm{cap}_{p,q}(\overline{M_{at}},M_t)^{q/p})\frac{dt}{t} \leq \int_0^\infty \Phi(\frac{1}{(a-1)^q}||\chi_{M_t\setminus M_{at}}\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)})\frac{dt}{t}.$$

As before, we let γ denote a locally integrable function on $(0,\infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. We set

$$\gamma(t)=\Phi(\frac{1}{(a-1)^q}||\chi_{M_t}\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}).$$

Using the monotonicity and convexity of Φ together with Proposition 2.3 and the definition of γ, we see that

$$\Phi(\frac{1}{(a-1)^q}||\chi_{M_t\setminus M_{at}}\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}) \leq \gamma(t)-\gamma(at) \text{ for every } t>0.$$

Since

$$\gamma(0)=\Phi(\frac{1}{(a-1)^q}||\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}) \text{ and } \gamma(\infty)=0,$$

we get

$$\int_0^\infty \Phi(t^q \mathrm{cap}_{p,q}(\overline{M_{at}},M_t)^{q/p})\frac{dt}{t} \leq \log a\cdot\Phi(\frac{1}{(a-1)^q}||\nabla\varphi||^q_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)}).$$

This finishes the proof of the case $p<q<\infty$. The theorem is proved.

□

Choosing $\Phi(t)=t$, we arrive at the inequalities mentioned in the beginning of this paper.

Corollary 4.3. *Suppose that $1<p<\infty$ and $1\leq q<\infty$. Let $a>1$ be a constant. Then (2) and (3) hold for every $\varphi\in Lip_0(\Omega)$.*

5. **Necessary and sufficient conditions for two-weight embeddings**

Now, we derive necessary and sufficient conditions for Sobolev-Lorentz type inequalities involving two measures, generalizing results obtained in [M4] and [M5].

Theorem 5.1. *Let p, q, r, s be chosen such that $1 < p < \infty$, $1 \leq q < \infty$ and $1 < s \leq \max(p, q) \leq r < \infty$. Let Ω be an open set in $\mathbf{R}^n$ and let μ and ν be two nonnegative locally finite measures on Ω.*

(i) *Suppose that $1 \leq q \leq p$. The inequality*

$$||f||_{L^{r,p}(\Omega,\mu)} \leq A \left(||\nabla f||_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} + ||f||_{L^{s,p}(\Omega,\nu)} \right) \tag{13}$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (5) is valid for all open bounded sets g and G that are subject to $\overline{g} \subset G \subset \overline{G} \subset \Omega$.

(ii) *Suppose that $p < q < \infty$. The inequality*

$$||f||_{L^{r,q}(\Omega,\mu)} \leq A \left(||\nabla f||_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} + ||f||_{L^{s,q}(\Omega,\nu)} \right) \tag{14}$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (5) is valid for all open bounded sets g and G that are subject to $\overline{g} \subset G \subset \overline{G} \subset \Omega$.

Proof. We suppose first that $1 \leq q \leq p$. The case $q = p$ was studied in [M5]. Without loss of generality we can assume that $q < p$. We choose some bounded open sets g and G such that $\overline{g} \subset G \subset \overline{G} \subset \Omega$ and $f \in W(\overline{g}, G)$ with $0 \leq f \leq 1$. We have

$$\mu(g) \leq C(r, p) \, ||f||^r_{L^{r,p}(\Omega,\mu)}$$

and

$$||f||^s_{L^{s,p}(\Omega,\nu)} \leq C(s, p) \, \nu(G)$$

for every $f \in W(\overline{g}, G)$ with $0 \leq f \leq 1$. The necessity for $1 \leq q < p$ is obtained by taking the infimum over all such functions f that are admissible for the conductor $(\overline{g}, G)$.

We prove the sufficiency now when $1 \leq q < p$. Let $a \in (1, \infty)$. We have

$$a^p \int_0^\infty \mu(M_{at})^{p/r} d(t^p) \leq a^p K_1 \left(\int_0^\infty (\mathrm{cap}_{p,q}(\overline{M_{at}}, M_t) + \nu(M_t)^{p/s}) d(t^p) \right).$$

This and (2) yield the sufficiency for the case $1 \leq q < p$.

Now, suppose that $p < q < \infty$. We choose some bounded open sets g and G such that $\overline{g} \subset G \subset \overline{G} \subset \Omega$ and $f \in W(\overline{g}, G)$ with $0 \leq f \leq 1$. We have

$$\mu(g) \leq C(r, q) \, ||f||^r_{L^{r,q}(\Omega,\mu)}$$

and

$$||f||^s_{L^{s,q}(\Omega,\nu)} \leq C(s, q) \, \nu(G)$$

for every $f \in W(\overline{g}, G)$ with $0 \leq f \leq 1$. The necessity for $p < q < \infty$ is obtained by taking the infimum over all such functions f that are admissible for the conductor $(\overline{g}, G)$.

We prove the sufficiency now when $p < q < \infty$. Let $a \in (1, \infty)$. We have

$$a^q \int_0^\infty \mu(M_{at})^{q/r} d(t^q) \leq a^q K_2 \left(\int_0^\infty (\mathrm{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p} + \nu(M_t)^{q/s}) d(t^q) \right).$$

This and (3) yield the sufficiency for the case $p < q < \infty$. The proof is finished. $\square$

We look for a simplified necessary and sufficient two-weight imbedding condition when $n = 1$. Before we state and prove such a condition for the case $n = 1$, we need to obtain sharp estimates for the p, q-capacitance of conductors $([a, b], (A, B))$ with $A < a < b < B$. This is the goal of the following proposition.

Proposition 5.2. *Suppose that $n = 1$, $1 < p < \infty$ and $1 \leq q \leq \infty$. There exists a constant $C(p, q) \geq 1$ such that*

$$C(p,q)^{-1}(\sigma_1^{1-p} + \sigma_2^{1-p}) \leq \mathrm{cap}_{p,q}([a,b],(A,B)) \leq C(p,q)(\sigma_1^{1-p} + \sigma_2^{1-p}),$$

where $\sigma_1 = a - A$ and $\sigma_2 = B - b$.

Proof. By the behaviour of the Lorentz p, q-quasinorm in q (see for example [BS, Proposition IV.4.2]), it suffices to find the upper bound for the $p, 1$-capacitance and the lower bound for the p, ∞-capacitance of the conductor $([a, b], (A, B))$. We start with the upper bound for the $p, 1$-capacitance of this conductor.

We use the function $u : (A, B) \to \mathbf{R}$ defined by

$$u(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ \frac{x-A}{\sigma_1} & \text{if } A < x < a \\ \frac{B-x}{\sigma_2} & \text{if } b < x < B. \end{cases}$$

Then from Lemma 3.1 it follows that $u \in W_1([a, b], (A, B))$ with

$$|u'(x)| = \begin{cases} 0 & \text{if } a < x < b \\ {\sigma_1}^{-1} & \text{if } A < x < a \\ {\sigma_2}^{-1} & \text{if } b < x < B. \end{cases}$$

We want to compute an upper estimate for $||u'||_{L^{p,1}((A,B),m_1)}$. We have

$$\begin{aligned} ||u'||_{L^{p,1}((A,B),m_1)} &\leq ||{\sigma_1}^{-1}||_{L^{p,1}((A,a),m_1)} + ||{\sigma_2}^{-1}||_{L^{p,1}((b,B),m_1)} \\ &= p\left(\sigma_1^{-1+1/p} + \sigma_2^{-1+1/p}\right). \end{aligned} \tag{15}$$

Therefore

$$\mathrm{cap}_{p,1}([a,b],(A,B)) \leq C(p)(\sigma_1^{1-p} + \sigma_2^{1-p}).$$

We try to get lower estimates for the p, ∞-capacitance of this conductor. Let $v \in W([a, b], (A, B))$ be an arbitrary admissible function. We let v_1 be the restriction of v to (A, a) and v_2 be the restriction of v to (b, B) respectively. We note that v' is supported in $(A, a) \cup (b, B)$. Therefore, since v' coincides with v_1' on (A, a) and with v_2' on (b, B), we have that

$$||v'||_{L^{p,\infty}((A,B),m_1)} \geq \max(||v_1'||_{L^{p,\infty}((A,a),m_1)}, ||v_2'||_{L^{p,\infty}((b,B),m_1)}). \tag{16}$$

From ([Cos, Corollary 2.4]) we have

$$||v_1'||_{L^{p,\infty}((A,a),m_1)} \geq 1/p' \cdot \sigma_1^{-1/p'} ||v_1'||_{L^1((A,a),m_1)}$$

and

$$||v_2'||_{L^{p,\infty}((b,B),m_1)} \geq 1/p' \cdot \sigma_2^{-1/p'} ||v_2'||_{L^1((b,B),m_1)}.$$

Since

$$||v_1'||_{L^1((A,a),m_1)} = \int_A^a |v_1'(x)| dx \geq 1,$$

we obtain

$$(17)\qquad ||v_1'||_{L^{p,\infty}((A,a),m_1)} \geq 1/p' \cdot \sigma_1^{-1/p'}.$$

Similarly, since

$$||v_2'||_{L^1((b,B),m_1)} = \int_b^B |v_2'(x)|dx \geq 1,$$

we obtain

$$(18)\qquad ||v_2'||_{L^{p,\infty}((b,B),m_1)} \geq 1/p' \cdot \sigma_2^{-1/p'}.$$

From (16), (17) and (18) we get the desired lower bound for the p,∞-capacitance. This finishes the proof.

□

Now we state and prove a necessary and sufficient two-weight imbedding condition for the case $n = 1$.

Theorem 5.3. *Suppose that $n = 1$. Let p, q, r, s be chosen such that $1 < p < \infty$, $1 \leq q < \infty$ and $1 < s \leq \max(p, q) \leq r < \infty$. Let Ω be an open set in $\mathbf{R}$ and let μ and ν be two nonnegative locally finite measures on Ω.*

(i) *Suppose that $1 \leq q \leq p$. The inequality*

$$(19)\qquad ||f||_{L^{r,p}(\Omega,\mu)} \leq A\left(||f'||_{L^{p,q}(\Omega,m_1)} + ||f||_{L^{s,p}(\Omega,\nu)}\right)$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (7) is valid whenever x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

(ii) *Suppose that $p < q < \infty$. The inequality*

$$(20)\qquad ||f||_{L^{r,q}(\Omega,\mu)} \leq A\left(||f'||_{L^{p,q}(\Omega,m_1)} + ||f||_{L^{s,q}(\Omega,\nu)}\right)$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (7) is valid whenever x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

Proof. We only have to prove that the sufficiency condition for intervals implies the sufficiency condition for general bounded and open sets g and G with $\overline{g} \subset G \subset \overline{G} \subset \Omega$. Let G be the union of nonoverlapping intervals G_i and let $g_i = G \cap g_i$. We denote by h_i the smallest interval containing g_i and by τ_i the minimal distance from h_i to $\mathbf{R} \setminus G_i$. We also denote by H_i the open interval concentric with h_i such that the minimal distance from h_i to $\mathbf{R} \setminus H_i$ is τ_i. Then $H_i \subset G_i$. From Remark 3.3 we have that $\mathrm{cap}_{p,q}(\overline{g_i}, G_i) = \mathrm{cap}_{p,q}(\overline{h_i}, G_i)$. Moreover, from Theorem 3.2 (ii) and Proposition 5.2 we have

$$C(p,q)^{-1}\tau_i^{1-p} \leq \mathrm{cap}_{p,q}(\overline{h_i}, G_i) \leq \mathrm{cap}_{p,q}(\overline{h_i}, H_i) \leq 2\,C(p,q)\tau_i^{1-p}$$

for some constant $C(p,q) \geq 1$. Since $\overline{g}$ is compact lying in $\cup_{i\geq 1} G_i$, it follows that $\overline{g}$ is covered by only finitely many of the sets G_i. This and Theorem 3.2 (ix) allow us to assume that G is in fact written as a finite union of disjoint intervals G_i. Now the proof splits, depending on whether $1 \leq q \leq p$ or $p < q < \infty$.

We assume first that $1 \leq q \leq p$. Then

$$(21)\qquad \mathrm{cap}_{p,q}(\overline{g}, G) \geq \sum_i \mathrm{cap}_{p,q}(\overline{g_i}, G_i) = \sum_i \mathrm{cap}_{p,q}(\overline{h_i}, G_i).$$

Using (7), we obtain

$$\begin{aligned}\mu(g_i)^{p/r} &\leq \mu(h_i)^{p/r} \leq K_1(\tau_i^{1-p} + \nu(H_i)^{p/s}) \\ &\leq K_1\, C(p,q)(\operatorname{cap}_{p,q}(\overline{g_i}, G_i) + \nu(G_i)^{p/s})\end{aligned}$$

where K_1 is a positive constant independent of g and G. Since $s \leq p \leq r < \infty$, we have

$$\mu(g)^{p/r} \leq \sum_i \mu(g_i)^{p/r}$$

and

$$\sum_i \nu(G_i)^{p/s} \leq \nu(G)^{p/s}.$$

This and (21) prove the claim when $1 \leq q \leq p$.

We assume now that $p < q < \infty$. Then

$$\operatorname{cap}_{p,q}(\overline{g}, G)^{q/p} \geq \sum_i \operatorname{cap}_{p,q}(\overline{g_i}, G_i)^{q/p} = \sum_i \operatorname{cap}_{p,q}(\overline{h_i}, G_i)^{q/p}. \tag{22}$$

Using (7), we obtain

$$\begin{aligned}\mu(g_i)^{q/r} &\leq \mu(h_i)^{q/r} \leq K_2(\tau_i^{q(1-p)/p} + \nu(H_i)^{q/s}) \\ &\leq K_2\, C(p,q)^{q/p}(\operatorname{cap}_{p,q}(\overline{g_i}, G_i)^{q/p} + \nu(G_i)^{q/s})\end{aligned}$$

where K_2 is a positive constant independent of g and G. Since $s \leq q \leq r < \infty$, we have

$$\mu(g)^{q/r} \leq \sum_i \mu(g_i)^{q/r}$$

and

$$\sum_i \nu(G_i)^{q/s} \leq \nu(G)^{q/s}.$$

This and (22) prove the claim when $p < q < \infty$. The theorem is proved. $\square$

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