Carleson Measures and Interpolating Sequences for Besov Spaces on Complex Balls

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Abstract

We characterize Carleson measures for the analytic Besov spaces B_p on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ in terms of a discrete tree condition on the associated Bergman tree $\mathcal{T}_n$. We also characterize the pointwise multipliers on B_p in terms of Carleson measures. We then apply these results to characterize the interpolating sequences in $\mathbb{B}_n$ for B_p and their multiplier spaces M_{B_p}, generalizing a theorem of Böe in one dimension. The interpolating sequences for B_p and for M_{B_p} are precisely those sequences satisfying a separation condition and a Carleson embedding condition. These results hold for $1 < p < \infty$ with the exceptions that for $2 + \frac{1}{n-1} \leq p < \infty$, the necessity of the tree condition for the Carleson embedding is left open, and for $2 + \frac{1}{n-1} \leq p \leq 2n$, the sufficiency of the separation condition and the Carleson embedding for multiplier interpolation is left open; the separation and tree conditions are however sufficient for multiplier interpolation. Novel features of our proof of the interpolation theorem for M_{B_p} include the crucial use of the discrete tree condition for sufficiency, and a new notion of holomorphic Besov space on a Bergman tree, one suited to modeling spaces of holomorphic functions defined by the size of higher order derivatives, for necessity.

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CARLESON MEASURES AND INTERPOLATING SEQUENCES

1. Introduction

In this paper we consider the analytic Besov spaces $B_p(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, consisting of those holomorphic functions f on the ball such that

$$\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^m f^{(m)}(z) \right|^p d\lambda_n(z) < \infty,$$

where $m > \frac{n}{p}$, $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dz$ is invariant measure on the ball with dz Lebesgue measure on $\mathbb{C}^n$, and $f^{(m)}$ is the m^{th} order complex derivative of f. We characterize their Carleson measures (except for p in an exceptional range $\left[2 + \frac{1}{n-1}, \infty\right)$), pointwise multipliers and interpolating sequences. We also characterize interpolating sequences for the corresponding pointwise multiplier spaces $M_{B_p(\mathbb{B}_n)}$ (except for p in the smaller exceptional range $\left[2 + \frac{1}{n-1}, 2n\right]$). Finally, in order to obtain the characterization of interpolating sequences for $M_{B_p(\mathbb{B}_n)}$ in the difficult range $1 + \frac{1}{n-1} \leq p < 2$, we introduce "holomorphic" Besov spaces $HB_p(\mathcal{T}_n)$ on the Bergman trees $\mathcal{T}_n$, and develop the necessary part of the analogous theory of Carleson measures, pointwise multipliers and interpolating sequences. The main feature of these holomorphic Besov spaces on Bergman trees is that they provide a martingale-like analogue of the analytic Besov space on the ball. They also enjoy properties not found in the actual Besov spaces, such as reproducing kernels with a positivity property for *all* p in the range $1 < p < \infty$ (Lemma 8.11 below). Various solutions to the problems mentioned here in one dimension can be found in [**Car**], [**MaSu**], [**Wu**], [**Boe**] and our earlier paper [**ArRoSa**]. We remark that the one-dimensional methods for characterizing multiplier interpolation generalize for $n > 1$ to prove necessity at most in the ranges $1 < p < 1 + \frac{1}{n-1}$ and $p > 2$, and sufficiency at most in the range $p > 2n$, resulting in a common range of only $p > 2n$ for all $n > 1$.

1.1. History. We begin with an informal discussion of the context in which our results can be viewed. For more details on this background (at least the part having to do with Hilbert spaces) we refer to the beautiful recent monograph of K. Seip [**Sei**].

The theory of Carleson measures and interpolating sequences has its roots in Lennart Carleson's 1958 paper [**Car**], the first of his papers motivated by the corona problem for the Banach algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions in the unit disk $\mathbb{D}$: if $\{f_j\}_{j=1}^J$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$\sum_{j=1}^J |f_j(z)| \geq c > 0, \qquad z \in \mathbb{D},$$

are there are functions $\{g_j\}_{j=1}^J$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^J f_j(z)\, g_j(z) = 1, \qquad z \in \mathbb{D},$$

i.e., is every multiplicative linear functional on $H^\infty(\mathbb{D})$ in the closure of the point evaluations, so that there is no "corona"? In [**Car**], Carleson observed the following connection between the corona problem and interpolating sequences. A Blaschke product B_0 has the "baby corona" property,

$$\begin{aligned}&\text{For all } f_1 \in H^\infty(\mathbb{D}) \text{ satisfying } \inf_{z\in\mathbb{D}} \{|B_0(z)| + |f_1(z)|\} > 0, \\ &\text{there are } g_0, g_1 \in H^\infty(\mathbb{D}) \text{ with } B_0 g_0 + f_1 g_1 \equiv 1,\end{aligned} \tag{1.1}$$

if the zero set

$$Z_0 = \{z \in \mathbb{D} : B_0(z) = 0\} = \{z_j\}_{j=1}^\infty$$

of B_0 is an interpolating sequence for $H^\infty(\mathbb{D})$:

$$\text{The map } f \to \{f(z_j)\}_{j=1}^\infty \text{ takes } H^\infty(\mathbb{D}) \text{ boundedly into and onto } \ell^\infty(Z_0). \tag{1.2}$$

Carleson solved this latter problem completely by showing that a sequence $Z = \{z_j\}_{j=1}^\infty$ is an interpolating sequence for $H^\infty(\mathbb{D})$ if and only if

$$\prod_{j:j\neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \geq c > 0, \qquad k = 1, 2, 3, ... \tag{1.3}$$

The proof made crucial use of Blaschke products and duality. In the same paper he showed implicitly that the characterizing condition (1.3) can be rephrased in modern language as

$$\begin{aligned}&\left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \geq c > 0 \text{ for } j \neq k, \text{ and} \\ &\sum_{j=1}^\infty \left(1 - |z_j|^2\right) \delta_{z_j} \text{ is a Carleson measure for } H^p(\mathbb{D}),\end{aligned}$$

where a positive Borel measure μ on the disk $\mathbb{D}$ is now said to be a Carleson measure for $H^p(\mathbb{D})$ if the embedding $H^p(\mathbb{D}) \subset L^p(d\mu)$ holds. Carleson later solved the corona problem affirmatively in [**Car2**] by demonstrating the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$.

In 1961, H. Shapiro and A. Shields [**ShSh**] demonstrated that the interpolation property (1.2) is equivalent to weighted interpolation for Hardy spaces $H^p(\mathbb{D})$; that is, the map

$$f \to \left\{ \left(1 - |z_j|^2\right)^{\frac{1}{p}} f(z_j) \right\}_{j=1}^\infty$$

maps $H^p(\mathbb{D})$ boundedly into and onto $\ell^p(Z_f)$.The factor $\left(1-|z_j|^2\right)^{\frac{1}{p}}$ forces the map to be into $\ell^\infty(Z_f)$; the Carleson measure condition ensures that it is into $\ell^p(Z_f)$.

We now recast the case $p=2$ of this result in a way that will emphasize the analogy with what comes later. The Hardy space $H^2(\mathbb{D})$ is a Hilbert space with reproducing kernel. This means that for each $z\in\mathbb{D}$, there is $k_z\in H^2(\mathbb{D})$, the reproducing kernel for z, which is characterized by the fact that for any $f\in H^2(\mathbb{D})$ we have $f(z)=\langle f,k_z\rangle$. A sequence $Z=\{z_j\}_{j=1}^\infty$ is an interpolating sequence for $H^2(\mathbb{D})$ if one can freely assign the values of an $H^2(\mathbb{D})$ function on Z, subject only to the natural size restriction. More precisely, Hilbert space basics ensure that if $f\in H^2(\mathbb{D})$, then the function

$$z_i\to\frac{f(z_i)}{\|k_{z_i}\|}=\left(1-|z_i|^2\right)^{\frac{1}{2}}f(z_i)$$

is a bounded function on Z. The sequence Z is called an interpolating sequence for $H^2(\mathbb{D})$ if all the functions on Z which are obtained in this way are in $\ell^2(Z)$, and if furthermore, every function in $\ell^2(Z)$ can be obtained in this way. For any $z\in\mathbb{D}$, set $\widetilde{k_z}=\frac{k_z}{\|k_z\|}$, and note that by the Cauchy-Schwarz inequality,

$$\left|\left\langle\widetilde{k_z},\widetilde{k_w}\right\rangle\right|\le 1,\qquad z,w\in\mathbb{D}.$$

If Z is an interpolating sequence, then it must be possible, given any i and j, to find $f\in H^2(\mathbb{D})$ so that

$$\left(1-|z_i|^2\right)^{\frac{1}{2}}f(z_i)=0,\qquad\left(1-|z_j|^2\right)^{\frac{1}{2}}f(z_j)=1;$$

and to do this with control on the size of $\|f\|$. This implies a weak separation condition on the points of Z which is necessary for Z to be an interpolating sequence: there is $\varepsilon>0$ so that for all $i\ne j$, $\left|\left\langle\widetilde{k_{z_i}},\widetilde{k_{z_j}}\right\rangle\right|\le 1-\varepsilon$. An equivalent geometric statement is that there is a uniform lower bound on the hyperbolic distances $\beta(z_i,z_j)$. The result of Shapiro and Shields can now be restated as saying that interpolating sequences for $H^2(\mathbb{D})$ are characterized by the following two conditions:

$$\text{There is }\varepsilon>0\text{ so that }\left|\left\langle\widetilde{k_{z_i}},\widetilde{k_{z_j}}\right\rangle\right|\le 1-\varepsilon\text{ for all }i\ne j,\tag{1.4}$$

and

$$\sum_{j=1}^\infty\left\|k_{z_j}\right\|^{-2}\delta_{z_j}\text{ is a Carleson measure for }H^2(\mathbb{D}).\tag{1.5}$$

Interpolation problems, multiplier questions and Carleson measure characterizations were then studied by various authors in other classical function spaces on the disk, including certain of the spaces $B_p^\alpha(\mathbb{D})$ normed by

$$\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|+\left\{\int_{\mathbb{D}}\left|\left(1-|z|^2\right)^{m+\alpha}f^{(m)}(z)\right|^p d\lambda(z)\right\}^{\frac{1}{p}},$$

where $d\lambda(z)=\left(1-|z|^2\right)^{-2}dz$ is invariant measure on the disk, and for fixed α and p, the norms are equivalent for $(m+\alpha)p>1$. This scale of spaces includes the Hardy space $H^2(\mathbb{D})=B_2^{\frac{1}{2}}(\mathbb{D})$ with $\alpha=\frac{1}{2}$, the weighted Bergman spaces with $\alpha > \frac{1}{p}$, and the weighted Dirichlet-type spaces with $0 < \alpha < \frac{1}{p}$. See for example the recent book by K. Seip [**Sei**], which contains an in depth discussion of the history of interpolating sequences for Hilbert spaces of functions of a single variable. Interpolation proved more difficult for the family of analytic Besov spaces $B_p(\mathbb{D}) = B_p^0(\mathbb{D})$ on the disk, the prototypical Möbius invariant spaces, which do not admit *any* infinite Blaschke products - the Dirichlet norm $\|f\|_{B_2(\mathbb{D})}$ measures the square root of the area of the range of f counting multiplicities, and so is infinite for every infinite Blaschke product f. These Besov spaces are also distinguished by being the limit of those spaces $B_p^\alpha(\mathbb{D})$ with $\alpha < 0$ that are too smooth (they admit continuous extensions to the closed disk $\overline{\mathbb{D}}$) to contain any infinite interpolating sequences.

In a revolutionary paper in 1994, D. Marshall and C. Sundberg [**MaSu**] used Hilbert space methods (and independently C. Bishop [**Bis**] used different techniques) to characterize interpolating sequences for the Dirichlet space $B_2(\mathbb{D})$ and its multiplier space $M_{B_2(\mathbb{D})}$ (note the connection $H^\infty(\mathbb{D}) = M_{H^2(\mathbb{D})}$) by the condition

$$\begin{aligned}&\beta(z_i, 0) \leq C\beta(z_i, z_j) \text{ for } i \neq j \text{ and}\\ &\sum_{j=1}^\infty \left(1 + \log \frac{1}{1 - |z_j|^2}\right)^{-1} \delta_{z_j} \text{ is a } B_2(\mathbb{D})\text{-Carleson measure,}\end{aligned}$$

where β is the Bergman metric, and a positive Borel measure μ is a $B_2(\mathbb{D})$-Carleson measure if the embedding $B_2(\mathbb{D}) \subset L^2(d\mu)$ holds:

$$\int |f(z)|^2 d\mu(z) \leq C \|f\|_{B_2(\mathbb{D})}^2 .$$

In fact, these two conditions can be rewritten in exactly the same form as (1.4) and (1.5) with only the natural changes; the $\widetilde{k}'s$ must now be normalized reproducing kernels for the Dirichlet space and the measure must be a Dirichlet space Carleson measure.

More recently, in 2002 in [**Boe**], B. Böe has extended this theorem to all $1 < p < \infty$ by a long and clever construction involving Carleson measures, that was in turn based on an earlier construction in [**MaSu**] (see also the analogous construction on trees in Section 6 of [**ArRoSa**]), together with, in Böe's words, a "curious lemma" on unconditional basic sequences $\{f_j\}_{j=1}^\infty$ of positive functions in a Lebesgue space $L^q(d\mu)$ (Lemma 5.11 below):

$$\left\|\sum_{j=1}^\infty |a_j f_j|\right\|_{L^q(d\mu)} \approx C_q \left\|\sup_{j\geq 1} |a_j f_j|\right\|_{L^q(d\mu)} .$$

In this paper, we extend Böe's results to the analytic Besov spaces $B_p(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ for $n > 1$. As far as we know this represents the first solution of its kind in dimension greater than one. We note that the corresponding questions for the Hardy spaces on the ball remain open in higher dimensions, due in part to the lack of Blaschke products, but also since the relevant separation condition fails to be sparse enough to accommodate the "hands-on" type of construction used by Böe. We do not treat the endpoint spaces $B_1(\mathbb{B}_n)$ and $B_\infty(\mathbb{B}_n)$, which are respectively the minimal and maximal Möbius invariant spaces on the ball. Interesting work on interpolating sequences for $B_\infty(\mathbb{D})$ is in [**BoNi**] and is reported in [**Sei**].

At least two difficulties arise immediately in higher dimensions. Böe makes use of Stegenga's 1980 characterization [**Ste**] of $B_2(\mathbb{D})$-Carleson measures by a capacity condition, as well as later extensions to $p>1$:

$$\mu(T(E)) \leq C\, cap_p(E),$$

for all compact subsets E (or equivalently finite unions of arcs) of the circle $\mathbb{T}$, and where $T(E)$ denotes the Carleson tent associated to E, and

$$cap_p(E) = \inf\left\{\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)^p d\theta : f \geq 0 \text{ and } \int_{-\pi}^{\pi} f\left(e^{i(\phi-\theta)}\right)|\theta|^{-\frac{1}{2}} d\theta \geq \chi_E(\phi)\right\}.$$

This characterization is not yet available in higher dimensions, and as indicated in [**Boe**], seems difficult to check even in certain one-dimensional situations. Instead, we will extend our characterization in [**ArRoSa**] involving the discrete Bergman tree condition (Here and throughout p' is the conjugate exponent to p: $\frac{1}{p}+\frac{1}{p'}=1$.),

$$\sum_{\beta\in\mathcal{T}:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}:\gamma\geq\beta}\mu(\gamma)\right)^{p'} \leq C^{p'}\sum_{\beta\in\mathcal{T}:\beta\geq\alpha}\mu(\alpha)<\infty, \qquad \alpha\in\mathcal{T}, \tag{1.6}$$

to higher dimensions where it will play a crucial role both as a substitute for a capacity condition, and in generalizing the clever Carleson measure construction of Böe in [**Boe**]. We also remark that for $p>\widehat{n}=\begin{cases} 1 & \text{if} \quad n=1 \\ 2n & \text{if} \quad n>1 \end{cases}$, we characterize multiplier interpolation without recourse to any capacity or tree condition.

The second difficulty runs deeper. It is connected to the fact that the reproducing kernel $k_w(z)=\log\frac{1}{1-\overline{w}z}$ for $B_p(\mathbb{D})$ has derivative $\overline{w}\frac{1}{1-\overline{w}z}$ where $\frac{1}{1-\overline{w}z}$ has positive real part, and that this positivity played a crucial role in part of Böe's argument when $p<2$. In particular his "curious lemma", which deals with positive functions, is applied to those real parts. This property persists in dimension n only for $1<p<1+\frac{1}{n-1}$, where the analogous derivative $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}}$ of the reproducing kernel $k^{\alpha,p}_w(z)$ is

$$\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}}k^{\alpha,p}_w(z) = (1-\overline{w}\cdot z)^{-\frac{n+1+\alpha}{p'}}, \qquad \alpha>-1,$$

which has positive real part only when $\frac{n+1+\alpha}{p'}\leq 1$ for some $\alpha>-1$, i.e. $p<1+\frac{1}{n-1}$ (see (2.13) below).

As a consequence, the aforementioned "curious lemma" of Böe only generalizes to prove the necessity of the discrete tree condition for $M_{B_p(\mathbb{B}_n)}$ interpolation in the thin range $1<p<1+\frac{1}{n-1}$ (where reproducing kernels for $B_p(\mathbb{B}_n)$ have the requisite positivity property). To combat the failure of this positivity property for larger p, we introduce "holomorphic" Besov spaces $HB_p(\mathcal{T}_n)$ on Bergman trees $\mathcal{T}_n$ whose reproducing kernels *do* enjoy a suitable positivity property, and such that the restriction map from $B_p(\mathbb{B}_n)$ to $HB_p(\mathbb{B}_n\mathcal{T}_n)$, as well as the restriction map between their multiplier spaces, is bounded. This requires a great deal of effort and is accomplished in the latter half of the paper. Another consequence is that our one-dimensional proof of the characterization of Carleson measures by the discrete tree condition extends to dimension n only in the thin range of p

given by $1 < p < 1 + \frac{1}{n-1}$. A TT^* argument lifts the proof to the larger range $1 < p < 2 + \frac{1}{n-1}$, beyond which we are unable to proceed at this time.

1.2. Plan of the paper. In Section 2 we introduce a tree structure for the unit ball $\mathbb{B}_n$ by choosing a set $\mathcal{T}_n$ of points in the ball at roughly a fixed distance apart in the Bergman metric, and declaring a point $\beta \in \mathcal{T}_n$ to be a child of another point $\alpha \in \mathcal{T}_n$ if the Bergman ball around β lies just "beyond" the Bergman ball around α. This simple construction suffices for dealing with Carleson measures and sufficient conditions for interpolation in Sections 3 through 5. The construction must be significantly refined in order to deal with the holomorphic Besov spaces on trees in Section 8, and this is carried out in Subsection 8.5. The refinement allows us to develop an effective discrete version of passing from spaces defined by a single derivative to spaces of functions defined using higher derivatives.

This tree structure $\mathcal{T}_n$ is then used in Section 3 to characterize Carleson embeddings for Besov spaces $B_p(\mathbb{B}_n)$ on the ball,

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \leq C \, \|f(z)\|^p_{B_p(\mathbb{B}_n)},$$

in terms of a discrete condition on the Bergman tree (1.6), or in different notation,

$$\sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} I^*\mu(\beta)^{p'} \leq C^{p'} I^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n,$$

where

$$I^*\mu(\beta) = \sum_{\gamma \in \mathcal{T}_n : \gamma \geq \beta} \mu(\gamma).$$

However we are unable to obtain the necessity of this tree condition when $2 + \frac{1}{n-1} \leq p < \infty$. It turns out that the one-dimensional methods in [**ArRoSa**], using a positivity property of the reproducing kernels, generalize to obtain the characterization in the thin range $1 < p < 1 + \frac{1}{n-1}$. A standard TT^* argument can be used to obtain the case $p = 2$ since the kernel $K(z, w)$ of TT^* turns out to have appropriate "derivative" $\log\left(\frac{1}{1 - \overline{w} \cdot z}\right)$, whose real part is positive. We combine the two techniques to obtain the larger range $1 < p < 2 + \frac{1}{n-1}$ in Theorem 3.1.

Pointwise multipliers of $B_p(\mathbb{B}_n)$ are characterized in Theorem 4.2 in terms of Carleson embeddings in the short Section 4, where no use is made of the underlying tree structure.

Interpolating sequences for both Besov spaces $B_p(\mathbb{B}_n)$ and their multiplier spaces $M_{B_p(\mathbb{B}_n)}$ are considered in the lengthy Section 5, where for the most part, we follow the development in Böe [**Boe**]. In Theorem 5.1, weighted $B_p(\mathbb{B}_n)$ interpolation is characterized by separation and Carleson embedding conditions for all $1 < p < \infty$:

$$\beta(z_i, 0) \leq C\beta(z_i, z_j), \, i \neq j \text{ and}$$
$$\sum_{j=1}^{\infty} \left\| k^{\alpha,p}_{z_j} \right\|^{-p}_{B_{p'}} \delta_{z_j} \text{ is a } B_p\text{-Carleson measure,}$$

where $k_w^{\alpha,p}(z)$ is a reproducing kernel for B_p. In Theorem 5.2, the separation and tree conditions,

$$d(\alpha_i, o) \leq C d(\alpha_i, \alpha_j), i \neq j \text{ and}$$
$$\sum_{j=1}^{\infty}\left(1+\log\frac{1}{1-\left|c_{\alpha_j}\right|^2}\right)^{1-p}\delta_{c_{\alpha_j}} \text{ satisfies the tree condition (3.2),}$$

are proved sufficient for $M_{B_p(\mathbb{B}_n)}$ interpolation, and the separation and Carleson embedding conditions above are proved necessary, for all $1<p<\infty$. As well, in the range $p>2n$, we prove that the separation and Carleson embedding conditions are sufficient. The necessity of the Carleson embedding condition is proved first for p in the two ranges $\left(1,1+\frac{1}{n-1}\right)$ and $[2,\infty)$. The first range exploits the positivity of a reproducing kernel on the ball, and the second range exploits the embedding of ℓ^q spaces in connection with Khinchine's inequality.

Necessity of the Carleson embedding condition for $M_{B_p(\mathbb{B}_n)}$ interpolation in the remaining range $1+\frac{1}{n-1}\leq p<2$ is much more difficult, and is the subject of the remaining Sections 6, 7, 8 and 9 of the paper. In Section 6 we begin by introducing an alternate characterization of Besov spaces using almost invariant holomorphic derivatives,

$$D_a f(z) = -f'(z)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\},$$

in order to obtain global and local oscillation inequalities for $B_p(\mathbb{B}_n)$ in Proposition 6.5. Global oscillation inequalities are given in Peloso [**Pel**], but we need local versions,

$$\sup_{\xi\in K_\alpha}\left|f(\xi)-\sum_{k=0}^{m-1}\frac{((\xi-a_\alpha)\cdot')^k f(a_\alpha)}{k!}\right| \leq C\left(\int_{K_\alpha^*}\left|D_{c_\alpha}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}},$$

as well in order to estimate Carleson measures.

In Sections 7 and 8 we define Besov spaces $B_p(\mathcal{T})$ on abstract trees $\mathcal{T}$, and holomorphic Besov spaces $HB_p(\mathcal{T}_n)$ on Bergman trees $\mathcal{T}_n$ respectively. One main point is that Besov spaces built on trees typically have reproducing kernels with appropriate positivity properties that result from the "unidirectional arrow" of the tree structure (the descendents of a given tree element α lie in an appropriate nonisotropic cone with vertex at α). The positivity lets us use techniques similar to Böe's "curious lemma". While the simple abstract Besov spaces $B_p(\mathcal{T})$ do indeed have the appropriate positivity property, when $n>1$ they fail to capture enough of the holomorphic structure of the ball to permit the corresponding restriction map

$$f \to \{f(\alpha)\}_{\alpha\in\mathcal{T}_n}$$

to be bounded from $B_p(\mathbb{B}_n)$ to $B_p(\mathcal{T})$, except when p is large, $p>2n$. In Remark 7.13, it is shown that even linear functions on the ball fail to restrict boundedly to abstract trees in the range $p\leq 2n$.

To get a better tree model of the $B_p(\mathbb{B}_n)$ we need to take fuller account of the complex structure $\mathcal{T}_n$ inherits as a subset of the ball $\mathbb{B}_n$, including the nonisotropic nature of the ball's geometry. Here is the fundamental difference. Suppose that the function $\{f(\alpha)\}_{\alpha\in\mathcal{T}_n}$ on the tree is obtained by restricting a holomorphic function

f. For a typical $\alpha \in \mathcal{T}_n$ the set of forward differences

$$\triangle_\alpha = \{(\beta, f(\beta) - f(\alpha))\}_{\beta \text{ is a child of } \alpha}$$

has cardinality $N \approx 2^n$. However, to the accuracy of the linear Taylor approximation of f at α, those numbers are determined by the n-tuple of numbers $f'(\alpha)$ and the set of complex numbers $\{\beta - \alpha\}_{\beta \text{ is a child of } \alpha}$, a set which is determined by the tree and doesn't depend on f. Thus the set of forward differences $\triangle_\alpha$, *a priori* in N-space, lives approximately in an n-dimensional subspace determined by $f'(\alpha)$. Hence when we define the norm of f in $HB_p(\mathcal{T}_n)$, for f a function defined only on the tree, we will measure the size of a quantity, which we also denote by $f'(\alpha)$, an n-tuple which will be our proxy for the derivative. We then also measure the extent to which the N-tuple of differences are approximated by the N-tuple with β^{th} entry $f'(\alpha)(\beta - \alpha)$. More precisely, given f on the tree, we project the family of differences $\{f(\beta) - f(\alpha)\}_\beta$, associated to a point α and its children β, onto the finite linear space M_α of differences generated by linear functions on the ball at the point α. This permits the association of a "complex derivative" $f'(\alpha)$ to f at the point α defined by

$$\{f'(\alpha)(\beta - \alpha)\}_\beta = \textit{projection } \{f(\beta) - f(\alpha)\}_\beta .$$

In order to pass to p with $\frac{n}{p}$ large, and hence in defining $B_p(\mathbb{B}_n)$ using higher order derivatives, we must extend our definition of derivative on a tree to higher order "tree derivatives". This requires the introduction of tensors, and other constructions on the Bergman tree, that mimic the development of complex geometry on manifolds. Finally, we norm these spaces by taking ℓ^p norms of the higher order "derivatives", weighted nonisotropically according to radial and tangential directions, and by taking a large norm of the "nonholomorphic" part of the differences, that is, of the residue after f is approximated by an "m^{th} order tree Taylor theorem". More precisely, we use the unweighted ℓ^p norm of the projections of $\{f(\beta) - f(\alpha)\}_\beta$ onto the orthogonal complement of the linear space of differences generated by polynomials of degree at most m on the ball at the point α.

The global and local oscillation inequalities mentioned above are then used in Theorem 8.14 to prove that the restriction map from the ball to the Bergman tree is bounded from $B_p(\mathbb{B}_n)$ to $HB_p(\mathcal{T}_n)$, and provided $1 < p < 2 + \frac{1}{n-1}$, from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_p(\mathcal{T}_n)}$ as well:

$$\begin{aligned} \|f\|_{HB_p(\mathcal{T}_n)} &\leq C \|f\|_{B_p(\mathbb{B}_n)} , \\ \|f\|_{M_{HB_p(\mathcal{T}_n)}} &\leq C \|f\|_{M_{B_p(\mathbb{B}_n)}} . \end{aligned}$$

These restrictions also require a delicate refinement of the construction of the Bergman tree, and as we mentioned above, this is carried out in Subsection 8.5. In Section 9 we use multiplier restriction to transfer questions of multiplier interpolation from the ball to the tree in this range of p, where many obstructions disappear, and in particular we can exploit the positivity property (Lemma 8.11) of reproducing kernels for $HB_p(\mathcal{T}_n)$, valid for *all* $1 < p < \infty$.

Curiously, one property proves more difficult to obtain on the Bergman tree $\mathcal{T}_n$ than on the ball $\mathbb{B}_n$ - the property that pointwise multipliers are characterized by Carleson embeddings. Compare the easy proof of Theorem 4.2 on the ball to the lengthy proof of Lemma 8.17 on the tree. The main difficulties arise from the fact that "derivatives" on the tree are not derivations, resulting in an error term in the corresponding Leibniz formula on the tree.

Open problems.

(1) Does the tree condition characterize Carleson measures on the ball in the missing range $2+\frac{1}{n-1}\leq p<\infty$? The situation for the somewhat similar Hardy-Sobolev spaces on the ball is known to be complicated. See e.g. [**Ahe**] and [**CaOr**].
(2) Find a characterization of $M_{B_p(\mathbb{B}_n)}$ interpolating sequences on the ball for the missing range $2+\frac{1}{n-1}\leq p\leq 2n$. We know that separation and the tree condition are sufficient, and that separation and the Carleson embedding are necessary.
(3) It may be that the restriction map of $B_p(\mathbb{B}_n)$ to the sequence space maps *onto* the weighted ℓ^p space even though it is not a *bounded map into*. What is a geometric characterization of the sequences for which this happens? Such a phenomenon does not occur for the Hardy spaces.
(4) There are Hilbert space considerations that apply to both the Hardy and Dirichlet space, which when applicable, ensure that a Hilbert space of functions and its multiplier algebra have the same interpolating sequences. For details see [**AgMc**] or [**Boe**]. The results here and those being generalized suggest that there may be a similar result for some class of Banach spaces of functions.

1.3. Earlier results. We now recall some of our earlier results on Carleson measures for analytic Besov spaces $B_p(\mathbb{D})$ on the unit disk $\mathbb{D}$, as well as for certain $B_p(\mathcal{T})$ spaces on trees $\mathcal{T}$. By a tree we mean a connected loopless graph $\mathcal{T}$ with a root o and a partial order $\leq$ defined by $\alpha\leq\beta$ if α belongs to the geodesic $[o,\beta]$. See for example [**ArRoSa**] for more details. We define $B_p(\mathbb{D})$ and $B_p(\mathcal{T})$ respectively by the norms

$$\|f\|_{B_p(\mathbb{D})}=\left(\int_{\mathbb{D}}\left|\left(1-|z|^2\right)f'(z)\right|^p\frac{dz}{\left(1-|z|^2\right)^2}\right)^{\frac{1}{p}}+|f(0)|,$$

for f holomorphic on $\mathbb{D}$, and

$$\|f\|_{B_p(\mathcal{T})}=\left(\sum_{\alpha\in\mathcal{T}:\alpha\neq o}|f(\alpha)-f(A\alpha)|^p\right)^{\frac{1}{p}}+|f(o)|,$$

for f on the tree $\mathcal{T}$. Here $A\alpha$ denotes the immediate predecessor of α in the tree $\mathcal{T}$. For $1<q<\infty$, we also define the weighted Lebesgue space $L^q_\mu(\mathcal{T})$ on the tree by the norm

$$\|f\|_{L^q_\mu(\mathcal{T})}=\left(\sum_{\alpha\in\mathcal{T}}|f(\alpha)|^q\mu(\alpha)\right)^{\frac{1}{q}},$$

for f and μ on the tree $\mathcal{T}$. We say that μ is a $(B_p(\mathcal{T}),q)$-Carleson measure on the tree $\mathcal{T}$ if $B_p(\mathcal{T})$ imbeds continuously into $L^q_\mu(\mathcal{T})$, i.e.

$$\left(\sum_{\alpha\in\mathcal{T}}If(\alpha)^q\mu(\alpha)\right)^{1/q}\leq C\left(\sum_{\alpha\in\mathcal{T}}f(\alpha)^p\right)^{1/p},\qquad f\geq 0,\tag{1.7}$$

or equivalently, by duality,

$$\left(\sum_{\alpha\in\mathcal{T}} I^*\left(g\mu\right)\left(\alpha\right)^{q'}\right)^{1/q'} \leq C\left(\sum_{\alpha\in\mathcal{T}} g\left(\alpha\right)^{p'}\mu\left(\alpha\right)\right)^{1/p'}, \quad g\geq 0, \tag{1.8}$$

where

$$If\left(\alpha\right)=\sum_{\beta\in\mathcal{T}:\beta\leq\alpha} f\left(\beta\right),$$

and

$$I^*\left(g\mu\right)\left(\alpha\right)=\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} g\left(\beta\right)\mu\left(\beta\right).$$

Of special interest is the case when μ satisfies (1.7) with $p=q$,

$$\left(\sum_{\alpha\in\mathcal{T}} If\left(\alpha\right)^p\mu\left(\alpha\right)\right)^{1/p} \leq C\left(\sum_{\alpha\in\mathcal{T}} f\left(\alpha\right)^p\right)^{1/p}, \quad f\geq 0, \tag{1.9}$$

If (1.9) is satisfied, we say that μ is a $B_p\left(\mathcal{T}\right)$-Carleson measure on the tree $\mathcal{T}$. A necessary and sufficient condition for (1.9) given in [**ArRoSa**] is the discrete tree condition

$$\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} I^*\mu\left(\beta\right)^{p'} \leq C^{p'} I^*\mu\left(\alpha\right)<\infty, \quad \alpha\in\mathcal{T}, \tag{1.10}$$

which is obtained by testing (1.8) over $g=\chi_{S_\alpha}$, $\alpha\in\mathcal{T}$. We note that a simpler necessary condition for (1.7) is

$$d\left(\alpha\right)^{p-1} I^*\mu\left(\alpha\right)\leq C^p, \tag{1.11}$$

or equivalently

$$d\left(\alpha\right) I^*\mu\left(\alpha\right)^{p'-1}\leq C^{p'}, \tag{1.12}$$

which one obtains by testing (1.7) over $f=I^*\delta_\alpha=\chi_{[0,\alpha]}$. However, condition (1.11) is not in general sufficient for (1.7) as evidenced by certain Cantor-like measures μ.

We also have the more general two-weight tree theorem from [**ArRoSa**], if $q\geq p$, and its extension to $q<p$ from [**Ar**].

Theorem 1.1. *Let w and v be nonnegative weights on a tree $\mathcal{T}$. Then,*

$$\left(\sum_{\alpha\in\mathcal{T}} Ig\left(\alpha\right)^q w\left(\alpha\right)\right)^{1/q} \leq C\left(\sum_{\alpha\in\mathcal{T}} g\left(\alpha\right)^p v\left(\alpha\right)\right)^{1/p}, \quad g\geq 0,$$

if and only if

$(p=q)$ *in the case $p=q$,*

$$\sum_{\beta\geq\alpha} I^*w\left(\beta\right)^{p'} v\left(\beta\right)^{1-p'} \leq CI^*w\left(\alpha\right)<\infty, \quad \alpha\in\mathcal{T}.$$

$(p<q)$ *in the case $p<q$,*

$$\left(\sum_{\beta\geq\alpha} w(\beta)\right)^{\frac{1}{q}}\left(\sum_{\beta\leq\alpha} v(\beta)^{1-p'}\right)^{\frac{1}{p'}} \leq C, \quad \alpha\in\mathcal{T}.$$

$(p > q)$ *in the case* $p > q$,

$$\sum_{\alpha \in T} w(\alpha) \left(W_v(w)(\alpha)\right)^{\frac{q(p-1)}{p-q}} < \infty$$

where

$$W_v(w)(\alpha) = \sum_{\beta \leq \alpha} v(\beta)^{1-p'} w(\{\gamma : \ \gamma \geq \beta\})^{p'-1}$$

is the Wolff potential.

We now specialize the tree $\mathcal{T}$ to the dyadic tree associated with the usual decomposition of the unit disk into Carleson boxes or Bergman "kubes" K_α. See [**ArRoSa**] or Section 2.2 below for details.

The main theorem in [**ArRoSa**] is the characterization of $(B_p(\mathbb{D}), q)$-Carleson measures on the unit disk $\mathbb{D}$, which can be rephrased as follows (see [**Ar**]). We will only use the case $p = q$ in the remainder of this paper. Given a positive measure μ on the disk, we denote by $\widehat{\mu}$ the associated measure on the dyadic tree $\mathcal{T}$ given by $\widehat{\mu}(\alpha) = \int_{K_\alpha} d\mu$ for $\alpha \in \mathcal{T}$.

Theorem 1.2. *Suppose* $1 < p, q < \infty$ *and that* μ *is a nonnegative measure on the unit disk* $\mathbb{D}$*. Then the following conditions are equivalent:*

(1) μ *is a* $(B_p(\mathbb{D}), q)$*-Carleson measure on* $\mathbb{D}$*, i.e. there is* $C < \infty$ *such that*

$$\left(\int_{\mathbb{D}} |f(z)|^q \, d\mu(z)\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{D}} \left|\left(1 - |z|^2\right) f'(z)\right|^p d\lambda(z)\right)^{\frac{1}{p}} + |f(0)|, \tag{1.13}$$

for all $f \in B_p(\mathbb{D})$*, where* $d\lambda(z) = \left(1 - |z|^2\right)^{-2} dz$.

(2) $\widehat{\mu}$ *is a* $(B_p(\mathcal{T}), q)$ *Carleson measure on the binary tree* $\mathcal{T} = \{\alpha\}$ *associated to the unit disk, i.e. (1.7) holds with* μ *replaced by* $\widehat{\mu}$.

Together, Theorems 1.1 and 1.2 give a geometric characterization of $(B_p(\mathbb{D}), q)$-Carleson measures on $\mathbb{D}$.

Remark 1.3. The proof of this theorem relies on the use of the inequality

$$\mathrm{Re}\left(\frac{1 - |z|^2}{1 - \overline{w}z}\right) > 0, \qquad z, w \in \mathbb{D}.$$

Much of our effort in extending Theorem 1.2 to higher dimensions will involve finding a way to effectively use the analogous inequality in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. Even more effort will be expended in circumventing this inequality when it is unavailable for characterizing interpolating sequences for multiplier spaces in higher dimensions. This is where we pass to "holomorphic" Besov spaces on trees.

2. A tree structure for the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$

In order to extend the above characterization of Carleson measures to higher dimensions, we will need to define a tree $\mathcal{T}_n$ appropriately related to the Bergman metric on the ball. But first we need to recall some of the basic theory of holomorphic functions in the unit ball $\mathbb{B}_n$.

2.1. Invariant metrics, measures and derivatives. We recall some basic definitions and properties from W. Rudin's book [**Rud**] and K. Zhu's book [**Zhu**]. For $a \in \mathbb{B}_n$ let P_a denote orthogonal projection onto the one-dimensional complex subspace $\mathbb{C}a$ generated by a, i.e.

$$P_a z = \frac{z \cdot \overline{a}}{|a|^2} a, \tag{2.1}$$

and let $Q_a = I - P_a$ denote orthogonal projection onto the orthogonal complement of $\mathbb{C}a$. Define an involutive automorphism of the ball $\mathbb{B}_n$ by ([**Rud**], page 25)

$$\begin{aligned} \varphi_a(z) &= \frac{a - P_a z - \left(1 - |a|^2\right)^{\frac{1}{2}} Q_a z}{1 - z \cdot \overline{a}}, \\ &= \frac{a - \frac{z \cdot \overline{a}}{|a|^2} a - \left(1 - |a|^2\right)^{\frac{1}{2}} \left(z - \frac{z \cdot \overline{a}}{|a|^2} a\right)}{1 - z \cdot \overline{a}}, \end{aligned} \tag{2.2}$$

for $z \in \mathbb{B}_n$. Then $Aut(\mathbb{B}_n)$, the group of automorphisms of $\mathbb{B}_n$, consists of all maps $U\varphi_a$ where U is a unitary transformation and $a \in \mathbb{B}_n$.We have $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a \circ \varphi_a = I$. We also have the following identities ([**Rud**], Theorem 2.2.2),

$$\begin{aligned} \varphi_a'(0) &= -\left(1 - |a|^2\right) P_a - \left(1 - |a|^2\right)^{\frac{1}{2}} Q_a, \\ \varphi_a'(a) &= -\left(1 - |a|^2\right)^{-1} P_a - \left(1 - |a|^2\right)^{-\frac{1}{2}} Q_a, \\ 1 - \overline{\varphi_a(w)} \cdot \varphi_a(z) &= \frac{(1 - \overline{a} \cdot a)(1 - \overline{w} \cdot z)}{(1 - \overline{w} \cdot a)(1 - \overline{a} \cdot z)}, \\ 1 - |\varphi_a(z)|^2 &= \frac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{|1 - \overline{a} \cdot z|^2}, \end{aligned} \tag{2.3}$$

and ([**Rud**], Theorem 2.2.6)

$$J\varphi_a(z) = |\det \varphi_a'(z)|^2 = \left(\frac{1 - |a|^2}{|1 - \overline{a} \cdot z|^2}\right)^{n+1},$$

where $J\varphi_a(z)$ denotes the real Jacobian of φ_a at z.

An invariant measure on $\mathbb{B}_n$ is given by ([**Rud**], Theorem 2.2.6)

$$d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dz.$$

The invariance of $d\lambda_n$ follows from the above Jacobian formula and the last identity in (2.3).

An invariant metric on $\mathbb{B}_n$ is the Bergman metric $\beta(z,w)$ given by ([**Zhu**], Proposition 1.21)

$$\beta(z,w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \qquad z, w \in \mathbb{B}_n. \tag{2.4}$$

By invariance, the Bergman metric balls $B_\beta(a,r)$ of radius r at the point $a \in \mathbb{B}_n$ satisfy

$$B_\beta(a,r) = \varphi_a(B_\beta(0,r)),$$

and if $t > 0$ is such that $B_\beta(0,r) = B(0,t)$ (note from (2.4) that Bergman metric balls centered at the origin are Euclidean balls), then the β-balls are the ellipsoids ([**Rud**], page 29)

$$B_\beta(a,r) = \left\{ z \in \mathbb{B}_n : \frac{|P_a z - c_a|^2}{t^2 \rho_a^2} + \frac{|Q_a z|^2}{t^2 \rho_a} < 1 \right\},$$

where

$$c_a = \frac{(1-t^2)\, a}{1 - t^2 |a|^2}, \ \rho_a = \frac{1-|a|^2}{1-t^2|a|^2}.$$

We have the reproducing formula of Bergman ([**Rud**], Theorem 3.1.3),

$$f(z) = \frac{n!}{\pi^n} \int_{\mathbb{B}_n} \frac{f(w)}{(1-\overline{w}\cdot z)^{n+1}} dw, \qquad f \in L^1(d\lambda_n) \cap H(\mathbb{B}_n), \tag{2.5}$$

and the following variants ([**Rud**], Theorem 7.1.2)

$$f(z) = \frac{n!}{\pi^n} \binom{n+s}{n} \int_{\mathbb{B}_n} \frac{\left(1-|w|^2\right)^s}{(1-\overline{w}\cdot z)^{s+n+1}} f(w)\, dw, \qquad \operatorname{Re} s > -1, \tag{2.6}$$

valid for all $f \in H(\mathbb{B}_n)$ for which the integrand is in L^1.

We now recall the invertible "radial" operators $R^{\gamma,t} : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ given in [**Zhu**] by

$$R^{\gamma,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\gamma)\,\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\,\Gamma(n+1+k+\gamma)} f_k(z),$$

provided neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f. If the inverse of $R^{\gamma,t}$ is denoted $R_{\gamma,t}$, then Proposition 1.14 of [**Zhu**] yields

$$R^{\gamma,t}\left(\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma}} \right) = \frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma+t}}, \tag{2.7}$$

$$R_{\gamma,t}\left(\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma+t}} \right) = \frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma}},$$

for all $w \in \mathbb{B}_n$. Thus for any γ, $R^{\gamma,t}$ is approximately differentiation of order t. From Theorem 6.1 and Theorem 6.4 of [**Zhu**] we have that the derivatives $R^{\gamma,m} f(z)$ are "L^p norm equivalent" to $\sum_{k=0}^{m-1} \left|\nabla^k f(0)\right| + \nabla^m f(z)$ for m large enough.

Proposition 2.1 (Theorem 6.1 and Theorem 6.4 of [**Zhu**]). *Suppose that $0 < p < \infty$, $n+\gamma$ is not a negative integer, and $f \in H(\mathbb{B}_n)$. Then the following four conditions are equivalent:*

$$\left(1-|z|^2\right)^m \nabla^m f(z) \in L^p(d\lambda_n) \textit{ for } \text{some } m > \frac{n}{p}, m \in \mathbb{N},$$

$$\left(1-|z|^2\right)^m \nabla^m f(z) \in L^p(d\lambda_n) \textit{ for } \text{all } m > \frac{n}{p}, m \in \mathbb{N},$$

$$\left(1-|z|^2\right)^m R^{\gamma,m} f(z) \in L^p(d\lambda_n) \textit{ for } \text{some } m > \frac{n}{p}, m+n+\gamma \notin -\mathbb{N},$$

$$\left(1-|z|^2\right)^m R^{\gamma,m} f(z) \in L^p(d\lambda_n) \textit{ for } \text{all } m > \frac{n}{p}, m+n+\gamma \notin -\mathbb{N}.$$

Moreover, with $\sigma(z) = 1 - |z|^2$, we have for $1 < p < \infty$,

$$\begin{aligned} &C^{-1} \|\sigma^{m_1} R^{\gamma,m_1} f\|_{L^p(d\lambda_n)} \\ &\leq \sum_{k=0}^{m_2-1} \left|\nabla^k f(0)\right| + \left(\int_{\mathbb{B}_n} |\sigma(z)^{m_2} \nabla^{m_2} f(z)|^p \, d\lambda_n(z)\right)^{\frac{1}{p}} \\ &\leq C \|\sigma^{m_1} R^{\gamma,m_1} f\|_{L^p(d\lambda_n)} \end{aligned} \tag{2.8}$$

for all $m_1, m_2 > \frac{n}{p}$, $m_1 + n + \gamma \notin -\mathbb{N}$, $m_2 \in \mathbb{N}$, and where the constant C depends only on m_1, m_2, n, γ and p.

Definition 2.2. *We define the analytic Besov spaces $B_p(\mathbb{B}_n)$ on the ball $\mathbb{B}_n$ by taking $\gamma = 0$ and $m = \left[\frac{n}{p}\right] + 1$ and setting*

$$B_p = B_p(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \left\|\sigma^m R^{0,m} f\right\|_{L^p(d\lambda_n)} < \infty \right\}. \tag{2.9}$$

We will indulge in the usual abuse of notation by using $\|f\|_{B_p(\mathbb{B}_n)}$ to denote any of the norms appearing in (2.8).

2.1.1. *Duality and reproducing kernels.* For $\alpha > -1$, let $\langle \cdot, \cdot \rangle_\alpha$ denote the inner product for the weighted Bergman space A^2_α:

$$\langle f, g \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \overline{g(z)} d\nu_\alpha(z), \quad f, g \in A^2_\alpha,$$

where $d\nu_\alpha(z) = \left(1 - |z|^2\right)^\alpha dz$. Recall that

$$K^\alpha_w(z) = K^\alpha(z, w) = (1 - \overline{w} \cdot z)^{-n-1-\alpha}$$

is the reproducing kernel for A^2_α (Theorem 2.7 in [**Zhu**]):

$$f(w) = \langle f, K^\alpha_w \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \overline{K^\alpha_w(z)} d\nu_\alpha(z), \quad f \in A^2_\alpha.$$

This formula continues to hold as well for $f \in A^p_\alpha$, $1 < p < \infty$, since the polynomials are dense in A^p_α.

Corollary 6.5 of [**Zhu**] states that $R^{\gamma, \frac{n+1+\alpha}{p}}$ is a bounded invertible operator from B_p onto A^p_α, provided that neither $n + \gamma$ nor $n + \gamma + \frac{n+1+\alpha}{p}$ is a negative integer. It turns out to be convenient to take $\gamma = \alpha - \frac{n+1+\alpha}{p}$ here (with this choice we can explicitly compute certain derivatives and $B_{p'}$ norms of our reproducing kernels - see (2.13) and (5.9) below), and thus we single out the special operators

$$\mathcal{R}^\alpha_t = R^{\alpha-t,t}.$$

Note that the operators $\mathcal{R}^\alpha_t$ and their inverses $(\mathcal{R}^\alpha_t)^{-1} = R_{\alpha-t,t}$ are self-adjoint with respect to $\langle \cdot, \cdot \rangle_\alpha$ since the monomials are orthogonal with respect to $\langle \cdot, \cdot \rangle_\alpha$ (see (1.21) and (1.23) in [**Zhu**]), and the operators act on the homogeneous expansion of f by multiplying the homogeneous coefficients of f by certain positive constants. The next definition is motivated by the fact that $\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}$ is a bounded invertible operator from B_p onto A^p_α, and that $\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}$ is a bounded invertible operator from $B_{p'}$ onto $A^{p'}_\alpha$, provided that neither $n + \alpha$, $n + \alpha - \frac{n+1+\alpha}{p}$ nor $n + \alpha - \frac{n+1+\alpha}{p'}$ is a negative integer. Note that this proviso holds in particular for $\alpha > -1$.

DEFINITION 2.3. *For $\alpha > -1$ and $1 < p < \infty$, we define a pairing $\langle \cdot, \cdot \rangle_{\alpha,p}$ for B_p and $B_{p'}$ using $\langle \cdot, \cdot \rangle_\alpha$ as follows:*

$$\begin{aligned}\langle f, g\rangle_{\alpha,p} &= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g \right\rangle_\alpha = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z) \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g(z)} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} (1-|z|^2)^{\frac{n+1+\alpha}{p}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z) \overline{(1-|z|^2)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g(z)} d\lambda_n(z).\end{aligned}$$

Clearly we have

$$\left| \langle f, g\rangle_{\alpha,p} \right| \leq \|f\|_{B_p} \|g\|_{B_{p'}}$$

by Hölder's inequality. By Theorem 2.12 of [**Zhu**], we also have that every continuous linear functional Λ on B_p is given by $\Lambda f = \langle f, g\rangle_{\alpha,p}$ for a unique $g \in B_{p'}$ satisfying

$$\|g\|_{B_{p'}} = \sup_{\|f\|_{B_p}=1} \left| \langle f, g\rangle_{\alpha,p} \right|. \tag{2.10}$$

Indeed, if $\Lambda \in (B_p)^*$, then $\Lambda \circ \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} \right)^{-1} \in (A^p_\alpha)^*$, and by Theorem 2.12 of [**Zhu**], there is $G \in A^{p'}_\alpha$ with $\|G\|_{A^{p'}_\alpha} = \|\Lambda\|$ such that $\Lambda \circ \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} \right)^{-1} F = \langle F, G\rangle_\alpha$ for all $F \in A^p_\alpha$. If we set $g = \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} \right)^{-1} G$, then we have $\|g\|_{B_{p'}} = \|G\|_{A^{p'}_\alpha} = \|\Lambda\|$ and with $F = \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f$, we also have

$$\Lambda f = \Lambda \circ \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} \right)^{-1} F = \langle F, G\rangle_\alpha = \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g \right\rangle_\alpha = \langle f, g\rangle_{\alpha,p}$$

for all $f \in B_p$. Then (2.10) follows from

$$\|g\|_{B_{p'}} = \|\Lambda\| = \sup_{\|f\|_{B_p}=1} |\Lambda(f)| = \sup_{\|f\|_{B_p}=1} \left| \langle f, g\rangle_{\alpha,p} \right|.$$

REMARK 2.4. The Besov space pairing $\langle \cdot, \cdot \rangle$ introduced in [**Zhu**] is given by

$$\langle f, g\rangle = \left\langle R^{0,n+1} f, R^{0,n+1} g \right\rangle_{n+1} = \langle f, g\rangle_{n+1,2},$$

and so coincides with our pairing for B_2 with the choice $\alpha = n+1$. However, for $p \neq 2$, our pairing $\langle \cdot, \cdot \rangle_{\alpha,p}$ uses the operators $\mathcal{R}^\alpha_t = R^{\alpha - t, t}$ with $t = \frac{n+1+\alpha}{p}$ and $t = \frac{n+1+\alpha}{p'}$, and so does not coincide with $\langle \cdot, \cdot \rangle$ in [**Zhu**].

With $K^\alpha_w(z)$ the reproducing kernel for A^2_α, we now claim that the kernel

$$k^{\alpha,p}_w(z) = \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} \right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} \right)^{-1} K^\alpha_w(z) \tag{2.11}$$

satisfies the following reproducing formula for B_p:

$$f(w) = \langle f, k^{\alpha,p}_w\rangle_{\alpha,p} = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z) \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} k^{\alpha,p}_w(z)} d\nu_\alpha(z), \qquad f \in B_p. \tag{2.12}$$

Indeed, for f a polynomial, we have

$$\begin{aligned}
f(w) &= \langle f, K_w^\alpha\rangle_\alpha \\
&= \left\langle \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, K_w^\alpha \right\rangle_\alpha \\
&= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} K_w^\alpha \right\rangle_\alpha \\
&= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} K_w^\alpha \right\rangle_\alpha \\
&= \left\langle f, \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} K_w^\alpha \right\rangle_{\alpha,p}.
\end{aligned}$$

We now obtain the claim since the polynomials are dense in B_p and the kernels $k_w^{\alpha,p}$ are in $B_{p'}$ for each fixed $w \in \mathbb{B}_n$. Thus we have proved the following theorem.

THEOREM 2.5. *Let $1 < p < \infty$ and $\alpha > -1$. Then the dual space of B_p can be identified with $B_{p'}$ under the pairing $\langle \cdot, \cdot \rangle_{\alpha,p}$, and the reproducing kernel $k_w^{\alpha,p}$ for this pairing is given by (2.11).*

From (2.11) and (2.7) we have

$$\begin{aligned}
\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} k_w^{\alpha,p}(z) &= \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} K_w^\alpha(z) \\
&= R_{\alpha-\frac{n+1+\alpha}{p},\frac{n+1+\alpha}{p}} \left((1-\overline{w}\cdot z)^{-(n+1+\alpha)}\right) \\
&= (1-\overline{w}\cdot z)^{-\frac{n+1+\alpha}{p'}}.
\end{aligned} \tag{2.13}$$

Using this formula we will show in (5.9) below that the $B_{p'}$ norm of the reproducing kernel $k_w^{\alpha,p}$ is comparable to $\left(1+\log\frac{1}{1-|w|^2}\right)^{\frac{1}{p'}}$.

2.2. Carleson boxes. In order to facilitate the imposition of a tree structure, we give a more refined construction of Carleson boxes than that given in Theorem 2.23 of [**Zhu**]. Let β be the Bergman metric on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. The metric balls of radius 1 will essentially play the role of an upper half of a Carleson tent, or Carleson box. Note that the set

$$\mathcal{S}_r = \partial B_\beta(0,r) = \{z \in \mathbb{B}_n : \beta(0,z) = r\}$$

is a Euclidean sphere (with different radius) centered at the origin for each $r > 0$. In fact, by (1.40) in [**Zhu**] we have $\beta(0,z) = \tanh^{-1}|z|$, and so

$$\begin{aligned}
1-|z|^2 &= 1-\tanh^2\beta(0,z) \\
&= \frac{4}{e^{2\beta(0,z)}+2+e^{-2\beta(0,z)}} \\
&\approx 4e^{-2\beta(0,z)}
\end{aligned} \tag{2.14}$$

for $\beta(0,z)$ large. We will apply the following abstract construction to the spheres $\mathcal{S}_r$ for $r > 0$.

LEMMA 2.6. *Let* (X, d) *be a separable metric space and* $\lambda > 0$. *There is a denumerable set of points* $E = \{x_j\}_{j=1}^{\infty \text{ or } J}$ *and a corresponding set of Borel subsets* Q_j *of* X *satisfying*

$$
\begin{aligned}
(2.15) \qquad & X = \cup_{j=1}^{\infty \text{ or } J} Q_j, \\
& Q_i \cap Q_j = \phi, \quad i \neq j, \\
& B(x_j, \lambda) \subset Q_j \subset B(x_j, 2\lambda), \quad j \geq 1.
\end{aligned}
$$

We refer to the sets Q_j as unit *qubes* centered at x_j.

PROOF. Let $E = \{x_j\}_{j=1}^{\infty \text{ or } J}$ be a maximal λ-separated subset of X so that

$$
\begin{aligned}
d(x_i, x_j) &\geq \lambda, \quad i \neq j, \\
d(x, E) &< \lambda, \quad x \in X.
\end{aligned}
$$

Let $A = \cup_{j=1}^{\infty \text{ or } J} B(x_j, \lambda)$ and define inductively

$$
Q_j = B(x_j, 2\lambda) \setminus \{A \cup (\cup_{i<j} Q_i)\}, \quad j \geq 1.
$$

It is routine to verify that these qubes Q_j satisfy (2.15).

2.2.1. *Construction of the Bergman tree.* Now fix $\theta, \lambda > 0$ (various choices of θ and λ will be used below), which we will refer to as *structural constants* for the Bergman tree. For $N \in \mathbb{N}$, apply the lemma to the metric space $(\mathcal{S}_{N\theta}, \beta)$ to obtain points $\left\{z_j^N\right\}_{j=1}^J$ and unit qubes $\left\{Q_j^N\right\}_{j=1}^J$ in $\mathcal{S}_{N\theta}$ satisfying (2.15). For $z \in \mathbb{B}_n$, let $P_r z$ denote the radial projection of z onto the sphere $\mathcal{S}_r$ (not to be confused with the orthogonal projection P_a defined above). We now define subsets K_j^N of $\mathbb{B}_n$ by $K_1^0 = \{z \in \mathbb{B}_n : \beta(0, z) < \theta\}$ and

$$
K_j^N = \left\{z \in \mathbb{B}_n : N\theta \leq d(0, z) < (N+1)\theta,\ P_{N\theta} z \in Q_j^N\right\}, \quad N \geq 1, j \geq 1.
$$

We define corresponding points $c_j^N \in K_j^N$ by

$$
c_j^N = P_{\left(N+\frac{1}{2}\right)\theta}\left(z_j^N\right).
$$

We will refer to the subset K_j^N of $\mathbb{B}_n$ as a unit *kube* centered at c_j^N (while K_1^0 is centered at 0).

Define a tree structure on the collection of unit kubes

$$
\mathcal{T}_n = \left\{K_j^N\right\}_{N \geq 0, j \geq 1}
$$

by declaring that K_i^{N+1} is a child of K_j^N, written $K_i^{N+1} \in \mathcal{C}\left(K_j^N\right)$, if the projection $P_{N\theta}\left(z_i^{N+1}\right)$ of z_i^{N+1} onto the sphere $\mathcal{S}_{N\theta}$ lies in the qube Q_j^N. In the case $N = 0$, we declare every kube K_j^1 to be a child of the root kube K_1^0. We will typically write α, β, γ etc. to denote elements K_j^N of the tree $\mathcal{T}_n$ when the correspondence with the unit ball $\mathbb{B}_n$ is immaterial. We will write K_α for the kube K_j^N and c_α for its center c_j^N when the correspondence matters. Sometimes we will further abuse notation by using α to denote the center $c_\alpha = c_j^N$ of the kube $K_\alpha = K_j^N$, especially in the section on interpolating sequences below.

Finally, we define the *dimension* $n(\mathcal{T})$ of an arbitrary tree $\mathcal{T}$.

DEFINITION 2.7. *The* upper dimension $\overline{n}(\mathcal{T})$ *of a tree* $\mathcal{T}$ *is given by*

$$
\overline{n}(\mathcal{T}) = \limsup_{\ell \to \infty} \log_2 (N_\ell)^{\frac{1}{\ell}},
$$

where

$$N_\ell = \sup_{\alpha\in\mathcal{T}} \operatorname{card}\left\{\beta\in\mathcal{T} : \beta > \alpha \text{ and } d(\beta) = d(\alpha)+\ell\right\},$$

along with a similar definition for the lower dimension $\underline{n}(\mathcal{T})$ *using* $\liminf_{\ell\to\infty}$ *in place of* $\limsup_{\ell\to\infty}$. *If the upper and lower dimensions coincide, we denote their common value, called the* dimension *of* $\mathcal{T}$, *by* $n(\mathcal{T})$.

Note that if $\mathcal{T}$ is a homogeneous tree with branching number N, then $N = (N_\ell)^{\frac{1}{\ell}}$ for all $\ell \geq 1$. The choice of base 2 for the logarithm then yields the relationship $N = 2^n$, consistent with the familiar interpretation that the dyadic tree has dimension 1 and the linear tree has dimension 0.

Lemma 2.8. *The tree* $\mathcal{T}_n$, *constructed above with positive parameters* λ *and* θ, *and the unit ball* $\mathbb{B}_n$ *satisfy the following properties.*

1. *The ball* $\mathbb{B}_n$ *is a pairwise disjoint union of the kubes* K_α, $\alpha\in\mathcal{T}_n$, *and there are positive constants* C_1 *and* C_2 *depending on* λ *and* θ *such that*
$$B_\beta(c_\alpha, C_1) \subset K_\alpha \subset B_\beta(c_\alpha, C_2), \qquad \alpha\in\mathcal{T}_n, n\geq 1.$$
2. $\bigcup_{\beta\geq\alpha} K_\beta$ *is "comparable" to the Carleson tent* V_{c_α} *associated to the point* c_α, *where*
$$V_z = \left\{w\in\mathbb{B}_n : |1-\overline{w}\cdot Pz| \leq 1-|z|\right\},$$
and Pz *denotes radial projection of* z *onto the sphere* $\partial\mathbb{B}_n$.
3. *The invariant volume of* K_α *is bounded between positive constants depending on* λ *and* θ, *but independent of* $\alpha\in\mathcal{T}_n$.
4. *The dimension* $n(\mathcal{T}_n)$ *of the tree* $\mathcal{T}_n$ *is* $\frac{2\theta}{\ln 2}n$.
5. *For any* $R>0$, *the balls* $B_\beta(c_\alpha,R)$ *satisfy the finite overlap condition*
$$\sum_{\alpha\in\mathcal{T}_n} \chi_{B_\beta(c_\alpha,R)}(z) \leq C_R, \qquad z\in\mathbb{B}_n.$$

Proof. Property 1 follows easily from the construction of K_j^N, and property 2 is then a consequence of the formula for the metric β.

Property 3 follows since using (2.14), K_α is comparable to a rectangle, two of whose side lengths (those in the complex radial directions) are $e^{-2\theta d(\alpha)}$, while the remaining side lengths (those in the complex tangential directions) are $e^{-\theta d(\alpha)}$.

The final two properties follow from volume counting. Indeed, given $\alpha\in\mathcal{T}_n$ and $\ell\geq 1$, let $\{\beta_j\}_{j=1}^{N_\ell}$ be an enumeration of the descendents ℓ generations beyond α:

$$\mathcal{C}^{(\ell)}(\alpha) = \{\beta\in\mathcal{T}_n : \beta>\alpha \text{ and } d(\beta) = d(\alpha)+\ell\} = \{\beta_j\}_{j=1}^{N_\ell}.$$

From property 3 and (2.14) we have that $1\approx |K_\gamma|_{\lambda_n} \approx e^{2d(\gamma)\theta(n+1)}|K_\gamma|$, and since $\left|\cup_{j=1}^N K_{\beta_j}\right| \approx e^{-2\ell\theta}|K_\alpha|$, we obtain

$$e^{-2\ell\theta}e^{-2d(\alpha)\theta(n+1)} \approx e^{-2\ell\theta}|K_\alpha| \approx \left|\cup_{j=1}^{N_\ell} K_{\beta_j}\right| = \sum_{j=1}^{N_\ell}\left|K_{\beta_j}\right|$$
$$\approx \sum_{j=1}^{N_\ell} e^{-2d(\beta_j)\theta(n+1)} = N_\ell e^{-2(d(\alpha)+\ell)\theta(n+1)}.$$

Thus there are positive constants c and C depending only on n and θ such that

$$c \le \frac{N_\ell}{e^{2n\ell\theta}} \le C, \qquad \alpha \in \mathcal{T}_n, \ell \ge 1,$$

and it follows that $n(\mathcal{T}) = \log_2 e^{2n\theta}$, completing the proof of property 4.

The finite overlap property 5 is obtained as follows. Let $z \in K_\alpha$. If $z \in \cap_{j=1}^N B_\beta(c_{\alpha_j}, R)$, then since β is a metric,

$$\bigcup_{j=1}^N B_\beta(c_{\alpha_j}, C_1) \subset B_\beta(c_\alpha, R + 2C_1).$$

Since the balls $B_\beta(c_{\alpha_j}, C_1)$ are pairwise disjoint by property 1, we thus have

$$N \approx \sum_{j=1}^N \left| B_\beta(c_{\alpha_j}, C_1) \right|_{\lambda_n} \le \left| B_\beta(c_\alpha, R + 2C_1) \right|_{\lambda_n} \approx C_R,$$

where C_R is a positive constant depending on n and θ, but independent of $\alpha \in \mathcal{T}_n$, hence also of $z \in \mathbb{B}_n$. We will often use the notation $|E|_\mu$ for the μ-measure of a set E.

Remark 2.9. The choice $\theta = \frac{\ln 2}{2}$ yields $\dim(\mathcal{T}_n) = n$ and the convenient equivalence $1 - |z|^2 \approx 2^{-d(\alpha)}$ for $z \in K_\alpha$. In one dimension with $\lambda \approx 1$, this identifies $B_p(\mathcal{T}_1)$ with the one-dimensional Besov space $B_p(\mathcal{T})$ defined above on the dyadic tree $\mathcal{T}$, and in higher dimensions with the abstract Besov spaces $B_p(\mathcal{T}_n)$ defined below on $\mathcal{T}_n$. However, Corollary 3.2 on monotonicity of Carleson measures requires both θ and λ to be small (to invoke the atomic decomposition of Besov spaces), while the more refined holomorphic Besov spaces $HB_p(\mathcal{T}_n)$ on $\mathcal{T}_n$ considered in Section 8 will require λ small and θ large. The structural inequality (8.37), used to define the spaces $HB_p(\mathcal{T}_n)$ and prove the restriction Theorem 8.14, requires θ sufficiently large, while the positivity property (8.43) in Lemma 8.11 requires in addition that λ is sufficiently small. The above construction simplifies greatly when $n = 1$ since then the spheres $\mathcal{S}_r$ are circles, and the qubes Q_j^N can be taken to be circular arcs of equal length.

2.2.2. *Discretization of Carleson measures.* Let $B_p = B_p(\mathbb{B}_n)$. Given $1 < p, q < \infty$ and a positive Borel measure μ on the ball $\mathbb{B}_n$, we say that μ is a (B_p, q)-Carleson measure on $\mathbb{B}_n$ if there is a positive constant C such that

$$\left(\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \right)^{\frac{1}{q}} \le C \|f\|_{B_p}, \tag{2.16}$$

for all $f \in B_p$. We wish to show that μ is a (B_p, q)-Carleson measure if and only if its averaged version $\widetilde{\mu}$ is a (B_p, q)-Carleson measure, where $\widetilde{\mu}$ is defined by

$$d\widetilde{\mu}(z) = \sum_{\alpha \in \mathcal{T}_n} \left(\int_{K_\alpha} d\mu \right) \lambda_n(K_\alpha)^{-1} \chi_{K_\alpha}(z) \, d\lambda_n(z). \tag{2.17}$$

It is convenient to introduce as well the discretized version $\mu^\natural$ of μ given by

$$\mu^\natural = \sum_\alpha \left(\int_{K_\alpha} d\mu \right) \delta_{c_\alpha}, \tag{2.18}$$

where c_α is the center of K_α. We will need the inequality

$$|f(z)-f(w)| \le C\|f\|_{B_p}\beta(z,w)^{\frac{1}{p'}}, \qquad z,w\in\mathbb{B}_n, \tag{2.19}$$

which is proved below in (5.11), and also given as Exercise 21 on page 220 of [**Zhu**].

We make no restriction here on the structural constants θ and λ.

PROPOSITION 2.10. *Let $1<p,q<\infty$, μ be a positive Borel measure on the ball $\mathbb{B}_n$ and $\widetilde{\mu},\mu^\natural$ be defined as in (2.17) and (2.18) respectively. Then μ is a (B_p,q)-Carleson measure on $\mathbb{B}_n$ if and only if $\widetilde{\mu}$ is a (B_p,q)-Carleson measure on $\mathbb{B}_n$ if and only if $\mu^\natural$ is a (B_p,q)-Carleson measure on $\mathbb{B}_n$.*

PROOF. Note that Carleson measures are *a priori* bounded and $\mu(\mathbb{B}_n)=\widetilde{\mu}(\mathbb{B}_n)=\mu^\natural(\mathbb{B}_n)$. Fix $f\in B_p$ for the moment and let $\{w_\alpha\}_{\alpha\in\mathcal{T}_n}$ and $\{\xi_\alpha\}_{\alpha\in\mathcal{T}_n}$ be sequences with $w_\alpha,\xi_\alpha\in\overline{K_\alpha}$ satisfying

$$|f(w_\alpha)| = \max_{w\in\overline{K_\alpha}}|f(w)|, \qquad |f(\xi_\alpha)| = \min_{w\in\overline{K_\alpha}}|f(w)|, \qquad \alpha\in\mathcal{T}_n.$$

We then have the inequalities

$$\begin{aligned}
c|f(\xi_\alpha)|^p\mu(K_\alpha) &\le \int_{K_\alpha}|f(z)|^p\,d\mu(z) \le C|f(w_\alpha)|^p\mu(K_\alpha),\\
c|f(\xi_\alpha)|^p\mu(K_\alpha) &\le \int_{K_\alpha}|f(z)|^p\,d\widetilde{\mu}(z) \le C|f(w_\alpha)|^p\mu(K_\alpha),\\
c|f(\xi_\alpha)|^p\mu(K_\alpha) &\le \int_{K_\alpha}|f(z)|^p\,d\mu^\natural(z) \le C|f(w_\alpha)|^p\mu(K_\alpha).
\end{aligned}$$

Now if ν_1 and ν_2 denote any two of the three measures $\mu,\widetilde{\mu},\mu^\natural$, then

$$\|f\|_{L^p(\nu_1)} \le \left|\|f\|_{L^p(\nu_1)}-\|f\|_{L^p(\nu_2)}\right| + \|f\|_{L^p(\nu_2)},$$

and by Minkowski's inequality and (2.19),

$$\begin{aligned}
\left|\|f\|_{L^p(\nu_1)}-\|f\|_{L^p(\nu_2)}\right| &\le \left(\sum_{\alpha\in\mathcal{T}_n}\left|\left(\int_{K_\alpha}|f|^p\,d\nu_1\right)^{\frac{1}{p}}-\left(\int_{K_\alpha}|f|^p\,d\nu_2\right)^{\frac{1}{p}}\right|^p\right)^{\frac{1}{p}}\\
&\le \left(\sum_{\alpha\in\mathcal{T}_n}|f(w_\alpha)-f(\xi_\alpha)|^p\mu(K_\alpha)\right)^{\frac{1}{p}}\\
&\le C\left(\sum_{\alpha\in\mathcal{T}_n}\|f\|_{B_p}^p\mu(K_\alpha)\right)^{\frac{1}{p}}\\
&\le C\mu(\mathbb{B}_n)^{\frac{1}{p}}\|f\|_{B_p}.
\end{aligned}$$

Thus

$$\|f\|_{L^p(\nu_1)} \le C\mu(\mathbb{B}_n)\|f\|_{B_p} + \|f\|_{L^p(\nu_2)}$$

and the conclusion of Proposition 2.10 follows.

3. Carleson measures

Given a positive measure μ on the ball, we denote by $\widehat{\mu}$ the associated measure on the Bergman tree $\mathcal{T}_n$ given by $\widehat{\mu}(\alpha) = \int_{K_\alpha} d\mu$ for $\alpha \in \mathcal{T}_n$. Let $1 < p, q < \infty$. In this section we show that μ is a (B_p, q)-Carleson measure on $\mathbb{B}_n$ if $\widehat{\mu}$ is a $(B_p(\mathcal{T}_n), q)$-Carleson measure, i.e. if

$$\left(\sum_{\alpha \in \mathcal{T}_n} If(\alpha)^q \, \widehat{\mu}(\alpha) \right)^{1/q} \leq C \left(\sum_{\alpha \in \mathcal{T}_n} f(\alpha)^p \right)^{1/p}, \qquad f \geq 0. \tag{3.1}$$

In the case $p = q$, that arises in section 5 below on interpolation, Theorem 1.1 shows that (3.1) is equivalent to the tree condition

$$\sum_{\beta \geq \alpha} I^* \widehat{\mu}(\beta)^{p'} \leq C I^* \widehat{\mu}(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n. \tag{3.2}$$

Conversely, in the range $1 < p < 2 + \frac{1}{n-1}$, $1 < q < \infty$, we show that $\widehat{\mu}$ is $(B_p(\mathcal{T}_n), q)$-Carleson if μ is a $(B_p(\mathbb{B}_n), q)$-Carleson measure on $\mathbb{B}_n$ (necessity in the range $p \in \left[2 + \frac{1}{n-1}, \infty\right)$ is left open). We have the following generalization of Theorem 1.2. We often write $\mu(\alpha)$ for $\widehat{\mu}(\alpha)$ when there is no chance of confusion.

Theorem 3.1. *Suppose $1 < p, q < \infty$ and that $0 < \lambda, \theta < \infty$ are the structural constants in the construction of $\mathcal{T}_n$ in Subsubsection 2.2.1. Let μ be a positive measure on the unit ball $\mathbb{B}_n$. Then with constants depending only on p, λ, θ, n, conditions 2 and 3 below are equivalent, condition 3 is sufficient for condition 1, and provided $1 < p < 2 + \frac{1}{n-1}$, condition 3 is necessary for condition 1:*

(1) *μ is a $(B_p(\mathbb{B}_n), q)$-Carleson measure on $\mathbb{B}_n$, i.e. (2.16) holds.*
(2) *$\widehat{\mu} = \{\mu(\alpha)\}_{\alpha \in \mathcal{T}_n}$ is a $(B_p(\mathcal{T}_n), q)$-Carleson measure on the Bergman tree $\mathcal{T}_n$, i.e. (1.7) holds with $\mu(\alpha) = \int_{K_\alpha} d\mu$ and $\mathcal{T}$ replaced by $\mathcal{T}_n$.*
(3) *There is $C < \infty$ such that*
    (i) in the case $p = q$,

$$\sum_{\beta \geq \alpha} I^* \mu(\beta)^{p'} \leq C I^* \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n.$$

(ii) in the case $p < q$,

$$\left(\sum_{\beta \geq \alpha} \mu(\beta) \right)^{\frac{1}{q}} \left(\sum_{\beta \leq \alpha} 1 \right)^{\frac{1}{p'}} \leq C, \qquad \alpha \in \mathcal{T}_n.$$

(iii) in the case $p > q$,

$$\sum_{\alpha \in_n} \mu(\alpha) \left(W(\mu)(\alpha) \right)^{\frac{q(p-1)}{p-q}} \leq C < \infty$$

where

$$W(\mu)(\alpha) = \sum_{\beta \leq \alpha} \mu(\{\gamma : \ \gamma \geq \beta\})^{p'-1}.$$

Theorem 3.1 yields the following monotonicity property for $B_p(\mathbb{B}_n)$-Carleson measures on $\mathbb{B}_n$.

Corollary 3.2. *If μ is a $B_{p_0}(\mathbb{B}_n)$-Carleson measure for some $1 < p_0 < \infty$, then μ is a $B_p(\mathbb{B}_n)$-Carleson measure for all $1 < p < p_0$. Moreover, μ satisfies the tree condition (3.2) with*

$$p = \begin{cases} p_0 & \textit{if} \quad p_0 < 2 + \frac{1}{n-1} \\ 2 + \frac{1}{n-1} - \varepsilon,\ \varepsilon > 0 & \textit{if} \quad 2 + \frac{1}{n-1} \le p_0 < 4 + \frac{2}{n-1} \\ \frac{p_0}{2} & \textit{if} \quad p_0 \ge 4 + \frac{2}{n-1} \end{cases} .$$

Proof. For $1 < p_0 < 2 + \frac{1}{n-1}$, we simply use Theorem 3.1 together with the fact that inequality (1.9) has the corresponding monotonicity property: for $f \ge 0$ and $1 < p < p_0$,

$$\begin{aligned} \left(\sum_{\alpha \in \mathcal{T}} If(\alpha)^p \mu(\alpha) \right)^{1/p} &= \left(\sum_{\alpha \in \mathcal{T}} \left[\sum_{\beta \le \alpha} f(\beta) \right]^{\frac{p}{p_0} p_0} \mu(\alpha) \right)^{1/p} \\ &\le \left(\sum_{\alpha \in \mathcal{T}} \left[\sum_{\beta \le \alpha} f(\beta)^{\frac{p}{p_0}} \right]^{p_0} \mu(\alpha) \right)^{1/p} \\ &\le \left(C^{p_0} \sum_{\alpha \in \mathcal{T}} \left[f(\beta)^{\frac{p}{p_0}} \right]^{p_0} \right)^{1/p} = C^{\frac{p_0}{p}} \left(\sum_{\alpha \in \mathcal{T}} f(\beta)^p \right)^{1/p}, \end{aligned}$$

if (1.9) holds for the exponent p_0.

For $p_0 > 2$, we use the atomic decomposition of $B_{p_0}(\mathbb{B}_n)$ (Theorem 6.6 in [**Zhu**]), Khinchine's inequality for the Rademacher functions $r_k(t)$, and complex interpolation of the Besov spaces $B_{p_0}(\mathbb{B}_n)$ (Theorem 6.12 in [**Zhu**]). If the measure μ is $B_{p_0}(\mathbb{B}_n)$-Carleson, and $a > \frac{n}{p_0'}$, then with $f(z) = \sum_{\alpha \in \mathcal{T}_n} f_\alpha \left(\frac{1 - |c_\alpha|^2}{1 - \overline{c_\alpha} \cdot z} \right)^a$ and $p = q = p_0$ in (2.16), we have

$$\int_{\mathbb{B}_n} \left| \sum_{\alpha \in \mathcal{T}_n} f_\alpha \left(\frac{1 - |c_\alpha|^2}{1 - \overline{c_\alpha} \cdot z} \right)^a \right|^{p_0} d\mu(z) \le C \sum_{\alpha \in \mathcal{T}_n} |f_\alpha|^{p_0}$$

for all $\{f_\alpha\}_{\alpha \in \mathcal{T}_n} \in \ell^{p_0}$ by the atomic decomposition of $B_{p_0}(\mathbb{B}_n)$. Here we use the atomic decomposition Theorem 6.6 in [**Zhu**], which requires that we choose both parameters λ and θ sufficiently small in the construction of the Bergman tree $\mathcal{T}_n$. Using Khinchine's inequality for the Rademacher functions, we obtain

$$\begin{aligned} &\int_{\mathbb{B}_n} \left\{ \sum_{\alpha \in \mathcal{T}_n} \left| f_\alpha \left(\frac{1 - |c_\alpha|^2}{1 - \overline{c_\alpha} \cdot z} \right)^a \right|^2 \right\}^{\frac{p_0}{2}} d\mu(z) \\ \approx &\int_{\mathbb{B}_n} \int_0^1 \left| \sum_{\alpha \in \mathcal{T}_n} r_\alpha(t) f_\alpha \left(\frac{1 - |c_\alpha|^2}{1 - \overline{c_\alpha} \cdot z} \right)^a \right|^{p_0} dt d\mu(z) \\ \le &\int_0^1 C \sum_{\alpha \in \mathcal{T}_n} |c_k r_\alpha(t) f_\alpha|^{p_0} dt = C \sum_{\alpha \in \mathcal{T}_n} |f_\alpha|^{p_0}. \end{aligned}$$

Now we observe that for $z \in K_\beta$,

$$\sum_{\alpha \in \mathcal{T}_n} \left| f_\alpha \left(\frac{1 - |c_\alpha|^2}{1 - \overline{c_\alpha} \cdot z} \right)^a \right|^2 \geq c \sum_{\alpha \leq \beta} |f_\alpha|^2 = cIg(\beta),$$

where $g(\alpha) = |f_\alpha|^2$. Thus we have the tree inequality

$$\sum_{\beta \in \mathcal{T}_n} Ig(\beta)^{\frac{p_0}{2}} \mu(\beta) \leq C \sum_{\beta \in \mathcal{T}_n} g(\beta)^{\frac{p_0}{2}}, \qquad g \geq 0,$$

and hence by Theorem 3.1, μ satisfies the tree condition (3.2) with exponent $p = \frac{p_0}{2} > 1$. As above, we conclude that μ is $B_p(\mathbb{B}_n)$-Carleson for all $1 < p < \frac{p_0}{2}$. Since μ is also $B_{p_0}(\mathbb{B}_n)$-Carleson, complex interpolation shows that μ is $B_p(\mathbb{B}_n)$-Carleson for all $1 < p < p_0$. The assertion regarding the tree condition now follows easily, and this completes the proof of Corollary 3.2.

The tree theorem in [**ArRoSa**] mentioned above, together with the extension to $q < p$ in [**Ar**] (not used in the remainder of this paper), yields the equivalence of conditions 2 and 3 in Theorem 3.1, and we will consider the necessity and sufficiency of condition 3 for condition 1 separately in the next two subsections. For convenience we prove only the case $\lambda = 1$ and $\theta = \frac{\ln 2}{2}$, so that by Lemma 2.8, the dimension of $\mathcal{T}_n$ is n, and

$$1 - |z|^2 \approx e^{-2\beta(0,z)} \approx e^{-2\theta d(\alpha)} = 2^{-d(\alpha)} \tag{3.3}$$

for $z \in K_\alpha$ by (2.14). The proof of the general case is similar. But first we dualize the Carleson embedding by computing its adjoint relative to the pairing $\langle \cdot, \cdot \rangle_{\alpha,p}$ introduced above.

Let $\langle f, g \rangle_\mu = \int_{\mathbb{B}_n} f(z) \overline{g(z)} d\mu(z)$ be the usual pairing between $L^q(\mu)$ and $L^{q'}(\mu)$, and suppose that $1 < p < \infty$, $\alpha > -1$. We claim that for a polynomial $f \in B_p(\mathbb{B}_n)$ and a simple function $g \in L^{q'}(\mu)$, we have

$$\langle f, g \rangle_\mu = \langle f, \Theta g \rangle_{\alpha,p}$$

where $\Theta g \in B_{p'}(\mathbb{B}_n)$ is given by the formula

$$\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} \Theta g(w) = \int_{\mathbb{B}_n} \left(\frac{1}{1 - \overline{z} \cdot w} \right)^{\frac{n+1+\alpha}{p'}} g(z)\, d\mu(z).$$

Indeed, by (2.12) and (2.13) we have

$$\begin{aligned}
\langle f, g \rangle_\mu &= \int_{\mathbb{B}_n} f(z) \overline{g(z)} d\mu(z) = \int_{\mathbb{B}_n} \langle f, k_z^{\alpha,p} \rangle_{\alpha,p} \overline{g(z)} d\mu(z) \\
&= \int_{\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}} f(w) \overline{\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} k_z^{\alpha,p}(w)} d\nu_\alpha(w) \right\} \overline{g(z)} d\mu(z) \\
&= \int_{\mathbb{B}_n} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}} f(w) \overline{\left\{ \int_{\mathbb{B}_n} \left(\frac{1}{1 - \overline{z} \cdot w} \right)^{\frac{n+1+\alpha}{p'}} g(z)\, d\mu(z) \right\}} d\nu_\alpha(w).
\end{aligned}$$

Using the density of polynomials in Besov spaces, and simple functions in Lebesgue spaces, we can now dualize the Carleson embedding

$$\|f\|_{L^q(\mu)} = \left(\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{B_p(\mathbb{B}_n)}, \qquad f \in B_p(\mathbb{B}_n),$$

as equivalent to

$$\left\|\Theta g\right\|_{B_{p'}(\mathbb{B}_n)} \leq C\left\|g\right\|_{L^{q'}(\mu)}.$$

Since we have

$$\begin{aligned}\left\|\Theta g\right\|_{B_{p'}(\mathbb{B}_n)} &= \left(\int_{\mathbb{B}_n}\left|\int_{\mathbb{B}_n}\left(\frac{1}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)\right|^{p'} d\nu_\alpha(w)\right)^{\frac{1}{p'}} \\ &= \left(\int_{\mathbb{B}_n}\left|\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)\right|^{p'} d\lambda_n(w)\right)^{\frac{1}{p'}},\end{aligned}$$

we can restate the dual inequality as

$$\left\|S_\mu^* g\right\|_{L^{p'}(\lambda_n)} \leq C\left\|g\right\|_{L^{q'}(\mu)}, \qquad g\in L^{q'}(\mu), \tag{3.4}$$

where the operator S_μ^* is given by

$$S_\mu^* g(w) = \int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z).$$

Remark 3.3. The implication 1. implies 2. of Theorem 3.1 is equivalent to the implication that boundedness of S_μ^* from $L^{q'}(\mu)$ to $L^{p'}(\lambda_n)$ (as in (3.4)) implies the boundedness of T_μ^* from $L^{q'}(\mu)$ to $L^{p'}(\lambda_n)$, where

$$T_\mu^* g(w) = \int_{\mathbb{B}_n}\left|\frac{1-|w|^2}{1-\overline{z}\cdot w}\right|^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)$$

has kernel equal to the modulus of the kernel of S_μ^*. Roughly speaking, the Carleson embedding implies the tree condition if and only if we can take absolute values inside the operator S_μ^* without destroying the boundedness in (3.4). This claim follows easily from the argument in the next subsection.

3.1. Necessity in case $1<p<1+\frac{1}{n-1}$**.** Suppose that μ is a $(B_p(\mathbb{B}_n), q)$-Carleson measure on $\mathbb{B}_n$ where $1<p<1+\frac{1}{n-1}$ and $1<q<\infty$. Choose $\alpha>-1$ so that $\frac{n+1+\alpha}{p'}=1$. We then obtain from (3.4) that

$$\begin{aligned}\int_{\mathbb{B}_n}\left|\int_{\mathbb{B}_n}\mathrm{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right) g(z)\,d\mu(z)\right|^{p'} d\lambda_n(w) &\leq \left\|S_\mu^* g\right\|_{L^{p'}(\lambda_n)}^{p'} \\ &\leq C\left(\int_{\mathbb{B}_n}|g|^{q'}\,d\mu\right)^{p'/q'}\end{aligned}$$

for all $g\geq 0$. The tree inequality (3.1) now follows as in the one-dimensional case in [**ArRoSa**], [**Ar**]. Indeed, fix $\alpha\in\mathcal{T}_n$ and let $g=\sum_{\alpha\in_n} g(\alpha)\chi_{K_\alpha}$. Here g is constant on K_α with value $g(\alpha)$. Then since

$$\mathrm{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)\geq c>0, \qquad w\in K_\beta, z\in S_\beta$$

for $\beta \geq \alpha$, and $\mathrm{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right) \geq 0$ otherwise, we obtain

$$\begin{aligned} c^{p'} \|I^* \widehat{g\mu}\|^{p'}_{L^{p'}(\mathcal{T}_n)} &= \sum_{\alpha \in \mathcal{T}_n} \left(\sum_{\beta \geq \alpha} c\widehat{g\mu}(\beta) \right)^{p'} \\ &\leq C \int_{\mathbb{B}_n} \left(\int \mathrm{Re} \left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right) g(z)\, d\mu(z) \right)^{p'} d\lambda_n(w) \\ &\leq C \left(\int_{\mathbb{B}_n} |g|^{q'} d\mu \right)^{p'/q'} \\ &= C \left(\sum_{\alpha \in \mathcal{T}_n} |g(\alpha)|^{q'} \mu(\alpha) \right)^{\frac{p'}{q'}} = C\|g\|^{p'}_{L^{q'}(\mathcal{T}_n)}, \end{aligned}$$

which yields the dual of inequality (3.1).

Unfortunately, this elegant proof breaks down for $p \geq 1 + \frac{1}{n-1}$, since we can no longer choose $\alpha > -1$ so that $\theta = \frac{n+1+\alpha}{p'} \in (0, 1]$, thus forcing $\mathrm{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\theta} > 0$.

3.2. Sufficiency. Suppose that $\widehat{\mu}$ satisfies the tree inequality (3.1). Since $\widehat{\mu} = \widehat{\widetilde{\mu}}$, we now replace μ by $\widetilde{\mu}$ in the dual inequality (3.4) and consider

$$S^*_{\widetilde{\mu}} g(w) = \int_{\mathbb{B}_n} \left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right)^{\frac{n+1+\alpha}{p'}} g(z)\, d\widetilde{\mu}(z), \qquad w \in \mathbb{B}_n,$$

We will show that the positive operator $T^*_{\widetilde{\mu}}$ given by

$$T^*_{\widetilde{\mu}} g(w) = \int_{\mathbb{B}_n} \left(\frac{1-|w|^2}{|1-w\cdot\overline{z}|} \right)^{\frac{n+1+\alpha}{p'}} g(z)\, d\widetilde{\mu}(z), \qquad w \in \mathbb{B}_n,$$

is bounded from $L^{q'}(\widetilde{\mu})$ to $L^{p'}(\lambda_n)$, i.e.

$$\left(\int_{\mathbb{B}_n} \left| T^*_{\widetilde{\mu}} g(w) \right|^{p'} d\lambda_n(w) \right)^{\frac{1}{p'}} \leq C \left(\int_{\mathbb{B}_n} g^{q'} d\widetilde{\mu} \right)^{\frac{1}{q'}}, \qquad g \geq 0. \tag{3.5}$$

With this done, it follows that $\widetilde{\mu}$ is a (B_p, q)-Carleson measure, and hence also μ by Proposition 2.10.

From Lemma 3.4 below with $\sigma = 0$, $s = n+1+\alpha$, $r = p'$ and

$$f(z) = g(z) \frac{d\widetilde{\mu}}{dz}(z) = g(z) \sum_{\alpha \in \mathcal{T}_n} \mu(K_\alpha)\, \lambda_n(K_\alpha)^{-1} \left(1-|z|^2\right)^{-n-1} \chi_{K_\alpha}(z),$$

we obtain with $\widehat{T}$ as in Lemma 3.4,

$$\left\| T^*_{\widetilde{\mu}} g \right\|_{L^{p'}(d\lambda_n)} = \left\| \widehat{T}(g\widetilde{\mu}) \right\|_{L^{p'}(d\lambda_n)} \leq C_{p'} \left(\sum_{\alpha \in \mathcal{T}_n} I^* \widehat{g\widetilde{\mu}}(\alpha)^{p'} \right)^{\frac{1}{p'}},$$

where

$$I^* \widehat{g\widetilde{\mu}}(\alpha) = \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \int_{K_\beta} g\, d\widetilde{\mu} = \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \mu(K_\beta)\, \lambda_n(K_\beta)^{-1} \int_{K_\beta} g\, d\lambda_n$$

and $\widehat{g}(\alpha) = \int_{K_\alpha} g d\lambda_n$. The tree inequality (3.1) holds for $\widetilde{\mu}(\alpha) = \widetilde{\mu}(K_\alpha)$, and this in turn is equivalent to

$$\left(\sum_{\alpha \in \mathcal{T}_n} I^* \widehat{g\widetilde{\mu}}(\alpha)^{p'} \right)^{\frac{1}{p'}} \leq C \left(\sum_{\alpha \in \mathcal{T}_n} \widehat{g}(\alpha)^{q'} \widetilde{\mu}(\alpha) \right)^{\frac{1}{q'}},$$

Finally, since $\widetilde{\mu}$ is constant on K_α, we have

$$\widehat{g}(\alpha)^{q'} \widetilde{\mu}(\alpha) = \mu(K_\alpha) \left(\int_{K_\alpha} g d\lambda_n \right)^{q'} \leq C\mu(K_\alpha) \int_{K_\alpha} g^{q'} d\lambda_n \approx \int_{K_\alpha} g^{q'} d\widetilde{\mu},$$

for all $\alpha \in \mathcal{T}_n$, and hence

$$\left(\sum_{\alpha \in \mathcal{T}_n} \widehat{g}(\alpha)^{q'} \widetilde{\mu}(\alpha) \right)^{\frac{1}{q'}} \leq \left(\int_{\mathbb{B}_n} g^{q'} d\widetilde{\mu} \right)^{\frac{1}{q'}}.$$

Combining these inequalities establishes (3.5) as required.

LEMMA 3.4. *For $0 \leq \sigma \leq 1$, $1 < r < \infty$, $s + \sigma r > n$ and $f \in L^1$ define*

$$\widehat{T}f(w) = \int_{\mathbb{B}_n} \frac{\left(1 - |w|^2\right)^{\frac{s}{r}}}{|1 - w \cdot \overline{z}|^{\frac{s+\sigma r}{r}}} f(z)\, dz.$$

Then we have

$$\left\| \widehat{T}f \right\|_{L^r(d\lambda_n)} \leq C_r \left(\sum_{\alpha \in \mathcal{T}_n} \left[e^{2\theta\sigma d(\alpha)} I^* \widehat{f}(\alpha) \right]^r \right)^{\frac{1}{r}},$$

where $I^* \widehat{f}(\alpha) = \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \widehat{f}(\beta)$ and $\widehat{f}(\beta) = \int_{K_\beta} |f(z)|\, dz$.

Note that the discretization of f involves the measure dz, as compared to the discretization of g above that uses $d\lambda_n(z)$.

PROOF. Let $\theta = \frac{\ln 2}{2}$ for notational convenience. We compute for $f \geq 0$,

$$\begin{aligned} \left\| \widehat{T}f \right\|_{L^r(d\lambda_n)}^r &= \int \left(\int_{\mathbb{B}_n} \frac{\left(1 - |w|^2\right)^{\frac{s}{r}}}{|1 - w \cdot \overline{z}|^{\frac{s+\sigma r}{r}}} f(z)\, dz \right)^r d\lambda_n(w) \\ &\leq \int \left(\frac{1}{\left(1 - |w|^2\right)^\sigma} \sum_{m=0}^{\log_2\left(\frac{1}{1-|w|}\right)} 2^{-m\frac{s+\sigma r}{r}} \int_{S_{w_m}} f(z)\, dz \right)^r d\lambda_n(w) \end{aligned}$$

where S_{w_m} is the union of the Carleson box at w_m with its lower half, and the points $w_m = P_m(w)$ are positive multiples of w, but with modulus satisfying $1 - |w_m| = 2^m(1 - |w|)$. We now fix a constant μ such that $\frac{n}{s+\sigma r} < \mu < 1$ and continue the string of inequalities starting with Hölder's inequality:

$$\left\|\widehat{T}f\right\|_{L^r(d\lambda_n)}^r \leq \int \left\{ \sum_{m=0}^{\log_2\left(\frac{1}{1-|w|}\right)} 2^{-m\frac{s+\sigma r}{r}(1-\mu)r'} \right\}^{\frac{r}{r'}}$$

$$\times \sum_{m=0}^{\log_2\left(\frac{1}{1-|w|}\right)} 2^{-m(s+\sigma r)\mu} \left(\frac{1}{\left(1-|w|^2\right)^\sigma} \int_{S_{w_m}} f(z)\,dz \right)^r d\lambda_n(w)$$

$$\leq C_{s,r,\sigma,\mu} \sum_{m=0}^{\infty} 2^{-m(s+\sigma r)\mu} \int \left(\frac{1}{\left(1-|w|^2\right)^\sigma} \int_{S_{w_m}} f(z)\,dz \right)^r$$

$$\times \chi_{[m,\infty)}\left(\log_2\left(\frac{1}{1-|w|}\right)\right) d\lambda_n(w)$$

$$\leq C_{s,r,\sigma,\mu} \sum_{m=0}^{\infty} 2^{-m(s+\sigma r)\mu} \sum_{\alpha\in\mathcal{T}_n} \chi_{[m,\infty)}(d(\alpha))$$

$$\times \left(2^{\sigma d(\alpha)} \sum_{\beta\in\mathcal{T}_n:\beta\geq P^m(\alpha)} f(\beta) \right)^r,$$

with $C_{s,r,\sigma,\mu} = \left(\sum_{m=0}^{\infty} 2^{-m\frac{s+\sigma r}{r}(1-\mu)r'}\right)^{r-1}$, and where $P^m(\alpha)$ denotes the m^{th} predecessor of α. The final term above satisfies

$$\sum_{m=0}^{\infty} 2^{-m(s+\sigma r)\mu} \sum_{\alpha\in\mathcal{T}_n} \chi_{[m,\infty)}(d(\alpha)) \left(2^{\sigma d(\alpha)} \sum_{\beta\in\mathcal{T}_n:\beta\geq P^m(\alpha)} f(\beta) \right)^r$$

$$\leq \sum_{m=0}^{\infty} 2^{m(n-(s+\sigma r)\mu)} \sup_{m\geq 0} 2^{-nm} \sum_{\alpha\in\mathcal{T}_n} \chi_{[m,\infty)}(d(\alpha)) \left(2^{\sigma d(\alpha)} \sum_{\substack{\beta\in\mathcal{T}_n \\ \beta\geq P^m(\alpha)}} f(\beta) \right)^r$$

$$\leq CC'_{s,n,\sigma,\mu} \sum_{\alpha\in\mathcal{T}_n} \left(2^{\sigma d(\alpha)} \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha} f(\beta) \right)^r = \sum_{\alpha\in\mathcal{T}_n} \left(2^{\sigma d(\alpha)} I^* f(\alpha) \right)^r,$$

with $C'_{s,n,\sigma,\mu} = \sum_{m=0}^{\infty} 2^{m(n-(s+\sigma r)\mu)} < \infty$ since $\frac{n}{s+\sigma r} < \mu$, and where we have used the fact that

$$card\{\alpha\in\mathcal{T}_n : P^m(\alpha) = \gamma\} \leq C(2^n)^m,$$

which follows from Lemma 2.8 and our choice of $\theta = \frac{\ln 2}{2}$ in (3.3). This completes the proof of Lemma 3.4.

With this lemma proved, we have completed the proof that condition 3 is sufficient for condition 1 in Theorem 3.1.

3.3. Necessity in the extended range $1 < p < 2 + \frac{1}{n-1}$. Suppose that μ is a $(B_p(\mathbb{B}_n), q)$-Carleson measure on $\mathbb{B}_n$ where $1 < p < 2 + \frac{1}{n-1}$, $1 < q < \infty$. We have from (3.4) that

$$\int_{\mathbb{B}_n}\left|\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)\right|^{p'} d\lambda_n(w) \tag{3.6}$$
$$= \left\|S_\mu^* g\right\|_{L^{p'}(\lambda_n)}^{p'} \leq C\left(\int_{\mathbb{B}_n}|g|^{q'}\,d\mu\right)^{p'/q'},$$

where the operator S_μ^* is given by

$$S_\mu^* g(w) = \int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z).$$

The left side of (3.6) raised to the power $\frac{1}{p'}$ is

$$\left(\int_{\mathbb{B}_n}\left|\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)\right|^{p'} d\lambda_n(w)\right)^{\frac{1}{p'}} \tag{3.7}$$
$$= \sup_{\|F\|_{L^p(d\lambda_n)}\leq 1}\left|\int_{\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)\right\}\overline{F(w)}\,d\lambda_n(w)\right|$$
$$\geq \sup_{f\geq 0}\left|\int_{\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} g(z)\,d\mu(z)\right\}\frac{\overline{T_\eta f(w)}}{\|T_\eta f\|_{L^p(d\lambda_n)}}\,d\lambda_n(w)\right|,$$

where for $0<\eta<1$ we define the operator T_η by

$$T_\eta f(w) = \int_{\mathbb{B}_n}\left(1-|w|^2\right)^{\frac{n+1+\alpha}{p}}\left\{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\left(\frac{1}{1-\overline{z}\cdot w}\right)^\eta\right\} f(z)\,dz$$
$$= \left(1-|w|^2\right)^{\frac{n+1+\alpha}{p}}\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\int_{\mathbb{B}_n}\frac{f(z)}{(1-\overline{z}\cdot w)^\eta}\,dz.$$

From (2.7) we have

$$\left(\frac{1}{1-\overline{z'}\cdot w}\right)^\eta = R_{\eta-n-1,\frac{n+1+\alpha}{p}}\left(\frac{1}{1-\overline{z'}\cdot w}\right)^{\frac{n+1+\alpha}{p}+\eta},$$

and so with

$$\widetilde{T_\eta} f(w) = \int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{p}}}{(1-\overline{z}\cdot w)^{\frac{n+1+\alpha}{p}+\eta}} f(z)\,dz,$$
$$U_\eta f(w) = \frac{T_\eta f(w)}{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{p}}} = \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\int_{\mathbb{B}_n}\frac{f(z)}{(1-\overline{z}\cdot w)^\eta}\,dz,$$
$$\widetilde{U_\eta} f(w) = \frac{\widetilde{T_\eta} f(w)}{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{p}}} = \int_{\mathbb{B}_n}\frac{f(z)}{(1-\overline{z}\cdot w)^{\frac{n+1+\alpha}{p}+\eta}}\,dz,$$

we obtain

$$U_\eta f(w) = R^{\alpha - \frac{n+1+\alpha}{p}, \frac{n+1+\alpha}{p}} R_{\eta-n-1, \frac{n+1+\alpha}{p}} \widetilde{U_\eta} f(w).$$

We now use the fact that $R^{\gamma_1,t}R_{\gamma_2,t}$ is bounded on the Bergman space A_p^α provided $n+\gamma_i$, $n+\gamma_i+t \notin -\mathbb{N}$ (Corollary 6.5 of [**Zhu**]) to conclude that

$$\begin{aligned} \|T_\eta f\|_{L^p(d\lambda_n)} &= \|U_\eta f\|_{L^p(d\nu_\alpha)} \\ &= \left\| R^{\alpha - \frac{n+1+\alpha}{p}, \frac{n+1+\alpha}{p}} R_{\eta-n-1, \frac{n+1+\alpha}{p}} \widetilde{U_\eta} f \right\|_{L^p(d\nu_\alpha)} \\ &\leq C \left\| \widetilde{U_\eta} f \right\|_{L^p(d\nu_\alpha)} = C \left\| \widetilde{T_\eta} f \right\|_{L^p(d\lambda_n)}. \end{aligned} \tag{3.8}$$

REMARK 3.5. Note that in the special case $p=2$, $S_\mu^* g = \widetilde{T_0}(g\mu)$, so that the adaptation of the argument below to $\eta = 0$ reduces to a familiar TT^* argument.

The final line in (3.7) is

$$\sup_{f \geq 0} \frac{1}{\|T_\eta f\|_{L^p(d\lambda_n)}} \left| \int_{\mathbb{B}_n} S_\mu^* g(w) \overline{T_\eta f(w)} d\lambda_n(w) \right|. \tag{3.9}$$

To compute this supremum, we note that the integral in (3.9) for $f \geq 0$ is

(3.10)

$$\begin{aligned} &\int_{\mathbb{B}_n} S_\mu^* g(w) \overline{T_\eta f(w)} d\lambda_n(w) \\ &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \left(\frac{1}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} \left(\frac{1}{1-\overline{w}\cdot z'}\right)^\eta d\nu_\alpha(w) f(z') dz' g(z) d\mu(z) \\ &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \left(\frac{1}{1-\overline{z}\cdot z'}\right)^\eta f(z') dz' g(z) d\mu(z) \end{aligned}$$

where we obtained the final equality using (2.13). Writing $\ell^\eta_{z'}(z) = \left(\frac{1}{1-\overline{z'}\cdot z}\right)^\eta$ we have

$$\begin{aligned} \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} k_z^{\alpha,p}(w) \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} \ell^\eta_{z'}(w)} d\nu_\alpha(w) &= \overline{\langle \ell^\eta_{z'}, k_z^{\alpha,p}\rangle_{\alpha,p}} \\ &= \overline{\ell^\eta_{z'}(z)} = \left(\frac{1}{1-\overline{z}\cdot z'}\right)^\eta. \end{aligned}$$

We can discretize the last integral in (3.10) by breaking up the integrals over the balls into Bergman kubes K_α and using the fact that $\left(\frac{1}{1-\overline{z}\cdot z'}\right)^\eta$ is essentially constant on products of Bergman kubes. Observe also that by our choice of $\theta = \frac{\ln 2}{2}$ in (3.3),

$$\operatorname{Re}\left(\frac{1}{1-\overline{z}\cdot z'}\right)^\eta \geq c_\eta 2^{\eta d(\alpha\wedge\beta)}, \qquad z \in K_\alpha, z' \in K_\beta,$$

for a positive constant c_η, provided $0 < \eta < 1$. Note that c_η tends to 0 as $\eta \to 1$, so that we cannot use $\eta = 1$ even though Re $\frac{1}{1-\overline{z}\cdot z'} > 0$ on the ball. We obtain that

$$\begin{aligned}\left|\int_{\mathbb{B}_n} S_\mu^* g(w)\overline{T_\eta f(w)}d\lambda_n(w)\right| &\geq \left|\mathrm{Re}\int_{\mathbb{B}_n} S_\mu^* g(w)\overline{T_\eta f(w)}d\lambda_n(w)\right| \\ &\geq c_\eta \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \mathrm{Re}\left(\frac{1}{1-\overline{z}\cdot z'}\right)^\eta g(z)\,d\mu(z)\,\overline{f(z')}dz' \\ &\geq c_\eta \sum_{\alpha\in\mathcal{T}_n}\sum_{\beta\in\mathcal{T}_n} 2^{\eta d(\alpha\wedge\beta)} g\mu(\alpha)\,f(\beta),\end{aligned}$$

for $f, g \geq 0$ on the ball $\mathbb{B}_n$, and with $f(\beta) = \int_{K_\beta} f(z)\,dz$, $g\mu(\alpha) = \int_{K_\alpha} g(z)\,d\mu(z)$. We also observe that

$$\begin{aligned}\sum_{\gamma\in\mathcal{T}_n} 2^{\eta d(\gamma)} I^* g\mu(\gamma)\, I^* f(\gamma) &= \sum_{\gamma\in\mathcal{T}_n} 2^{\eta d(\gamma)} \sum_{\alpha\in\mathcal{T}_n:\alpha\geq\gamma} g\mu(\alpha) \sum_{\beta\in\mathcal{T}_n:\beta\geq\gamma} f(\beta) \\ &= \sum_{\alpha\in\mathcal{T}_n}\sum_{\beta\in\mathcal{T}_n}\left(\sum_{\gamma\in\mathcal{T}_n:\alpha,\beta\geq\gamma} 2^{\eta d(\gamma)}\right) g\mu(\alpha)\,f(\beta) \\ &\approx \sum_{\alpha\in\mathcal{T}_n}\sum_{\beta\in\mathcal{T}_n} 2^{\eta d(\alpha\wedge\beta)} g\mu(\alpha)\,f(\beta),\end{aligned}$$

and so altogether we obtain

$$\sup_{f\geq 0}\frac{1}{\|T_\eta f\|_{L^p(d\lambda_n)}}\sum_{\gamma\in\mathcal{T}_n} 2^{\eta d(\gamma)} I^* g\mu(\gamma)\, I^* f(\gamma) \leq C\left(\int_{\mathbb{B}_n} |g(z)|^{q'}\,d\mu(z)\right)^{\frac{1}{q'}}, \tag{3.11}$$

for $g \in L^{q'}(\mu)$. Provided g is nonnegative and essentially constant on the Bergman boxes K_α, we can discretize the final integral above as

$$\int_{\mathbb{B}_n} |g(w)|^{q'}\,d\mu(w) \approx \sum_{\alpha\in\mathcal{T}_n} g(\alpha)^{q'}\,\mu(\alpha). \tag{3.12}$$

Then (3.11) becomes approximately

$$\sup_{f\geq 0}\frac{1}{\|T_\eta f\|_{L^p(d\lambda_n)}}\sum_{\gamma\in\mathcal{T}_n} 2^{\eta d(\gamma)} I^* g\mu(\gamma)\, I^* f(\gamma) \leq C\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)^{q'}\,\mu(\alpha)\right)^{\frac{1}{q'}}.$$

With $\sigma = \eta$, $s = n + 1 + \alpha$ and $r = p$ in Lemma 3.4, we obtain

$$\left\|\widetilde{T_\eta} f\right\|_{L^p(d\lambda_n)} \leq C_p\left(\sum_{\alpha\in\mathcal{T}_n}\left(2^{\eta d(\alpha)} I^* f(\alpha)\right)^p\right)^{\frac{1}{p}},$$

which together with (3.8), yields the discretized inequality

$$\sum_{\gamma\in\mathcal{T}_n} I^* g\mu(\gamma)\, 2^{\eta d(\gamma)} I^* f(\gamma) \leq C\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)^{q'}\,\mu(\alpha)\right)^{\frac{1}{q'}}\left(\sum_{\alpha\in\mathcal{T}_n}\left(2^{\eta d(\alpha)} I^* f(\alpha)\right)^p\right)^{\frac{1}{p}}, \tag{3.13}$$

for $f, g \geq 0$ on $\mathcal{T}_n$. If we write the left side of (3.13) as

$$\sum_{\gamma\in\mathcal{T}_n} I^* g\mu(\gamma)\, 2^{\eta d(\gamma)} I^* f(\gamma) = \sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\, I 2^{\eta d} I^* f(\alpha)$$

and then take the supremum over all $g \geq 0$ we obtain the inequality
(3.14)
$$\left(\sum_{\alpha\in\mathcal{T}_n}\left[I2^{\eta d}I^*h(\alpha)\right]^q\mu(\alpha)\right)^{1/q}\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\left[2^{\eta d(\alpha)}I^*h(\alpha)\right]^p\right)^{1/p},\qquad h\geq 0 \text{ on } \mathcal{T}_n,$$
which is (1.7) tested over f of the form $f = 2^{\eta d}I^*h$, and with $\mu(\alpha) = \int_{K_\alpha} d\mu$ and $\mathcal{T}$ replaced by $\mathcal{T}_n$. We will however continue instead with the bilinear form (3.13). We may assume without loss of generality that μ has finite support on the tree $\mathcal{T}_n$. Indeed, if we simply restrict μ on the disk to a finite union $\mathcal{F}$ of Carleson boxes, then this restriction $\mu_{\mathcal{F}}$ is a Carleson measure with norm under control. If we can show that condition (3.2), and thus also (1.7), holds for $\mu_{\mathcal{F}}$ with a constant independent of $\mathcal{F}$, then obviously (1.7) holds for μ with the same constant.

If we are able to find $f \geq 0$ such that

$$2^{\eta d(\gamma)}I^*f(\gamma) = \left(I^*g\mu(\gamma)\right)^{p'-1},\qquad \gamma\in\mathcal{T}_n, \tag{3.15}$$

then from (3.13) we have

$$\begin{aligned}\sum_{\gamma\in\mathcal{T}_n}\left(I^*g\mu(\gamma)\right)^{p'} &\leq C\left(\sum_{\alpha\in\mathcal{T}_n}g(\alpha)^{q'}\mu(\alpha)\right)^{\frac{1}{q'}}\left(\sum_{\alpha\in\mathcal{T}_n}\left(I^*g\mu(\alpha)\right)^{(p'-1)p}\right)^{\frac{1}{p}}\\ &= C\left(\sum_{\alpha\in\mathcal{T}_n}g(\alpha)^{q'}\mu(\alpha)\right)^{\frac{1}{q'}}\left(\sum_{\alpha\in\mathcal{T}_n}\left(I^*g\mu(\alpha)\right)^{p'}\right)^{\frac{1}{p}},\end{aligned}$$

which yields (3.2) as required. So it remains to solve (3.15) for as large a range of p as we can, using $0 < \eta < 1$. To this end, we invoke the following elementary result on a tree $\mathcal{T}$.

LEMMA 3.6. *Given $G \geq 0$ on $\mathcal{T}$ such that $G(o) < \infty$, there is $h \geq 0$ on $\mathcal{T}$ satisfying*

$$I^*h(\alpha) = G(\alpha),\qquad \alpha\in\mathcal{T}, \tag{3.16}$$

if and only if both

$$\sum_{d(\alpha)=N}G(\alpha)\to 0 \text{ as } N\to\infty, \tag{3.17}$$

and

$$G(\alpha)-\sum_j G(\alpha_j)\geq 0,\qquad \alpha\in\mathcal{T}, \tag{3.18}$$

and where α_j are the children of α.

PROOF. For the necessity, (3.16) and $h \geq 0$ imply

$$0\leq h(\alpha) = I^*h(\alpha)-\sum_j I^*h(\alpha_j) = G(\alpha)-\sum_j G(\alpha_j),$$

which is (3.18), while (3.16) and $\sum_{\alpha\in\mathcal{T}}h(\alpha) = I^*h(o) = G(o) < \infty$ with $h \geq 0$ yield

$$\sum_{d(\alpha)=N}G(\alpha) = \sum_{d(\alpha)=N}I^*h(\alpha) = \sum_{\beta\in\mathcal{T}:d(\beta)\geq N}h(\beta)\to 0 \text{ as } N\to\infty,$$

which is (3.17).

Conversely, (3.18) and (3.17) yield (3.16):

$$\begin{aligned}
I^*h(\alpha) &= \sum_{\beta\in\mathcal{T}:\beta\geq\alpha} h(\beta) = \lim_{N\to\infty}\sum_{\beta\geq\alpha,d(\beta)<N} h(\beta) \\
&= \lim_{N\to\infty}\sum_{\beta\geq\alpha,d(\beta)<N}\left\{G(\beta)-\sum_j G(\beta_j)\right\} \\
&= \lim_{N\to\infty}\left\{G(\alpha)-\sum_{d(\gamma)=N} G(\gamma)\right\} \\
&= G(\alpha).
\end{aligned}$$

By Lemma 3.6, solving (3.15) for $f\geq 0$ is equivalent to the inequality

$$2^{-\eta d(\gamma)}\left(I^*g\mu(\gamma)\right)^{p'-1} \geq \sum_{\gamma_j\in\mathcal{C}(\gamma)} 2^{-\eta d(\gamma_j)}\left(I^*g\mu(\gamma_j)\right)^{p'-1}, \qquad \gamma\in\mathcal{T}_n,$$

where the sum is taken over the the set $\mathcal{C}(\gamma)$ of children γ_j of γ. This inequality is trivial if $p\leq 2$ since then $p'-1\geq 1$ implies

$$\left(I^*g\mu(\gamma)\right)^{p'-1} \geq \left(\sum_{\gamma_j\in\mathcal{C}(\gamma)} I^*g\mu(\gamma_j)\right)^{p'-1} \geq \sum_{\gamma_j\in\mathcal{C}(\gamma)} I^*g\mu(\gamma_j)^{p'-1},$$

and $d(\gamma_j)>d(\gamma)$ for $\gamma_j\in\mathcal{C}(\gamma)$. So we suppose that $p>2$. Since $d(\gamma_j)=d(\gamma)+1$, and j runs from 1 to the maximum branching number N of the Bergman tree $\mathcal{T}_n$, we have using Hölder's inequality with exponents $p-1$ and $\frac{p-1}{p-2}$,

$$\begin{aligned}
&\sum_{\gamma_j\in\mathcal{C}(\gamma)} 2^{-\eta d(\gamma_j)}\left(I^*g\mu(\gamma_j)\right)^{p'-1} \\
&\quad\leq 2^{-\eta d(\gamma)-\eta}\left(\sum_{\gamma_j\in\mathcal{C}(\gamma)} 1\right)^{\frac{p-2}{p-1}}\left(\sum_j \left(I^*g\mu(\gamma_j)\right)^{(p'-1)(p-1)}\right)^{\frac{1}{p-1}} \\
&\quad\leq 2^{-\eta d(\gamma)}2^{-\eta}(N)^{\frac{p-2}{p-1}}\left(\sum_j I^*g\mu(\gamma_j)\right)^{\frac{1}{p-1}} \\
&\quad\leq 2^{(\log_2 N)\frac{p-2}{p-1}-\eta}\left\{2^{-\eta d(\gamma)}\left(I^*g\mu(\gamma_j)\right)^{p'-1}\right\} \\
&\quad\leq 2^{-\eta d(\gamma)}\left(I^*g\mu(\gamma_j)\right)^{p'-1},
\end{aligned}$$

as required provided $(\log_2 N)\frac{p-2}{p-1}-\eta\leq 0$ or $p\leq\frac{2(\log_2 N)-\eta}{(\log_2 N)-\eta}=2+\frac{\eta}{(\log_2 N)-\eta}$.

In order to obtain the full range $1<p<2+\frac{1}{n-1}$, we will take η sufficiently close to 1, and use the device of splitting the sum $\sum_{\alpha\in\mathcal{T}_n}$ into "sparse" pieces $\sum_{\alpha\in\mathcal{T}_n:d(\alpha)\in\ell\mathbb{N}+m}$ for $0\leq m<\ell$, where ℓ is chosen so large in Definition 2.7 that (3.19)

$$\log_2(N_\ell)^{\frac{1}{\ell}} = \log_2\left(\sup_{\alpha\in\mathcal{T}} card\{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)=d(\alpha)+\ell\}\right)^{\frac{1}{\ell}} < \frac{p-1}{p-2}.$$

This can be done if $p < 2 + \frac{1}{n-1}$, or equivalently $n < \frac{p-1}{p-2}$, since the dimension $n(\mathcal{T}_n)$ of the tree $\mathcal{T}_n$ is n by Lemma 2.8 when $\theta = \frac{\ln 2}{2}$.

With ℓ chosen so that (3.19) holds, we consider solving the equation

$$2^{\eta d(\gamma)} I^* f(\gamma) = (I^* g\mu(\gamma))^{p'-1}, \quad \gamma \in \mathcal{T}_n, d(\gamma) \in \ell\mathbb{Z}_+, \tag{3.20}$$

for $0 < \eta < 1$ and with $f \geq 0$ on $\mathcal{T}_n$ and supported in

$$\Omega_\ell \equiv \{\gamma \in \mathcal{T}_n : d(\gamma) \in \ell\mathbb{Z}_+\}.$$

By Lemma 3.6 applied to the tree Ω_ℓ, this is equivalent to the inequality

$$2^{-\eta d(\gamma)} (I^* g\mu(\gamma))^{p'-1} \geq \sum_{\gamma_j \in \mathcal{C}_\ell(\gamma)} 2^{-\eta d(\gamma_j)} (I^* g\mu(\gamma_j))^{p'-1}, \quad \gamma \in \Omega_\ell,$$

where the sum is now taken over the the set $\mathcal{C}_\ell(\gamma)$ of grand$^{\ell-1}$-children γ_j of γ, i.e. those with $\gamma_j > \gamma$ and $d(\gamma_j) = \ell$. From Hölder's inequality with exponents $p-1$ and $\frac{p-1}{p-2}$ again,

$$\begin{aligned}
&\sum_{\gamma_j \in \mathcal{C}_\ell(\gamma)} 2^{-\eta d(\gamma_j)} (I^* g\mu(\gamma_j))^{p'-1} \\
&\qquad \leq 2^{-\eta d(\gamma) - \eta\ell} \left(\sum_{\gamma_j \in \mathcal{C}_\ell(\gamma)} 1 \right)^{\frac{p-2}{p-1}} \left(\sum_j (I^* g\mu(\gamma_j))^{(p'-1)(p-1)} \right)^{\frac{1}{p-1}} \\
&\qquad \leq 2^{-\eta d(\gamma)} 2^{-\eta\ell} (N_\ell)^{\frac{p-2}{p-1}} \left(\sum_j I^* g\mu(\gamma_j) \right)^{\frac{1}{p-1}} \\
&\qquad \leq 2^{(\log_2 N_\ell)\frac{p-2}{p-1} - \eta\ell} \left\{ 2^{-\eta d(\gamma)} (I^* g\mu(\gamma))^{p'-1} \right\} \\
&\qquad \leq 2^{-\eta d(\gamma)} (I^* g\mu(\gamma))^{p'-1},
\end{aligned}$$

since $\left(\log_2 (N_\ell)^{\frac{1}{\ell}}\right) \frac{p-2}{p-1} - \eta < 0$ for η sufficiently close to 1 by (3.19).

Thus we have solved (3.20) for $f \geq 0$ on $\mathcal{T}_n$, and supported in Ω_ℓ. We also have

$$\sum_{\beta > \gamma : 1 \leq d(\beta,\gamma) \leq \ell} \left[2^{\eta d(\beta)} I^* f(\beta) \right]^p \leq C_{p,\ell} \left[2^{\eta d(\gamma)} I^* f(\gamma) \right]^p \tag{3.21}$$

$$= C_{p,\ell} (I^* g\mu(\gamma))^{p'}, \quad \gamma \in \Omega_\ell. \tag{3.22}$$

Thus using first (3.20) and (3.13), and then (3.21), we have

$$\begin{aligned}
\sum_{\gamma \in \Omega_\ell} (I^* g\mu(\gamma))^{p'} &\leq C \left(\sum_{\alpha \in \mathcal{T}_n} g(\alpha)^{q'} \mu(\alpha) \right)^{\frac{1}{q'}} \left(\sum_{\alpha \in \mathcal{T}_n} \left[2^{\eta d(\alpha)} I^* f(\alpha) \right]^p \right)^{\frac{1}{p}} \\
&\leq C \left(\sum_{\alpha \in \mathcal{T}_n} g(\alpha)^{q'} \mu(\alpha) \right)^{\frac{1}{q'}} \left(C_{p,\ell} \sum_{\gamma \in \Omega_\ell} (I^* g\mu(\gamma))^{p'} \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\sum_{\alpha \in \mathcal{T}_n} (I^* g\mu(\alpha))^{p'} \leq C'_{p,\ell} \sum_{\gamma \in \Omega_\ell} (I^* g\mu(\gamma))^{p'}$, this yields the dual of (3.1), and hence (3.2) as required.

Remark 3.7. If the weight $2^{\eta d(\gamma)}$ in (3.15) were replaced by $2^{nd(\gamma)}$, then the argument above would solve (3.15) for all $1 < p < \infty$. However, this would require using $\ell^n_{z'}(z) = \left(\frac{1}{1-\overline{z'}\cdot z}\right)^n$ above, and the real part of $\ell^n_{z'}(z)$ would no longer be positive. We do not know if inequality (3.14) characterizes (1.7) when $p \geq 2 + \frac{1}{n-1}$.

4. Pointwise multipliers

Recall that given $1 < p < \infty$, a positive Borel measure μ on the ball $\mathbb{B}_n$ is a $B_p(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$ if there is $C < \infty$ such that

$$\left(\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z)\right)^{\frac{1}{p}} \leq C \|f\|_{B_p(\mathbb{B}_n)}, \qquad f \in B_p(\mathbb{B}_n),$$

where

$$\|f\|_{B_p(\mathbb{B}_n)} = \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^m \nabla^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} + \sum_{k=0}^{m-1} \left|\nabla^k f(0)\right|$$

for any $m > \frac{n}{p}$.

Definition 4.1. *We say that φ is a (pointwise) multiplier on B_p if $\varphi f \in B_p$ for all $f \in B_p$. By the closed graph theorem, this is equivalent to the existence of a constant $C < \infty$ such that*

$$\|\varphi f\|_{B_p} \leq C \|f\|_{B_p}, \qquad f \in B_p.$$

Standard arguments show that if φ is a multiplier on B_p, then $\varphi \in H^\infty(\mathbb{B}_n) \cap B_p$. Indeed, since $1 \in B_p$, we have $\varphi = \varphi 1 \in B_p$. The adjoint M^*_φ of the multiplier operator $M_\varphi f = \varphi f$ is also bounded on B_p, and if e_z is the point evaluation functional on B_p, then

$$\left\langle f, M^*_\varphi e_z\right\rangle = \langle M_\varphi f, e_z\rangle = \varphi(z) f(z) = \varphi(z)\langle f, e_z\rangle = \left\langle f, \overline{\varphi(z)} e_z\right\rangle, \qquad f \in B_p,$$

shows that $M^*_\varphi e_z = \overline{\varphi(z)} e_z$. Thus

$$|\varphi(z)| \, \|e_z\|_{B'_p} = \left\|\overline{\varphi(z)} e_z\right\|_{B'_p} = \left\|M^*_\varphi e_z\right\|_{B'_p} \leq \left\|M^*_\varphi\right\| \|e_z\|_{B'_p}$$

implies that $|\varphi(z)| \leq \left\|M^*_\varphi\right\| = \|M_\varphi\|$ since $\|e_z\|_{B'_p} < \infty$.

Theorem 4.2. *Let $\varphi \in H^\infty(\mathbb{B}_n) \cap B_p$ and $m > \frac{n}{p}$. Then φ is a multiplier on $B_p(\mathbb{B}_n)$ if and only if*

$$\left|\left(1-|z|^2\right)^m \nabla^m \varphi(z)\right|^p d\lambda_n(z)$$

is a $B_p(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$.

Proof. Fix $1 < p < \infty$ and $m > \frac{n}{p}$. Let $f, \varphi \in B_p$. Then

$$\|\varphi f\|_{B_p} = \sum_{k=0}^{m-1} \left|\nabla^k (\varphi f)(0)\right| + \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^m \nabla^m (\varphi f)(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}},$$

and

$$\nabla^m (\varphi f)(z) = \sum_{k=0}^{m} c_{m,k} \left(\nabla^{m-k}\varphi(z)\right)\left(\nabla^k f(z)\right). \tag{4.1}$$

show that

$$\begin{aligned}
&\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m\nabla^m(\varphi f)(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\
&\le C\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m\nabla^m\varphi(z)\right|^p|f(z)|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\
&+C\sum_{k=1}^{m-1}\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m-k}\nabla^{m-k}\varphi(z)\right|^p\left|\left(1-|z|^2\right)^k\nabla^k f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\
&+C\left(\int_{\mathbb{B}_n}|\varphi(z)|^p\left|\left(1-|z|^2\right)^m\nabla^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}.
\end{aligned}$$

Let

$$q_k=\frac{m}{m-k},q'_k=\frac{m}{k},\quad 1\le k\le m-1,$$

and apply Holder's inequality to obtain for each $1\le k\le m-1$,

$$\begin{aligned}
&\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m-k}\nabla^{m-k}\varphi(z)\right|^p\left|\left(1-|z|^2\right)^k\nabla^k f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\
&\qquad\le\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m-k}\nabla^{m-k}\varphi(z)\right|^{pq_k}d\lambda_n(z)\right)^{\frac{1}{pq_k}}\\
&\qquad\quad\times\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^k\nabla^k f(z)\right|^{pq'_k}d\lambda_n(z)\right)^{\frac{1}{pq'_k}}\\
&\le\|\varphi\|_{B_{pq_k}(\mathbb{B}_n)}\|f\|_{B_{pq'_k}(\mathbb{B}_n)}
\end{aligned}$$

since $m-k=\frac{m}{q_k}>\frac{n}{pq_k}$ and $k=\frac{m}{q'_k}>\frac{n}{pq'_k}$. Now the atomic decomposition of Besov spaces, Theorem 6.6. in [**Zhu**], implies in particular that the inclusions of the Besov spaces $B_p(\mathbb{B}_n)$ are determined by those of the ℓ^p spaces. Thus

$$B_p(\mathbb{B}_n)\subset B_q(\mathbb{B}_n),\quad 0<p<q<\infty,$$

and so we have

$$\|\varphi\|_{B_{pq_k}(\mathbb{B}_n)}\|f\|_{B_{pq'_k}(\mathbb{B}_n)}\le\|\varphi\|_{B_p(\mathbb{B}_n)}\|f\|_{B_p(\mathbb{B}_n)},\quad 1\le k\le m-1,$$

since $q_k,q'_k>1$. Also,

$$\begin{aligned}
\sum_{k=0}^{m-1}\left|\nabla^k(\varphi f)(0)\right|&\le C\sum_{k=0}^{m-1}\left|\sum_{j=0}^{k}\left(\nabla^{k-j}\varphi(0)\right)\left(\nabla^j f(0)\right)\right|\\
&\le C\left(\sum_{k=0}^{m-1}\left|\nabla^k\varphi(0)\right|\right)\left(\sum_{k=0}^{m-1}\left|\nabla^k f(0)\right|\right)\\
&\le\|\varphi\|_{B_p(\mathbb{B}_n)}\|f\|_{B_p(\mathbb{B}_n)},
\end{aligned}$$

and combining all of these inequalities, we obtain

(4.2)

$$\|\varphi f\|_{B_p(\mathbb{B}_n)}\le C\left\{\|f\|_{L^p(\mu)}+\|\varphi\|_{B_p(\mathbb{B}_n)}\|f\|_{B_p(\mathbb{B}_n)}+\|\varphi\|_{H^\infty(\mathbb{B}_n)}\|f\|_{B_p(\mathbb{B}_n)}\right\},$$

where

$$d\mu(z) = \left|\left(1-|z|^2\right)^m \nabla^m \varphi(z)\right|^p d\lambda_n(z).$$

Similarly, if we rewrite (4.1) as

$$c_{m,0}\left(\nabla^m \varphi(z)\right) f(z) = -\nabla^m(\varphi f)(z) + \sum_{k=1}^m c_{m,k}\left(\nabla^{m-k}\varphi(z)\right)\left(\nabla^k f(z)\right),$$

and multiply through by $\left(1-|z|^2\right)^m$, the above inequalities yield

(4.3)

$$\|f\|_{L^p(\mu)} \leq C\left\{\|\varphi f\|_{B_p(\mathbb{B}_n)} + \|\varphi\|_{B_p(\mathbb{B}_n)}\|f\|_{B_p(\mathbb{B}_n)} + \|\varphi\|_{H^\infty(\mathbb{B}_n)}\|f\|_{B_p(\mathbb{B}_n)}\right\}.$$

For $\varphi \in H^\infty(\mathbb{B}_n) \cap B_p(\mathbb{B}_n)$, inequalities (4.2) and (4.3) show that φ is a multiplier on B_p if and only if μ is a $B_p(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$.

5. Interpolating sequences

Let $\{z_j\}_{j=1}^\infty$ be a sequence of points in the unit ball $\mathbb{B}_n$, and $1<p<\infty$. In the present section we will prove that weighted ℓ^p interpolation for Besov spaces $B_p(\mathbb{B}_n)$ holds on the sequence $\{z_j\}_{j=1}^\infty$ if and only if the following separation condition and Carleson embedding hold;

$$\text{(5.1)}\qquad \beta(z_i,0) \leq C\beta(z_i,z_j) \text{ and}$$
$$\sum_{j=1}^\infty \left(1+\log\frac{1}{1-|z_j|^2}\right)^{1-p}\delta_{z_j} \text{ is a } B_p(\mathbb{B}_n)\text{-Carleson measure.}$$

We may assume without loss of generality that the points z_j occur as the centers c_{α_j} for a corresponding sequence $\{\alpha_j\}_{j=1}^\infty$ in the Bergman tree $\mathcal{T}_n$ (this requires only a much weaker notion of separation, $\beta(z_i,z_j) \geq c > 0$), and we take $\lambda = 1$ and $\theta = \frac{\ln 2}{2}$ in the construction. Note that (3.3) yields

$$\begin{aligned} d(\alpha_i,o) &\approx \beta(c_{\alpha_i},0) \\ &\approx \log\frac{1}{1-|c_{\alpha_i}|^2}, \end{aligned}$$

where d denotes distance in the Bergman tree $\mathcal{T}_n$. Furthermore, the separation condition $\beta(z_i,0) \leq C\beta(z_i,z_j)$ on the ball implies the tree separation condition $d(\alpha_i,o) \leq Cd(\alpha_i,\alpha_j)$, but not conversely. We then show that the analogue of condition (5.1) on the Bergman tree $\mathcal{T}_n$,

$$\text{(5.2)}\qquad \beta(z_i,0) \leq C\beta(z_i,z_j) \text{ and}$$
$$\sum_{j=1}^\infty (1+d(\alpha_j,o))^{1-p}\delta_{\alpha_j} \text{ is a } B_p(\mathcal{T}_n)\text{-Carleson measure,}$$

is sufficient for ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ on $\{z_j\}_{j=1}^\infty$ for all $1<p<\infty$, and necessary provided $1<p<2+\frac{1}{n-1}$. However, we leave the difficult subrange $\left[1+\frac{1}{n-1},2\right)$ to Section 8 where additional properties of a refined Bergman tree $\mathcal{T}_n$ are required, and a more delicate argument is then needed to show that we may assume the points z_j occur as the centers c_{α_j} of certain kubes K_{α_j} in the refined Bergman tree (see the final paragraph of subsection 8.5).

We are however able to show that (5.1) is sufficient for ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ for $p > 2n$, and that (5.1) is necessary for ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ for all $1 < p < \infty$. Since a measure μ is a $B_p(\mathcal{T}_n)$-Carleson measure if and only if it satisfies the tree condition (3.2), we see that one obstacle to obtaining a characterization of ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ in the exceptional range $\left[2+\frac{1}{n-1}, 2n\right]$ is our failure to find a proof for Theorem 3.1 when $p \geq 2+\frac{1}{n-1}$. For the remainder of this section, we consider mostly Besov spaces $B_p(\mathbb{B}_n)$ on the unit ball, and for convenience in notation, we will suppress the dependence on the ball by writing simply B_p for $B_p(\mathbb{B}_n)$.

For $\alpha > -1$ and $1 < p < \infty$, recall from Theorem 2.5 that $(B_p)^*$ and $B_{p'}$ are identified under the pairing

$$\begin{aligned}\langle f, g\rangle_{\alpha,p} &= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g\right\rangle_\alpha = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z)\, \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g(z)} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left(1-|z|^2\right)^{\frac{n+1+\alpha}{p}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z)\, \overline{\left(1-|z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g(z)} d\lambda_n(z),\end{aligned}$$

for $f \in B_p$, $g \in B_{p'}$, and that the reproducing kernel for B_p relative to this pairing is given by

$$\begin{aligned}k^{\alpha,p}_w(z) &= \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} K^\alpha_w(z) \\ &= \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}\right)^{-1} (1-\overline{w}\cdot z)^{-\frac{n+1+\alpha}{p'}},\end{aligned}$$

where the last formula above follows from (2.13).

We now state our analogue of Böe's interpolation theorem in two separate statements.

THEOREM 5.1. *Let $1 < p < \infty$, $\alpha > -1$ and $k^{\alpha,p}_w(z)$ be the reproducing kernel for B_p relative to the pairing $\langle\cdot,\cdot\rangle_{\alpha,p}$ given in Theorem 2.5 above. Let $\{z_j\}_{j=1}^\infty$ be a sequence in the unit ball $\mathbb{B}_n$. Then the following conditions are equivalent.*

(1) *$\{z_j\}_{j=1}^\infty$ interpolates B_p:*

$$\text{The map } f \to \left\{\frac{f(z_j)}{\left\|k^{\alpha,p}_{z_j}\right\|_{B_{p'}}}\right\}_{j=1}^\infty \text{ takes } B_p \text{ boundedly into and onto } \ell^p. \tag{5.3}$$

(2) *The following norm equivalence holds:*

$$\left\|\sum_{j=1}^\infty a_j \frac{k^{\alpha,p}_{z_j}}{\left\|k^{\alpha,p}_{z_j}\right\|_{B_{p'}}}\right\|_{B_{p'}} \approx \left(\sum_{j=1}^\infty |a_j|^{p'}\right)^{\frac{1}{p'}}. \tag{5.4}$$

(3) *The following separation condition and Carleson embedding hold:*

$$\begin{aligned}&\beta(z_i,0) \leq C\beta(z_i,z_j),\, i \neq j \text{ and} \\ &\sum_{j=1}^\infty \left\|k^{\alpha,p}_{z_j}\right\|^{-p}_{B_{p'}} \delta_{z_j} \text{ is a } B_p\text{-Carleson measure.}\end{aligned} \tag{5.5}$$

THEOREM 5.2. *Let $1 < p < \infty$, $\alpha > -1$ and $k_w^{\alpha,p}(z)$ be the reproducing kernel for B_p relative to the pairing $\langle \cdot, \cdot \rangle_{\alpha,p}$ given in Theorem 2.5 above. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in the unit ball $\mathbb{B}_n$. If $p \in \left(1, 2 + \frac{1}{n-1}\right)$, then each of conditions (5.6) and (5.8) below is equivalent to the three conditions in Theorem 5.1. In general, for $1 < p < \infty$, (5.8) implies (5.6) implies (5.7). For $p > 2n$ (5.5) implies (5.6). If $p \in \left(1, 1 + \frac{1}{n-1}\right) \cup [2, \infty)$, we also have that (5.7) implies (5.5):*

(1) *$\{z_j\}_{j=1}^{\infty}$ interpolates M_{B_p}:*

$$\text{The map } f \to \{f(z_j)\}_{j=1}^{\infty} \text{ takes } M_{B_p} \text{ boundedly into and onto } \ell^{\infty}. \tag{5.6}$$

(2) *$\left\{k_{z_j}^{\alpha,p}\right\}_{j=1}^{n}$ is an unconditional basic sequence in $B_{p'}$:*

$$\left\| \sum_{j=1}^{\infty} b_j k_{z_j}^{\alpha,p} \right\|_{B_{p'}} \leq C \left\| \sum_{j=1}^{\infty} a_j k_{z_j}^{\alpha,p} \right\|_{B_{p'}}, \quad \text{whenever } |b_j| \leq |a_j|. \tag{5.7}$$

(3) *$\{z_j\}_{j=1}^{\infty} = \left\{c_{\alpha_j}\right\}_{j=1}^{\infty}$ where $\{\alpha_j\}_{j=1}^{\infty}$ is a sequence in a Bergman tree $\mathcal{T}_n$ satisfying*

$$\begin{gathered} \beta(z_i, 0) \leq C\beta(z_i, z_j), i \neq j \text{ and} \\ \sum_{j=1}^{\infty} (1 + d(\alpha_j, o))^{1-p} \delta_{\alpha_j} \text{ satisfies the tree condition (3.2).} \end{gathered} \tag{5.8}$$

Note in particular that for $p \in \left(1, 2 + \frac{1}{n-1}\right) \cup (2n, \infty)$, multiplier interpolation (5.6) is characterized by the separation condition and Carleson embedding in (5.5).

The parameter $\alpha > -1$ appearing in condition (5.5) is not essential, as evidenced by the following calculation.

LEMMA 5.3. *For $\alpha > -1$ and $1 < p < \infty$, we have*

$$\|k_w^{\alpha,p}\|_{B_{p'}} \approx \left(1 + \log \frac{1}{1 - |w|^2}\right)^{\frac{1}{p'}} \approx (1 + \beta(0, w))^{\frac{1}{p'}}. \tag{5.9}$$

PROOF. Using (2.13) and $m = \frac{n+1+\alpha}{p'} > \frac{n}{p'}$, we compute that

$$\begin{aligned} \|k_w^{\alpha,p}\|_{B_{p'}} &= \left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} k_w^{\alpha,p}(z) \right|^{p'} d\lambda_n(z) \right)^{\frac{1}{p'}} \\ &= \left(\int_{\mathbb{B}_n} \left| \frac{1 - |z|^2}{1 - \overline{w} \cdot z} \right|^{n+1+\alpha} d\lambda_n(z) \right)^{\frac{1}{p'}} \\ &= \left(\int_{\mathbb{B}_n} \frac{\left(1 - |z|^2\right)^{\alpha}}{|1 - \overline{w} \cdot z|^{n+1+\alpha}} dz \right)^{\frac{1}{p'}} \\ &\approx \left(1 + \log \frac{1}{1 - |w|^2}\right)^{\frac{1}{p'}} \end{aligned}$$

by Theorem 1.12 of [**Zhu**].

Thus we can restate condition (5.5) in the equivalent form,

$$\beta(z_i, 0) \le C\beta(z_i, z_j) \text{ and} \tag{5.10}$$
$$\sum_{j=1}^{\infty}\left(1+\log\frac{1}{1-|z_j|^2}\right)^{1-p}\delta_{z_j} \text{ is a } B_p\text{-Carleson measure.}$$

We saw in Corollary 3.2 that if a *fixed* measure μ on the ball is a Carleson measure (that is it gives a Carleson embedding) for a fixed value of p, then it also gives a Carleson embedding for all $r < p$, thus propagating downward. In contrast, if a sequence $\{z_j\}_{j=1}^{\infty}$ satisfies (5.5) for a particular value of p, then it also satisfies the condition for all $r > p$, thus propagating upward. This fact lets us draw conclusions about the interpolation properties of $\{z_j\}_{j=1}^{\infty}$

LEMMA 5.4. (1) *Suppose* $1 < p < r < \infty$. *If* $\{z_j\}_{j=1}^{\infty}$ *is an interpolating sequence for* B_p, *i.e. if (5.3) holds, then* $\{z_j\}_{j=1}^{\infty}$ *is also an interpolating sequence for* B_r.

(2) *Suppose* $1 < p < r < \infty$ *and* $r \in \left(1, 2+\frac{1}{n-1}\right) \cup (2n, \infty)$. *If* $\{z_j\}_{j=1}^{\infty}$ *is an interpolating sequence for* M_{B_p}, *i.e. if (5.8) holds, then* $\{z_j\}_{j=1}^{\infty}$ *is also an interpolating sequence for* M_{B_r}.

PROOF. By the two previous theorems and the lemma, it will suffice to prove that if a sequence $\{z_j\}_{j=1}^{\infty}$ satisfies (5.10) for a particular value of p, then it also satisfies the condition for all $r > p$. Suppose now we are given a sequence $\{z_j\}_{j=1}^{\infty}$. We write the Carleson embedding in the following form,

$$\sum_{j=1}^{\infty}\left|\frac{f\left(c_{\alpha_j}\right)}{1+\log\frac{1}{1-\left|c_{\alpha_j}\right|^2}}\right|^{p_0}\left(1+\log\frac{1}{1-\left|c_{\alpha_j}\right|^2}\right) \le C\,\|f\|_{B_{p_0}}^{p_0},$$

or

$$\|Tf\|_{L^{p_0}(\mu)} \le C\,\|f\|_{B_{p_0}},$$

where

$$Tf(z) = \left(1+\log\frac{1}{1-|z|^2}\right)^{-1} f(z),$$
$$\mu = \sum_{j=1}^{\infty}\left(1+\log\frac{1}{1-\left|c_{\alpha_j}\right|^2}\right)\delta_{c_{\alpha_j}}.$$

Since T is bounded from the Bloch space B_∞ to $L^\infty(\mu)$(Corollary 3.7 in [**Zhu**]), we have by interpolation (Theorem 6.12 in [**Zhu**]) that T is bounded from B_p to $L^p(\mu)$, which is equivalent to the Carleson embedding for exponent p in (5.1), for all $p_0 \le p < \infty$. We remark that in the proof of Theorem 6.12 in [**Zhu**], Theorem 3.4 substitutes for Theorem 6.7 when $p_1 = \infty$ in the conclusion

$$B_p = [B_{p_0}, B_{p_1}]_\theta, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

REMARK 5.5. It is not apparent to us how to reach the conclusion of the corollary without passing through the equivalence of interpolation with condition (5.5).

REMARK 5.6. For $n = 1$ it is interesting to contrast the result in the corollary with the situation in the Hardy and Bergman spaces. For the Hardy spaces a sequence is an interpolating sequence for H^p for some p if and only if it is an interpolating sequence for H^r for every r, $0 < r \leq \infty$. For the Bergman space, we know from the work of Seip [**Sei2**] and Schuster and Varolin [**ScVa**], that if a sequence is an interpolating sequence for the Bergman space A^p, then it is an interpolating sequence for A^r for $0 < r < p$. One should be cautious in interpreting this comparison; in contrast with the Besov scale, the Hardy and Bergman scales are defined using a measure that does not vary with p, thus producing scales of spaces that get *smaller* as p gets larger.

REMARK 5.7. Our proofs show that the interpolations in (5.6) and (5.3) can be taken to be linear, i.e. there are bounded linear maps $R : \ell^\infty \to M_{B_p}$ and $S : \ell^p \to B_p$ that yield right inverses to the restriction maps in (5.6) and (5.3) respectively. In dimension $n = 1$ Böe has shown [**Boe**] the stronger result that there are functions $f_k \in M_{B_p}$ such that $\|f_k\|_{M_{B_p}} \leq C$, $f_k(z_j) = \delta_k^j$ and $\sum_k |f_k(z)| \leq C$ for all $z \in \mathbb{D}$ (compare Theorem 2.1 in chapter 7 of [**Gar**]). It seems likely that this extends to $1 < p < 2 + \frac{1}{n-1}$ for $n > 1$, but we will not pursue this here.

REMARK 5.8. For $p \in \left[2 + \frac{1}{n-1}, \infty\right)$ we do not know if (5.5) is sufficient for (5.8). Note that (5.2) and (5.8) are equivalent by Theorem 3.1.

For later use we note the following.

LEMMA 5.9. *For $f \in B_p$ and $z, w \in \mathbb{B}_n$,*

$$|f(z) - f(w)| \leq C \|f\|_{B_p} \beta(z, w)^{\frac{1}{p'}} . \tag{5.11}$$

PROOF. To see (5.11), we first observe that by (5.9),

$$\begin{aligned}
&\|k_w^{\alpha,p} - k_0^{\alpha,p}\|_{B_{p'}} \\
&= \left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} \left(k_w^{\alpha,p} - k_0^{\alpha,p}\right)(z) \right|^{p'} d\lambda_n(z) \right)^{\frac{1}{p'}} \\
&= \left(\int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{\alpha} \left| \frac{1}{(1 - \overline{w} \cdot z)^{\frac{n+1+\alpha}{p'}}} - 1 \right|^{p'} dz \right)^{\frac{1}{p'}} \\
&\leq C|w| \left(\int_{\mathbb{B}_n} \frac{\left(1 - |z|^2\right)^{\alpha}}{|1 - \overline{w} \cdot z|^{n+1+\alpha}} dz \right)^{\frac{1}{p'}} \leq C\beta(0, w)^{\frac{1}{p'}} .
\end{aligned} \tag{5.12}$$

By the invariance of B_p and the Bergman metric β, we then get

$$\begin{aligned}
|f(z) - f(w)| &= |f \circ \varphi_w(\varphi_w(z)) - f \circ \varphi_w(0)| \\
&= \left| \left\langle f \circ \varphi_w, k_{\varphi_w(z)}^{\alpha,p} - k_0^{\alpha,p} \right\rangle_{\alpha,p} \right| \\
&\leq \|f \circ \varphi_w\|_{B_p} \left\| k_{\varphi_w(z)}^{\alpha,p} - k_0^{\alpha,p} \right\|_{B_{p'}}
\end{aligned}$$

$$\leq C \left\| f \circ \varphi_w \right\|_{B_p} \beta\left(0, \varphi_w\left(z\right)\right)^{\frac{1}{p'}}$$
$$= C \left\| f \right\|_{B_p} \beta\left(z, w\right)^{\frac{1}{p'}} .$$

Proof. (of Theorems 5.1 and 5.2) We will see that the one-dimensional arguments used in Theorem 1.1 of Böe's paper [**Boe**] to prove that (5.6) implies (5.7), and that (5.3) implies (5.4) implies (5.5) extend to higher dimensions with any choice of pairing and corresponding reproducing kernel in Theorem 2.5. However, the one-dimensional proof of the implication (5.7) implies (5.4) does not extend in the range $1 + \frac{1}{n-1} \leq p < 2$. The proof of that implication is where Böe uses what he calls his "curious" Lemma 3.1. For this he needs to know that his reproducing kernel $k_w\left(z\right) = \log \frac{1}{1-\overline{w}z}$ satisfies

$$\operatorname{Re} \frac{1}{\overline{w}} k_w'\left(z\right) = \operatorname{Re} \frac{1}{1-\overline{w}z} > 0,$$

where he uses the pairing $\langle f, g\rangle = \int f' \overline{g'}$ to define the kernel $k_w\left(z\right)$ (and assumes his functions f satisfy $f\left(0\right) = 0$). With the choice $p = \frac{n+1+\alpha}{n+\alpha}$, it follows from (2.13) that the reproducing kernel $k_w^{\alpha,p}\left(z\right)$ given in Theorem 2.5 has the analogous property,

$$\operatorname{Re} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} k_w^{\alpha,p}\left(z\right) = \operatorname{Re} \mathcal{R}^{\alpha}_{1} k_w^{\alpha,p}\left(z\right) = \operatorname{Re} \frac{1}{1-\overline{w}\cdot z} > 0.$$

This does indeed lead to a proof that (5.7) implies (5.4), but only for the restricted range $1 < p < 1+\frac{1}{n-1}$, since we must have $\alpha > -1$. It is the failure of this argument for the remaining values of p, as well as our failure to obtain the necessity of the tree condition (3.2) for the Carleson embedding (2.16) when $p \geq 2 + \frac{1}{n-1}$, that forces us to proceed via a different logical route.

The proof of the theorems will take much of the rest of the paper and various of the arguments will only be valid for certain ranges of p. We will prove Theorems 5.1 and 5.2 by demonstrating the following implications:

(1) In Section 5.1, *Multiplier space necessity*, we prove that (5.6) implies (5.7), that (5.3) and (5.4) are equivalent, that (5.3) implies (5.5), and that if $p \in \left(1, 1 + \frac{1}{n-1}\right) \cup [2,\infty)$, then (5.7) implies (5.5).

(2) In Section 5.2, *Multiplier space sufficiency*, we prove that (5.8) implies (5.6), and we prove that if $p \in (\widehat{n}, \infty)$, $\widehat{n} = \begin{cases} 1 & \text{if} \quad n=1 \\ 2n & \text{if} \quad n>1 \end{cases}$, then (5.5) implies (5.6).

(3) In Section 5.3, *Besov space interpolation*, we prove that (5.10), which is equivalent to (5.5), implies (5.3).

(4) In Section 9, *Completing the multiplier loop*, we show that if p is in $\left(1, 2 + \frac{1}{n-1}\right)$, then (5.6) implies (5.8) (this implication already follows for p in $\left(1, 1 + \frac{1}{n-1}\right)$ or $\left[2, 2 + \frac{1}{n-1}\right)$ from the previous ones and Theorem 3.1).

The implications (5.3) implies (5.4) implies (5.5) implies (5.3) do not use our characterization of Carleson measures in Theorem 3.1, relying instead on only the embedding definition of Carleson measure (we do however use the simple condition (1.11) in conjunction with our general tree theorem from [**ArRoSa**]). The implication (5.8) implies (5.6) however, relies heavily on the use of the tree condition (3.2) to extend Böe's arguments to higher dimensions, and thus Theorem 3.1 plays a crucial role in the proof of this result. The remaining implications use only the embedding definition of Carleson measure together with one-dimensional techniques, except for the final implication (5.6) implies (5.8). For the range $p \in \left[1+\frac{1}{n-1},2\right)$, this is the most difficult implication, requiring a passage to multiplier interpolation on holomorphic Besov spaces on Bergman trees, and occupies the content of Sections 6, 7 and 8. We refer to the implications "(5.8) implies (5.6)", "(5.5) implies (5.6)" and "(5.5) implies (5.3)" as sufficiency implications, and the remaining ones as necessity implications.

5.1. Multiplier space necessity. We begin with the straightforward necessity implications; (5.6) implies (5.7), (5.3) implies (5.4), and (5.4) implies (5.5). For the most part, we follow Böe [**Boe**], who in turn generalized the Hilbert space arguments in Marshall and Sundberg [**MaSu**]. First, we have that condition (5.7) follows from (5.6) since if we choose $\varphi \in M_{B_p}$ so that $b_j = \overline{\varphi(z_j)} a_j$, then $M_\varphi^*\left(k_{z_j}^{\alpha,p}\right) = \overline{\varphi(z_j)} k_{z_j}^{\alpha,p}$ and so

$$\begin{aligned}\left\|\sum_{j=1}^\infty b_j k_{z_j}^{\alpha,p}\right\|_{B_{p'}} &= \left\|M_\varphi^*\left(\sum_{j=1}^\infty a_j k_{z_j}^{\alpha,p}\right)\right\|_{B_{p'}} \\ &\le \|M_\varphi\| \left\|\sum_{j=1}^\infty a_j k_{z_j}^{\alpha,p}\right\|_{B_{p'}} \\ &\le C \sup_j |\varphi(z_j)| \left\|\sum_{j=1}^\infty a_j k_{z_j}^{\alpha,p}\right\|_{B_{p'}}.\end{aligned}$$

Next we prove the equivalence of (5.3) and (5.4), the arguments being short and essentially reversible. First, if the map $Tf = \left\{\frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right\}_{j=1}^\infty$ in (5.3) maps B_p *into* ℓ^p, then we have

$$\begin{aligned}\left\|\sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right\|_{B_{p'}} &= \sup_{\|f\|_{B_p}=1} \left|\left\langle f, \sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right\rangle_{\alpha,p}\right| \\ &= \sup_{\|f\|_{B_p}=1} \left|\sum_{j=1}^\infty \frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \overline{a_j}\right| \\ &\le \sup_{\|f\|_{B_p}=1} \left(\sum_{j=1}^\infty \left|\frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right|^p\right)^{\frac{1}{p}} \left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}} \\ &\le C \left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}.\end{aligned}$$

If the map T is also *onto*, then its adjoint T^*, given by

$$T^*\left(\{a_j\}_{j=1}^\infty\right)=\sum_{j=1}^\infty \overline{a_j}\frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}},$$

satisfies

$$\left\|T^*\left(\{a_j\}_{j=1}^\infty\right)\right\|_{B_{p'}}\geq c\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}},$$

which is the opposite inequality in (5.4), and completes the proof that (5.3) implies (5.4). Conversely, if the inequality $\lesssim$ in (5.4) holds, then

$$\left(\sum_{j=1}^\infty\left|\frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right|^p\right)^{\frac{1}{p}} = \sup_{\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}=1}\left|\sum_{j=1}^\infty\frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\overline{a_j}\right| \tag{5.13}$$

$$= \sup_{\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}=1}\left|\sum_{j=1}^\infty\left\langle f,\frac{a_jk_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right\rangle_{\alpha,p}\right|$$

$$\leq \sup_{\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}=1}\|f\|_{B_p}\left\|\sum_{j=1}^\infty a_j\frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right\|_{B_{p'}} \tag{5.14}$$

$$\leq C\|f\|_{B_p}, \tag{5.15}$$

and thus the map T in (5.3) is *into*. If the reverse inequality $\gtrsim$ in (5.4) also holds, then

$$\left\|T^*\left(\{a_j\}_{j=1}^\infty\right)\right\|_{B_{p'}}=\left\|\sum_{j=1}^\infty\overline{a_j}\frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}}\right\|_{B_{p'}}$$

$$\geq c\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}},$$

which shows that T is also *onto*.

REMARK 5.10. We have shown in particular that the inequality $\lesssim$ in (5.4) implies that the map T in (5.3) is *into*. This will be used below.

The implication (5.4) implies (5.5) will now follow if we show that (5.3) implies (5.5). The Carleson embedding in (5.5) is a restatement that the map T in (5.3) is *into*. Indeed, the left side of (5.13) is $\|f\|_{L^p\left(\sum_{j=1}^\infty\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-p}\delta_{z_j}\right)}$, and thus shows that the Carleson embedding in (5.5) holds. To obtain the separation condition, fix i and use that T is *onto* to obtain $f\in B_p$ satisfying $f(z_i)=1$ and $f(z_j)=0$ for $i\neq j$. It now follows from the open mapping theorem and the Hölder estimate (5.11) that

$$\|f\|_{B_p}\leq C\|Tf\|_{\ell^p}=C\frac{|f(z_i)|}{\|k_{zi}^{\alpha,p}\|_{B_{p'}}}=C\frac{|f(z_i)-f(z_j)|}{\|k_{z_i}^{\alpha,p}\|_{B_{p'}}}\leq C\|f\|_{B_p}\frac{\beta(z_i,z_j)^{\frac{1}{p'}}}{\beta(z_i,0)^{\frac{1}{p'}}},$$

for all $i\neq j$.

5.1.1. *The necessity of separation and Carleson.* Now we turn to proving the more difficult necessity implication (5.7) implies (5.5). First we dispose of the easy part - namely that the separation condition in (5.5) follows from (5.7). Indeed, by (5.9), (5.7) and (5.11) we have

$$\begin{aligned}(1+\beta(0,z_i))^{\frac{1}{p'}} &\approx \left\|k_{z_i}^{\alpha,p}\right\|_{B_{p'}} \\ &\leq C\left\|k_{z_i}^{\alpha,p}-k_{z_j}^{\alpha,p}\right\|_{B_{p'}} \\ &= C\sup_{\|f\|_{B_p}=1}\left|\left\langle f,k_{z_i}^{\alpha,p}-k_{z_j}^{\alpha,p}\right\rangle_{\alpha,p}\right| \\ &= C\sup_{\|f\|_{B_p}=1}\left|f(z_i)-f(z_j)\right| \\ &\leq C\beta(z_i,z_j)^{\frac{1}{p'}}.\end{aligned}$$

It remains to prove that the Carleson embedding follows from (5.7). For this, we show that (5.7) implies (5.4) for both $1<p<1+\frac{1}{n-1}$ and $p=2$, and also that (5.7) implies the inequality $\lesssim$ of (5.4) for $p>2$. The note above then yields that the map T in (5.3) is *into*, which we showed above is a restatement of the Carleson embedding.

5.1.2. *The case* $1<p<1+\frac{1}{n-1}$. Here we prove the implication (5.7) implies (5.4) for the special case $1<p<1+\frac{1}{n-1}$. Given $1<p<1+\frac{1}{n-1}$, we make the choice $-1<\alpha<\infty$ to satisfy

$$p=\frac{n+1+\alpha}{n+\alpha}, \tag{5.16}$$

which accounts for our restriction $1<p<1+\frac{1}{n-1}$. Note that $p'=n+1+\alpha$, so that

$$\begin{aligned}\frac{n+1+\alpha}{p}&=n+\alpha,\\ \frac{n+1+\alpha}{p'}&=1.\end{aligned}$$

Thus in this case we have $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}}=\mathcal{R}^{\alpha}_{n+\alpha}$ and $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}}=\mathcal{R}^{\alpha}_{1}$ where α is as in (5.16), so that

$$\begin{aligned}\langle f,g\rangle_{\alpha,p}&=\left\langle\mathcal{R}^{\alpha}_{n+\alpha}f,\mathcal{R}^{\alpha}_{1}g\right\rangle_{A^2_\alpha}\\ &=\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{n+\alpha}\mathcal{R}^{\alpha}_{n+\alpha}f(z)\,\overline{\left(1-|z|^2\right)\mathcal{R}^{\alpha}_{1}g(z)}d\lambda_n(z).\end{aligned}$$

Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in the ball $\mathbb{B}_n$. We will need the following two results.

LEMMA 5.11 (Lemma 3.1 in [**Boe**]). *If $\{f_j\}_{j=1}^{\infty}$ is an unconditional basic sequence of positive functions in $L^q(d\mu)$, $1<q<\infty$, then*

$$\left\|\sum_{j=1}^{\infty}|a_jf_j|\right\|_{L^q(d\mu)}\approx C_q\left\|\sup_{j\geq1}|a_jf_j|\right\|_{L^q(d\mu)}\approx C_q\left(\sum_{j=1}^{\infty}|a_j|^q\|f_j\|^q_{L^q(d\mu)}\right)^{\frac{1}{q}}.$$

Proof. For convenience we sketch Böe's proof, which we will need to adapt in Subsection 9.1 anyway. Since the f_n are positive and unconditional in $L^q(d\mu)$, we have, letting $\{r_j(t)\}_{j=1}^{\infty}$ denote the Rademacher functions,

$$\left\|\sum_{j=1}^{\infty}|a_j f_j|\right\|_{L^q(d\mu)} \leq \left\|\sum_{j=1}^{\infty}|a_j| f_j\right\|_{L^q(d\mu)} \leq C\left\|\sum_{j=1}^{\infty}r_j(t)\, a_j f_j\right\|_{L^q(d\mu)}$$

for all $t\in[0,1]$ (since $|a_j|=|r_j(t)\,a_j|$). Now average the q^{th} power of this inequality over $t\in[0,1]$, and use Khinchine's inequality to obtain

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}|a_j f_j|\right\|_{L^q(d\mu)}^q &\leq C^q\int_0^1\left\|\sum_{j=1}^{\infty}r_j(t)\, a_j f_j\right\|_{L^q(d\mu)}^q dt\\ &= C^q\int\int_0^1\left|\sum_{j=1}^{\infty}r_j(t)\, a_j f_j\right|^q dt d\mu\\ &\leq C_q^q\int\left\|\{a_j f_j\}_{j=1}^{\infty}\right\|_{\ell^2}^q d\mu.\end{aligned}$$

Since $\left\|\{a_j f_j\}_{j=1}^{\infty}\right\|_{\ell^2}\leq\left\|\{a_j f_j\}_{j=1}^{\infty}\right\|_{\ell^1}^{\frac{1}{2}}\left\|\{a_j f_j\}_{j=1}^{\infty}\right\|_{\ell^\infty}^{\frac{1}{2}}$, by the Cauchy-Schwartz inequality we have,

$$\left\|\sum_{j=1}^{\infty}|a_j f_j|\right\|_{L^q(d\mu)}^q \leq C_q^q\left(\int\left\|\{a_j f_j\}_{j=1}^{\infty}\right\|_{\ell^1}^q d\mu\right)^{\frac{q}{2}}\left(\int\left\|\{a_j f_j\}_{j=1}^{\infty}\right\|_{\ell^\infty}^q d\mu\right)^{\frac{q}{2}},$$

which yields the inequality

$$\left\|\sum_{j=1}^{\infty}|a_j f_j|\right\|_{L^q(d\mu)} \leq C_q\left\|\sup_{j\geq 1}|a_j f_j|\right\|_{L^q(d\mu)}.$$

Thus the expressions

$$\left\|\left(\sum_{j=1}^{\infty}|a_j f_j|^r\right)^{\frac{1}{r}}\right\|_{L^q(d\mu)}$$

are all comparable for $1<r<\infty$, and the choice $r=q$ yields the final equivalence in the lemma.

Lemma 5.12. *For* $-1<\alpha<\infty$, $1<q<\infty$ *and* $F\in H(\mathbb{B}_n)$ *with* $\operatorname{Im}F(0)=0$,

$$\left(\int_{\mathbb{B}_n}|F(z)|^q\, d\nu_\alpha(z)\right)^{\frac{1}{q}}\approx\left(\int_{\mathbb{B}_n}|\operatorname{Re}F(z)|^q\, d\nu_\alpha(z)\right)^{\frac{1}{q}}. \tag{5.17}$$

Proof. The Koràny-Vagi theorem (Theorem 6.3.1 in [**Rud**]) shows the equivalence of the left and right hand sides in (5.17) when the measure $d\nu_\alpha(z)$ on the ball $\mathbb{B}_n$ is replaced by surface measure $d\sigma(z)$ on the sphere $\partial\mathbb{B}_n$ (and F is say a polynomial). Note that $d\sigma$ corresponds to $\lim_{\alpha\rightarrow-1}d\nu_\alpha$. This immediately yields (5.17) by an integration in polar coordinates.

Now suppose that (5.7) holds. Since $p' = n+1+\alpha$, we have from (2.13) that

$$\mathcal{R}_1^\alpha k_w^{\alpha,p}(z) = \frac{1}{1-\overline{w}\cdot z}. \tag{5.18}$$

We now compute that

$$\begin{aligned}
\left\| \sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}} &= \left\| \left(1-|z|^2\right) \mathcal{R}_1^\alpha \left(\sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right) \right\|_{L^{p'}(d\lambda_n)} \\
&= \left\| \left(1-|z|^2\right) \left(\sum_{j=1}^\infty a_j \frac{\mathcal{R}_1^\alpha k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right) \right\|_{L^{p'}(d\lambda_n)} \\
&= \left\| \sum_{j=1}^\infty a_j \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)}
\end{aligned}$$

since $p' = n+1+\alpha$. Now by the lemmas above, and using $p' = n+1+\alpha$ and

$$f_j = \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \operatorname{Re} \frac{1}{1-\overline{z_j}\cdot z} > 0,$$

we continue with

$$\begin{aligned}
\left\| \sum_{j=1}^\infty a_j \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} &\approx \left\| \sum_{j=1}^\infty a_j \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \operatorname{Re} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} \\
&\approx \left(\sum_{j=1}^\infty |a_j|^{p'} \left\|f_j\right\|_{L^{p'}(d\nu_\alpha)}^{p'} \right)^{\frac{1}{p'}} \\
&\approx \left(\sum_{j=1}^\infty |a_j|^{p'} \right)^{\frac{1}{p'}},
\end{aligned}$$

since

$$\begin{aligned}
\|f_j\|_{L^{p'}(d\nu_\alpha)} &= \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \left\| \operatorname{Re} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} \\
&\approx \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \left\| \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} \\
&= \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \left\| \left(1-|z|^2\right) \mathcal{R}_1 k_{z_j}^{\alpha,p} \right\|_{L^{p'}(d\lambda_n)} \\
&= \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{-1} \left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}} = 1
\end{aligned}$$

upon using the second lemma above once more. This completes the proof of condition (5.4) in the case $1 < p < 1 + \frac{1}{n-1}$.

5.1.3. *The case* $p \geq 2$. Here we show that (5.7) implies the inequality $\lesssim$ in (5.4) for $p > 2$, and also that (5.7) implies (5.4) for $p = 2$. First we claim that the unconditional basis condition (5.7) and Khinchine's inequality yield the inequality

$$\left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}} \leq C \left(\sum_{j=1}^{\infty} |a_j|^{p'} \right)^{\frac{1}{p'}}$$

for $p \geq 2$, and with equality in the case $p = 2$. To see this, we compute using first (5.7) and then Khinchine, that for any $m > \frac{n}{p'}$, we have

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}}^{p'} &\approx \int_0^1 \left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} r_j(t) \right\|_{B_{p'}}^{p'} dt \\ &= \int_0^1 \int_{\mathbb{B}_n} \left| \sum_{j=1}^{\infty} a_j \frac{\left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} r_j(t) \right|^{p'} d\lambda_n(z)\, dt \\ &\approx \int_{\mathbb{B}_n} \left(\sum_{j=1}^{\infty} \left| a_j \frac{\left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right|^2 \right)^{\frac{p'}{2}} d\lambda_n(z). \end{aligned}$$

Since $\frac{p'}{2} \leq 1$, we continue with

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}}^{p'} &\leq C \int_{\mathbb{B}_n} \sum_{j=1}^{\infty} \left| a_j \frac{\left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right|^{p'} d\lambda_n(z) \\ &= \sum_{j=1}^{\infty} \frac{|a_j|^{p'}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}^{p'}} \int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z) \right|^{p'} d\lambda_n(z) \\ &= \sum_{j=1}^{\infty} |a_j|^{p'}, \end{aligned}$$

which is the inequality $\lesssim$ in (5.4). In the case $p = 2$ we have equality, and so then (5.4).

5.2. Multiplier space sufficiency. Here we prove that (5.8) implies (5.6) for $1 < p < \infty$, and also that (5.5) implies (5.6) for $p > 2n$, beginning with the proof that the multiplier interpolation property (5.6) follows from (5.8). We generalize the main ideas in Böe's one-dimensional proof to the unit ball $\mathbb{B}_n$.

For $z \in \mathbb{B}_n$ and $\beta < 1$, define the region V_z^{β} by

$$V_z^{\beta} = \left\{ w \in \mathbb{B}_n : |1 - \overline{w} \cdot Pz| \leq (1-|z|)^{\beta} \right\},$$

where Pz denotes the radial projection of z onto the sphere $\partial\mathbb{B}_n$. The intersection of V_z^{β} with the complex line $\mathbb{C}z$ through z and the origin is

$$\left\{ w \in \mathbb{B}_n \cap \mathbb{C}z : |w - Pz| \leq (1-|z|)^{\beta} \right\},$$

and the intersection of V_z^β with the sphere $\partial\mathbb{B}_n$ is an "ellipse" with radius $(1-|z|)^\beta$ in the radial tangential direction, and radius $(1-|z|)^{\frac{\beta}{2}}$ in the complex tangential directions. Using arguments in Marshall and Sundberg [**MaSu**], the separation condition in (5.5) implies the following geometric separation conditions.

LEMMA 5.13. *Suppose the separation condition in (5.5) holds. Then there are constants* $0<\beta<1<\beta\eta<\eta$ *such that if* $V_{z_i}^\beta \cap V_{z_j}^\beta \neq \phi$ *and* $|z_j| \geq |z_i|$, *then* $z_i \notin V_{z_j}^\beta$ *and*

$$(1-|z_j|) \leq (1-|z_i|)^\eta . \tag{5.19}$$

PROOF. Fix z_i, z_j with $|z_j| \geq |z_i|$, suppose $w \in V_{z_i}^\beta \cap V_{z_j}^\beta$ and set $\omega = |z_j|\, w$ so that

$$1-|\omega|^2 = 1-|z_j|^2 |w|^2 \geq 1-|z_j|^2 .$$

Then the separation condition $\beta(z_i,z_j) \geq c\beta(z_j,0)$ (the dual use of β as a positive real number less than one, and as the Bergman metric, should not cause confusion) yields

$$c\beta(0,\omega) \leq c\beta(0,z_j) \leq \beta(z_i,z_j) \leq \beta(z_i,\omega) + \beta(\omega,z_j),$$

and so

$$c\beta(0,\omega) \leq 2\beta(z_k,\omega),$$

where k is either i or j. Now the identity

$$\rho(z,w) \equiv |\varphi_z(w)| = \tanh\beta(z,w),$$

yields for this k,

$$\begin{aligned} 1-\rho(\omega,z_k)^2 &= 1-\tanh^2\beta(\omega,z_k) \\ &\leq 1-\tanh^2(c\beta(\omega,0)) \\ &= \frac{4}{e^{2c\beta(\omega,0)}+2+e^{-2c\beta(\omega,0)}} \\ &\leq \frac{4}{e^{2c\beta(\omega,0)}} = 4\frac{(1-|\omega|)^c}{(1+|\omega|)^c} \\ &\leq 4(1-|\omega|)^c \leq \left(1-|\omega|^2\right)^\lambda , \end{aligned}$$

for some $0<\lambda<c$ provided z_i is large enough, which we may assume by discarding finitely many of the points z_i.

Then the final identity in (2.3) gives

$$\frac{\left(1-|\omega|^2\right)\left(1-|z_k|^2\right)}{|1-\overline{\omega}\cdot z_k|^2} = 1-\rho(\omega,z_k)^2 \leq \left(1-|\omega|^2\right)^\lambda .$$

Now we use that

$$1-\overline{\omega}\cdot z_k = 1-|z_k| + |z_k|\,(1-\overline{w}\cdot Pz_k + \overline{w-\omega}\cdot Pz_k)$$

implies

$$\begin{aligned} |1-\overline{\omega}\cdot z_k| &\leq (1-|z_k|) + \left(1-|z_k|^2\right)^\beta + (1-|z_j|) \\ &\leq 3\left(1-|z_k|^2\right)^\beta , \end{aligned}$$

to obtain

$$\left(1-|\omega|^2\right)\left(1-|z_k|^2\right) \le |1-\overline{\omega}\cdot z_k|^2\left(1-|\omega|^2\right)^{\lambda}$$
$$\le C\left(1-|z_k|^2\right)^{2\beta}\left(1-|\omega|^2\right)^{\lambda}.$$

Hence with $1<\eta<\frac{2\beta-1}{1-\lambda}$ and $|z_i|$ sufficiently large, we have

$$1-|z_j|^2 \le 1-|\omega|^2 \le C\left(1-|z_k|^2\right)^{\frac{2\beta-1}{1-\lambda}}$$
$$\le \left(1-|z_k|^2\right)^{\eta} \le \left(1-|z_i|^2\right)^{\eta},$$

which is (5.19). In particular, if $z_i \in V_{z_j}^{\beta}$, then

$$1-|z_i| \le |1-\overline{z_i}\cdot Pz_j| \le \left(1-|z_j|^2\right)^{\beta} \le \left(1-|z_i|^2\right)^{\beta\eta} \le C\left(1-|z_i|\right)^{\beta\eta}$$

yields a contradiction for $|z_i|$ sufficiently large if $\beta\eta>1$. This completes the proof of Lemma 5.13.

We now fix constants β and η as in Lemma 5.13, and write $V_z = V_z^{\beta}$.

Lemma 5.18 below is the key construction in the sufficiency proof and is motivated by the formula (1.35) in [**Zhu**]

$$R^{s-n,n}\left(\frac{1}{(1-\overline{w}\cdot z)^{1+s}}\right) = \frac{1}{(1-\overline{w}\cdot z)^{n+1+s}},$$

valid for s not a negative integer. The point is that if we define

$$\Gamma_s g(z) \equiv \int_{\mathbb{B}_n} \frac{g(w)\left(1-|w|^2\right)^s}{(1-\overline{w}\cdot z)^{1+s}}\,dw \tag{5.20}$$

for a given (not necessarily holomorphic) function g, then with $\varphi(z)=\Gamma_s g(z)$,

$$R^{s-n,n}\varphi(z) = R^{s-n,n}\Gamma_s g(z) = \int_{\mathbb{B}_n} \frac{g(w)\left(1-|w|^2\right)^s}{(1-\overline{w}\cdot z)^{n+1+s}}\,dw, \tag{5.21}$$

and by the reproducing formula (2.6), valid for $\operatorname{Re} s > -1$, we also have that

$$R^{s-n,n}\varphi(z) = c_{n,s}\int_{\mathbb{B}_n} \frac{R^{s-n,n}\varphi(w)\left(1-|w|^2\right)^s}{(1-\overline{w}\cdot z)^{n+1+s}}\,dw.$$

Thus $R^{s-n,n}\varphi(w)$ behaves morally like $g(w)$, and this provides flexibility in choosing g so that φ has desirable algebraic multiplier properties on the one hand, while controlling the multiplier norm of φ on the other hand. Indeed, by Theorem 4.2, the multiplier norm is equivalent to the Carleson norm of

$$\left|\left(1-|z|^2\right)^n \nabla^n\varphi(z)\right|^p d\lambda_n(z),$$

which can in turn be dominated by the "tree condition" norm of

$$\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

by the lemma in the next subsubsection

5.2.1. *Transformation of Carleson measures.* It is here that we first use the tree condition (3.2) in a significant way.

DEFINITION 5.14. *We say that a measure μ on $\mathbb{B}_n$ satisfies the* tree condition *(3.2) if its discretization $\widetilde{\mu}$ satisfies (3.2).*

LEMMA 5.15 (analogue of Lemma 2.4 in [**Boe**]). *Suppose that g satisfies the following reverse Hölder condition on Bergman kubes,*

$$\left(\int_{K_\alpha} |g(z)|^p \, d\lambda_n(z)\right)^{\frac{1}{p}} \leq C_0 \int_{K_\alpha} |g(z)| \, d\lambda_n(z), \qquad \alpha \in \mathcal{T}_n, \tag{5.22}$$

and that the measure

$$d\mu(z) = \left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

satisfies the tree condition (3.2) with norm C_1. Then for s sufficiently large, both

$$\left|\left(1-|z|^2\right)^n R^{s-n,n}\Gamma_s g(z)\right|^p d\lambda_n(z)$$

and

$$\left|\left(1-|z|^2\right)^n \nabla^n \Gamma_s g(z)\right|^p d\lambda_n(z)$$

satisfy the tree condition (3.2) with norms at most $C(C_0+C_1)$.

Note that it then follows by Theorems 1.1 and 1.2 that both measures in the conclusion of the lemma are B_p-Carleson measures by Theorem 3.1. Following Böe [**Boe**] one can prove the following alternate version of Lemma 5.15, where the tree condition is replaced by the B_p-Carleson measure condition, for the range $p > 2n$ with $s > n - \frac{1}{p'}$. This version will be instrumental in proving the implication (5.5) implies (5.6) below.

LEMMA 5.16 (another analogue of Lemma 2.4 in [**Boe**]). *Suppose that*

$$\sup_{\zeta \in \mathbb{B}_n} \left|\left(1-|\zeta|^2\right)^n g(\zeta)\right| \leq C_0, \tag{5.23}$$

and that the measure

$$d\mu(z) = \left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

is a Carleson measure for B_p with norm C_1. Then for $p > 2n$ and $s > n - \frac{1}{p'}$, both

$$\left|\left(1-|z|^2\right)^n R^{s-n,n}\Gamma_s g(z)\right|^p d\lambda_n(z)$$

and

$$\left|\left(1-|z|^2\right)^n \nabla^n \Gamma_s g(z)\right|^p d\lambda_n(z)$$

are also Carleson measures for B_p, and with norms at most $C(C_0+C_1)$.

We first prove the simpler Lemma 5.16 using the characterization of B_p given by Theorem 6.28 of [**Zhu**], which states that

$$\|f\|_{B_p}^p \approx \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\overline{w}\cdot z|^{2(n+1+t)}} d\nu_t(z)\, d\nu_t(w), \tag{5.24}$$

provided $t > -1$ and $p > \begin{cases} 1, & n=1 \\ 2n & n>1 \end{cases}$. In order to obtain the full range $1 < p < \infty$ when $n > 1$, we instead need to use the the tree condition (3.2) in Theorem 3.1

to obtain Lemma 5.15, which is proved immediately following the proof of Lemma 5.16.

PROOF. (of Lemma 5.16) We begin with the case $R^{s-n,n}\Gamma_s g$. Define $T_s g(z) = R^{s-n,n}\Gamma_s g(z)$. Then by (5.21) and Theorem 2.10 in [**Zhu**], we have that T_s, and in fact $\widehat{T_s}$, is bounded on $L^p(d\nu_\alpha)$, $\alpha > -1$, if and only if $0 < \alpha + 1 < p(s+1)$, where

$$\widehat{T_s}g(z) = c_{n,s}\int_{\mathbb{B}_n} \frac{g(w)\left(1-|w|^2\right)^s}{|1-\overline{w}\cdot z|^{n+1+s}}dw.$$

Thus with $\alpha = np - (n+1)$, we obtain that

$$\begin{aligned}\|T_s(hg)\|_{L^p(d\nu_\alpha)} &\leq C\|hg\|_{L^p(d\nu_\alpha)} \\ &\leq CC_1\|h\|_{B_p}\end{aligned}$$

for all $h \in B_p$ provided that $0 < np - n < p(s+1)$, i.e. $p > 1$ and

$$s > \frac{n}{p'} - 1.$$

The conclusion of the lemma is equivalent to the inequality

$$\|hT_s g\|_{L^p(d\nu_\alpha)} \leq CC_1\|h\|_{B_p}, \qquad h \in B_p,$$

and thus in particular, it suffices to show that

$$\|T_s(hg) - hT_s g\|_{L^p(d\nu_\alpha)} \leq CC_0\|h\|_{B_p}, \qquad h \in B_p. \tag{5.25}$$

We have

$$T_s(hg)(z) - h(z)T_s g(z) = c_{n,s}\int_{\mathbb{B}_n}(h(w)-h(z))\frac{g(w)\left(1-|w|^2\right)^s}{(1-\overline{w}\cdot z)^{n+1+s}}dw,$$

and so by the sup norm estimate (5.23) on g and Hölder's inequality,

$$|T_s(hg)(z) - h(z)T_s g(z)|$$

is dominated by

$$\begin{aligned}&CC_0\left(\int_{\mathbb{B}_n}\frac{|h(w)-h(z)|^p}{|1-\overline{w}\cdot z|^{2(n+1)}}dw\right)^{\frac{1}{p}}\left(\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{(s-n)p'}}{|1-\overline{w}\cdot z|^{\left[\left(1-\frac{2}{p}\right)(n+1)+s\right]p'}}dw\right)^{\frac{1}{p'}} \\ &\leq CC_0\left(1-|z|^2\right)^{-n+\frac{n+1}{p}}\left(\int_{\mathbb{B}_n}\frac{|h(w)-h(z)|^p}{|1-\overline{w}\cdot z|^{2(n+1)}}dw\right)^{\frac{1}{p}},\end{aligned}$$

by Theorem 1.12 of [**Zhu**] provided $(s-n)p' > -1$, i.e.

$$s > n - \frac{1}{p'}.$$

Now we compute that

$$\begin{aligned}
&\left\|T_s(hg)-hT_sg\right\|^p_{L^p(d\nu_\alpha)}\\
&\qquad\le C^pC_0^p\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{-pn+(n+1)}\int_{\mathbb{B}_n}\frac{|h(w)-h(z)|^p}{|1-\overline{w}\cdot z|^{2(n+1)}}dwd\nu_\alpha(z)\\
&\qquad= C^pC_0^p\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|h(w)-h(z)|^p}{|1-\overline{w}\cdot z|^{2(n+1)}}dwdz\\
&\qquad\le C^pC_0^p\left\|h\right\|^p_{B_p}
\end{aligned}$$

by the case $t=0$ of (5.24) if $p>2n$, which yields (5.25) as required. The case where we consider $\nabla^n\Gamma_s$ in place of $T_s=R^{s-n,n}\Gamma_s$ is handled just as above using the pointwise estimate

$$|\nabla^n\Gamma_sg(z)|\le C_{s,n}\int_{\mathbb{B}_n}\frac{|g(w)|\left(1-|w|^2\right)^s}{|1-\overline{w}\cdot z|^{n+1+s}}dw=C'_{s,n}\widehat{T_s}\,|g|\,(z)\,. \tag{5.26}$$

Proof. (of Lemma 5.15) By the pointwise inequality (5.26) for $\nabla^n\Gamma_sg$ and the formula (5.21) for T_sg, it is enough to show that

$$\left|\left(1-|z|^2\right)^n\widehat{T_s}g(z)\right|^p d\lambda_n(z)$$

satisfies the tree condition (3.2), where

$$\widehat{T_s}f(z)=c_{s,n}\int_{\mathbb{B}_n}\frac{f(w)\left(1-|w|^2\right)^s}{|1-\overline{w}\cdot z|^{n+1+s}}dw.$$

We now discretize our hypothesis and conclusion. To this end, we first define

$$d\mu(z)=\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

and

$$d\mu_s(z)=\left|\left(1-|z|^2\right)^n g_s(z)\right|^p d\lambda_n(z)$$

where

$$g_s(z)=\widehat{T_s}\,|g|\,(z)\,.$$

The discretization is then given by

$$\begin{aligned}
g(\alpha)&=\int_{K_\alpha}|g(z)|\,d\lambda_n(z)\,,\\
\mu(\alpha)&=\left(1-|c_\alpha|^2\right)^{np}g(\alpha)^p\,,\\
g_s(\alpha)&=\int_{K_\alpha}g_s(z)\,d\lambda_n(z)\,,\\
\mu_s(\alpha)&=\left(1-|c_\alpha|^2\right)^{np}g_s(\alpha)^p\,,
\end{aligned}\tag{5.27}$$

where $1-|c_\alpha|^2\approx 2^{-d(\alpha)}$ if $d(\alpha)$ denotes tree distance from the root o to α. Here we are again choosing $\theta=\frac{\ln 2}{2}$ in (3.3) for convenience in notation. More generally, we let $[\alpha,\beta]$ denote the geodesic (unique path of minimal length) joining α and β, and $d(\alpha,\beta) = \#[\alpha,\beta] - 1$ denote the tree distance between α and β. Note that by (5.22), we have

$$\mu(\alpha) = \left(1 - |c_\alpha|^2\right)^{np} g(\alpha)^p \approx \int_{K_\alpha} d\mu(z),$$

and so the hypothesis that $d\mu(z)$ satisfies the tree condition is equivalent to the assertion that $\{\mu(\alpha)\}_{\alpha\in\mathcal{T}_n}$ satisfies the tree condition,

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha} I^*\mu(\beta)^{p'} \leq C^{p'} I^*\mu(\alpha), \qquad \alpha\in\mathcal{T}_n,$$

which written out in full is, for $\alpha\in\mathcal{T}_n$,

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\mu(\gamma)\right)^{p'} \leq C^{p'}\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\mu(\beta). \tag{5.28}$$

We also have

$$\begin{aligned}
g_s(\alpha) &= \int_{K_\alpha}\left(\widehat{T_s}\,|g|(z)\right) d\lambda_n(z) \\
&= \int_{K_\alpha}\left(c_{s,n}\int_{\mathbb{B}_n}\frac{|g(w)|\left(1-|w|^2\right)^s}{|1-\overline{w}\cdot z|^{n+1+s}}dw\right) d\lambda_n(z) \\
&= c_{s,n}\sum_{\beta\in\mathcal{T}_n}\int_{K_\alpha}\int_{K_\beta}\frac{|g(w)|\left(1-|w|^2\right)^{n+1+s}}{|1-\overline{w}\cdot z|^{n+1+s}}d\lambda_n(w)\,d\lambda_n(z) \\
&\approx \sum_{\beta\in\mathcal{T}_n} g(\beta)\left|\frac{1-|c_\beta|^2}{1-\overline{c_\beta}\cdot c_\alpha}\right|^{n+1+s}.
\end{aligned}$$

Again using $\theta = \frac{\ln 2}{2}$ we see that the factor $\left|\frac{1-|c_\beta|^2}{1-\overline{c_\beta}\cdot c_\alpha}\right|$ looks essentially like $\frac{2^{-d(\beta)}}{2^{-d(\alpha\wedge\beta)}}$, and we now plan to replace the former with the latter in our analysis. Let $\mathcal{U}_n$ be the unitary group with Haar measure dU. For each $U\in\mathcal{U}_n$ select $\alpha(z),\beta(w)\in\mathcal{T}_n$ so that $z\in UK_{\alpha(z)}$ and $w\in UK_{\beta(w)}$. If we apply a unitary transformation U to the Bergman grid of kubes $\{K_\alpha\}_{\alpha\in\mathcal{T}_n}$ to obtain the grid $\{UK_\alpha\}_{\alpha\in\mathcal{T}_n}$, then we have the inequality

$$\left|\frac{1-|w|^2}{1-\overline{w}\cdot z}\right| \leq C\int_{\mathcal{U}_n}\frac{2^{-d(\beta(w))}}{2^{-d(\alpha(z)\wedge\beta(w))}}dU. \tag{5.29}$$

Inequality (5.29) is analogous to similar inequalities in Euclidean space used to control an operator by translations of its dyadic version. Thus we may replace the kernel $\left|\frac{1-|c_\beta|^2}{1-\overline{c_\beta}\cdot c_\alpha}\right|$ by $\frac{2^{-d(\beta)}}{2^{-d(\alpha\wedge\beta)}} = 2^{-d(\beta,\alpha\wedge\beta)}$, provided we obtain operator estimates that are independent of rotating the Bergman grid by a unitary transformation (our estimates below clearly have this property). If we now choose s really large, then we essentially have

$$\left(2^{-d(\beta,\alpha\wedge\beta)}\right)^{n+1+s} \approx \begin{cases} 1 & \text{if } \beta\leq\alpha \\ 0 & \text{otherwise}\end{cases} = \chi_{[0,\alpha]}(\beta). \tag{5.30}$$

This suggests we dominate the kernel by the decomposition,

$$\left(2^{-d(\beta,\alpha\wedge\beta)}\right)^{n+1+s} \leq \sum_{\ell=0}^{\infty} 2^{-\ell(n+1+s)} \chi_{[0,\alpha]_\ell}(\beta), \tag{5.31}$$

where $[0,\alpha]_\ell = \{\beta \in \mathcal{T}_n : d(\beta,[0,\alpha]) \leq \ell\}$ is the set of tree elements within distance ℓ of the geodesic $[0,\alpha]$. Note that (5.31) holds since $d(\beta,[0,\alpha]) = d(\beta,\alpha\wedge\beta)$. Now define the fattened operators

$$I_\ell g(\alpha) = \sum_{\beta\in[0,\alpha]_\ell} g(\beta), \qquad \alpha \in \mathcal{T}_n,$$

so that we have

$$\begin{aligned} g_s(\alpha) &\leq \sum_{\ell=0}^{\infty} 2^{-\ell(n+1+s)} I_\ell g(\alpha), \\ \mu_s(\alpha) &\leq C\left(1-|c_\alpha|^2\right)^{np}\left(\sum_{\ell=0}^{\infty} 2^{-\ell(n+1+s)} I_\ell g(\alpha)\right)^p. \end{aligned} \tag{5.32}$$

The desired conclusion that μ_s is a Carleson measure will follow if we can show that the tree measures

$$\mu_s^\ell = \left\{\left(1-|c_\alpha|^2\right)^{np}\left(I_\ell g(\alpha)\right)^p\right\}_{\alpha\in\mathcal{T}_n}$$

are Carleson measures with norm bounded by $C_s 2^{A\ell}$ for some large positive constant A. Indeed, we can then choose s sufficiently large depending on A and apply Minkowski's inequality. So we now fix ℓ and note that μ_s^ℓ is a Carleson measure with norm C if and only if the tree condition

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha} I^*\mu_s^\ell(\beta)^{p'} \leq C^{p'} I^*\mu_s^\ell(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n,$$

holds, which written out in full becomes, for all $\alpha \in \mathcal{T}_n$,

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta} 2^{-npd(\gamma)} I_\ell g(\gamma)^p\right)^{p'} \leq C^{p'} \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha} 2^{-npd(\beta)} I_\ell g(\beta)^p < \infty. \tag{5.33}$$

We consider the simplest case $\ell = 0$ first. We begin by rewriting the tree condition (5.33) in terms of $\mu(\alpha)$ rather than $g(\alpha)$. For this purpose, we use from the second line of (5.27) that

$$g(\alpha) = 2^{nd(\alpha)}\mu(\alpha)^{\frac{1}{p}}, \qquad \alpha \in \mathcal{T}_n,$$

as well as $Ig(\gamma) = \sum_{\delta\leq\gamma} g(\delta)$ and $d(\delta) - d(\gamma) = -d(\delta,\gamma)$ for $\delta \leq \gamma$. Then the case $\ell = 0$ of (5.33) can be rewritten as, for $\alpha \in \mathcal{T}_n$,

$$\sum_{\beta\geq\alpha}\left(\sum_{\gamma\geq\beta}\left(\sum_{\delta\leq\gamma} 2^{-nd(\delta,\gamma)}\mu(\delta)^{\frac{1}{p}}\right)^p\right)^{p'} \leq C^{p'} \sum_{\beta\geq\alpha}\left(\sum_{\gamma\leq\beta} 2^{-nd(\gamma,\beta)}\mu(\gamma)^{\frac{1}{p}}\right)^p. \tag{5.34}$$

We now show that (5.28) implies (5.34) for all $1 < p < \infty$. First observe that the right side of (5.34) satisfies

$$\begin{aligned}
(5.35) \qquad \sum_{\beta \geq \alpha} \left(\sum_{\gamma \leq \beta} 2^{-nd(\gamma,\beta)} \mu(\gamma)^{\frac{1}{p}} \right)^p &\geq \sum_{\beta \geq \alpha} \left(\sum_{\gamma \leq \beta} 2^{-npd(\gamma,\beta)} \mu(\gamma) \right) \\
&= \sum_{\gamma} \mu(\gamma) \sum_{\beta \geq \alpha \vee \gamma} 2^{-npd(\gamma,\beta)} \\
&\approx \sum_{\gamma \leq \alpha} \mu(\gamma) 2^{-npd(\gamma,\alpha)} + \sum_{\gamma \geq \alpha} \mu(\gamma) \\
&\equiv A + B.
\end{aligned}$$

Next, we estimate the inner sum on the left side of (5.34);

$$\begin{aligned}
\sum_{\gamma \geq \beta} \left(\sum_{\delta \leq \gamma} 2^{-nd(\delta,\gamma)} \mu(\delta)^{\frac{1}{p}} \right)^p &\leq C \sum_{\gamma \geq \beta} \left(\sum_{\delta \leq \beta} 2^{-nd(\delta,\gamma)} \mu(\delta)^{\frac{1}{p}} \right)^p \\
&\quad + C \sum_{\gamma \geq \beta} \left(\sum_{\beta \leq \delta \leq \gamma} 2^{-nd(\delta,\gamma)} \mu(\delta)^{\frac{1}{p}} \right)^p \\
&= C(I + II).
\end{aligned}$$

We use Hölder's inequality to estimate term I by

$$\begin{aligned}
I &= \sum_{\gamma \geq \beta} \left(\sum_{\delta \leq \beta} 2^{-nd(\delta,\gamma)} \mu(\delta)^{\frac{1}{p}} \right)^p \\
&= \sum_{\gamma \geq \beta} \left(\sum_{\delta \leq \beta} 2^{-nd(\delta,\beta)} \mu(\delta)^{\frac{1}{p}} \, 2^{-nd(\beta,\gamma)} \right)^p \\
&\leq \sum_{\gamma \geq \beta} \left(\sum_{\delta \leq \beta} 2^{-npd(\delta,\beta)} \mu(\delta) \right) \left(\sum_{\delta \leq \beta} 2^{-np'd(\delta,\gamma)} \right)^{p-1} \\
&\approx C \sum_{\gamma \geq \beta} \left(\sum_{\delta \leq \beta} 2^{-npd(\delta,\beta)} \mu(\delta) \right) \left(2^{-npd(\beta,\gamma)} \right) \\
&= C \sum_{\gamma \geq \beta} \sum_{\delta \leq \beta} 2^{-npd(\delta,\gamma)} \mu(\delta) \\
&\leq C \sum_{\delta \leq \beta} 2^{-npd(\delta,\beta)} \mu(\delta).
\end{aligned}$$

To estimate terms II and IV above, as well as in applications to trees below, we will use the following form of Schur's test:

Lemma 5.17 (Schur's test, Theorem 2.9 in [**Zhu**]). *Let* (X, μ) *be a measure space,* $1 < p < \infty$*, and let* $K(x, y)$ *be a nonnegative kernel on* X*. Suppose that* *there exists a positive function h on X and a positive constant C so that*

$$\int_X K(x,y)\, h(y)^{p'}\, d\mu(y) \le C h(x)^{p'}, \quad \mu - a.e. x \in X,$$
$$\int_X K(x,y)\, h(x)^{p}\, d\mu(x) \le C h(y)^{p}, \quad \mu - a.e. x \in X.$$

Then the operator

$$Tf(x) = \int_X K(x,y)\, f(y)\, d\mu(y)$$

is bounded on $L^p(\mu)$ with norm at most C.

We now claim that term II satisfies

$$II = \sum_{\gamma \ge \beta} \left(\sum_{\beta \le \delta \le \gamma} 2^{-nd(\delta,\gamma)} \mu(\delta)^{\frac{1}{p}} \right)^p \le C \sum_{\delta \ge \beta} \mu(\delta).$$

To see this, let

$$\nu(\delta) = \mu(\delta)^{\frac{1}{p}},$$
$$T\nu(\gamma) = \sum_{\gamma \in \mathcal{S}(\beta)} 2^{-nd(\delta,\gamma)} \chi_{\{\delta \le \gamma\}} \nu(\delta) = \sum_{\gamma \in \mathcal{S}(\beta)} K(\delta,\gamma)\, \nu(\delta),$$

where $K(\delta,\gamma) = 2^{-nd(\delta,\gamma)} \chi_{\{\beta \le \delta \le \gamma\}}$ for $\delta, \gamma \in \mathcal{S}(\beta)$. Then the desired inequality,

$$\sum_{\gamma \in \mathcal{S}(\beta)} |T\nu(\gamma)|^p \le C \sum_{\delta \in \mathcal{S}(\beta)} |\nu(\delta)|^p,$$

follows from Schur's test, Lemma 5.17, with auxiliary function $h(\delta) = 2^{td(\delta)}$ where $-\frac{n}{p'} < t < 0$ as follows:

$$\begin{aligned} \sum_{\delta \in \mathcal{S}(\beta)} K(\delta,\gamma)\, h(\delta)^{p'} &= \sum_{\delta : \delta \le \gamma} 2^{-nd(\delta,\gamma)} 2^{p'td(\delta)} \\ &= 2^{-nd(\gamma)} \sum_{\delta : \delta \le \gamma} 2^{\left(p't+n\right)d(\delta)} \approx 2^{p'td(\gamma)} \\ &= h(\gamma)^{p'} \end{aligned}$$

since $p't + n > 0$, and

$$\begin{aligned} \sum_{\gamma \in \mathcal{S}(\beta)} K(\delta,\gamma)\, h(\gamma)^{p} &= \sum_{\gamma : \gamma \ge \delta} 2^{-nd(\delta,\gamma)} 2^{ptd(\gamma)} \\ &= \sum_{m=d(\delta)}^{\infty} \sum_{\gamma : \gamma \ge \delta \text{ and } d(\gamma) = m} 2^{-nd(\delta,\gamma)} 2^{ptd(\gamma)} \\ &= \sum_{m=d(\delta)}^{\infty} (2^n)^{m-d(\delta)}\, 2^{-n(m-d(0,\delta))} 2^{ptd(\gamma)} \approx 2^{ptd(\delta)} \\ &= h(\delta)^p \end{aligned}$$

since $t<0$. Altogether we have proved that the inner sum on the left side of (5.34) satisfies

$$\sum_{\gamma\geq\beta}\left(\sum_{\delta\leq\beta}2^{-nd(\delta,\gamma)}\mu(\delta)^{\frac{1}{p}}\right)^{p}\leq C\sum_{\delta\leq\alpha}2^{-npd(\delta,\beta)}\mu(\delta)+C\sum_{\alpha\leq\delta\leq\beta}2^{-npd(\delta,\beta)}\mu(\delta)$$
$$+C\sum_{\delta\geq\beta}\mu(\delta).$$

Thus we can estimate the left side of (5.34) by

$$\begin{aligned}\sum_{\beta\geq\alpha}\left(\sum_{\gamma\geq\beta}\left(\sum_{\delta\leq\gamma}2^{-nd(\delta,\gamma)}\mu(\delta)^{\frac{1}{p}}\right)^{p}\right)^{p'} &\leq C\sum_{\beta\geq\alpha}\left(\sum_{\delta\leq\alpha}2^{-npd(\delta,\beta)}\mu(\delta)\right)^{p'}\\ &\quad+C\sum_{\beta\geq\alpha}\left(\sum_{\alpha\leq\delta\leq\beta}2^{-npd(\delta,\beta)}\mu(\delta)\right)^{p'}\\ &\quad+C\sum_{\beta\geq\alpha}\left(\sum_{\delta\geq\beta}\mu(\delta)\right)^{p'}\\ &=C\left(III+IV+V\right).\end{aligned}$$

By the hypothesis (5.28), the main term V satisfies

$$V=\sum_{\beta\geq\alpha}\left(\sum_{\delta\geq\beta}\mu(\delta)\right)^{p'}\leq C\sum_{\beta\geq\alpha}\mu(\beta)=CB,$$

which by (5.35) is dominated by the right side of (5.34) as required.

We again use Hölder's inequality on term III to obtain

$$\begin{aligned}III=\sum_{\beta\geq\alpha}\left(\sum_{\delta\leq\alpha}2^{-npd(\delta,\beta)}\mu(\delta)\right)^{p'} &=\sum_{\beta\geq\alpha}\left(\sum_{\delta\leq\alpha}2^{-npd(\delta,\alpha)}\mu(\delta)\,2^{-npd(\alpha,\beta)}\right)^{p'}\\ &\leq\sum_{\beta\geq\alpha}\left(\sum_{\delta\leq\alpha}2^{-npp'd(\delta,\alpha)}\mu(\delta)^{p'}\right)\left(\sum_{\delta\leq\alpha}2^{-np^2d(\alpha,\beta)}\right)^{p'-1}\\ &\approx C\sum_{\beta\geq\alpha}\left(\sum_{\delta\leq\alpha}2^{-npp'd(\delta,\alpha)}\mu(\delta)^{p'}\right)\left(2^{-npp'd(\alpha,\beta)}\right)\\ &=C\sum_{\beta\geq\alpha}\sum_{\delta\leq\alpha}2^{-npp'd(\delta,\beta)}\mu(\delta)^{p'}\\ &\leq C\sum_{\delta\leq\alpha}2^{-npp'd(\delta,\alpha)}\mu(\delta)^{p'},\end{aligned}$$

which yields

$$\begin{aligned} III &\leq C \sum_{\delta \leq \alpha} \left(2^{-npd(\delta,\alpha)} \mu(\delta)\right)^{p'} \\ &\leq C \left(\sum_{\delta \leq \alpha} 2^{-npd(\delta,\alpha)} \mu(\delta)\right)^{p'} \\ &= CA^{p'} \\ &\leq CA, \end{aligned}$$

since A is bounded, and by (5.35), this is dominated by the right side of (5.34) as required.

Finally, we again use Schur's test, Lemma 5.17, on term IV. With

$$\begin{aligned} T\mu(\beta) &= \sum_{\delta \in \mathcal{S}(\alpha)} 2^{-npd(\delta,\beta)} \chi_{\{\delta \leq \beta\}} \mu(\delta) \\ &= \sum_{\delta \in \mathcal{S}(\alpha)} K(\delta,\beta)\, \mu(\delta), \end{aligned}$$

where $K(\delta,\beta) = 2^{-npd(\delta,\beta)} \chi_{\{\alpha \leq \delta \leq \beta\}}$ for $\beta, \delta \in \mathcal{S}(\alpha)$, we obtain

$$IV = \sum_{\gamma \in \mathcal{S}(\alpha)} |T\mu(\beta)|^{p'} \leq C \sum_{\delta \in \mathcal{S}(\alpha)} |\mu(\delta)|^{p'} = C \sum_{\delta \geq \alpha} \mu(\delta)^{p'}$$

from Schur's test again (Theorem 2.9 in [**Zhu**]), but with auxiliary function $h \equiv 1$ this time:

$$\begin{aligned} \sum_{\delta \in \mathcal{S}(\alpha)} K(\delta,\beta) &= \sum_{\delta : \delta \leq \beta} 2^{-npd(\delta,\beta)} \leq C, \\ \sum_{\beta \in \mathcal{S}(\alpha)} K(\delta,\beta) &= \sum_{\beta : \beta \geq \delta} 2^{-npd(\delta,\beta)} \\ &= \sum_{m=d(\delta)}^{\infty} \sum_{\substack{\beta : \beta \geq \delta \\ \text{and } d(\beta)=m}} 2^{-npd(\delta,\beta)} \\ &= \sum_{m=d(\delta)}^{\infty} (2^n)^{(m-d(\delta))}\, 2^{-np(m-d(\delta))} \leq C \end{aligned}$$

since $p > 1$. Thus we have

$$IV \leq C \sum_{\delta \geq \alpha} \mu(\delta)^{p'} \leq C \left(\sum_{\delta \geq \alpha} \mu(\delta)\right)^{p'} = CB^{p'} \leq CB,$$

since B is bounded, and by (5.35), this is dominated by the right side of (5.34) as required.

We now consider the general case $\ell \geq 0$ of (5.33). Using $g(\alpha) = 2^{nd(\alpha)}\mu(\alpha)^{\frac{1}{p}}$ and $I_\ell g(\gamma) = \sum_{d(\delta,\gamma\wedge\beta)\leq\ell} g(\delta)$, (5.33) can be rewritten as

(5.36)

$$\sum_{\beta\geq\alpha}\left(\sum_{\gamma\geq\beta}\left(\sum_{d(\delta,\delta\wedge\gamma)\leq\ell} 2^{n[d(\delta)-d(\gamma)]}\mu(\delta)^{\frac{1}{p}}\right)^p\right)^{p'} \leq C^{p'}\sum_{\beta\geq\alpha}\left(\sum_{d(\gamma,\gamma\wedge\beta)\leq\ell} 2^{n[d(\gamma)-d(\beta)]}\mu(\gamma)^{\frac{1}{p}}\right)^p,$$

for $\alpha \in \mathcal{T}_n$. This time we have only

$$-d(\gamma,\delta\wedge\gamma) \leq d(\delta) - d(\gamma) = d(\delta,\delta\wedge\gamma) - d(\gamma,\delta\wedge\gamma) \leq \ell - d(\gamma,\delta\wedge\gamma),$$

for $d(\delta,\delta\wedge\gamma) \leq \ell$, and a similar inequality for $d(\gamma) - d(\beta)$. In particular then, recalling that our bound need only be established modulo an exponential in ℓ, it will suffice to prove

$$\sum_{\beta\geq\alpha}\left(\sum_{\gamma\geq\beta}\left(\sum_{d(\delta,\delta\wedge\gamma)\leq\ell} 2^{-nd(\gamma,\delta\wedge\gamma)}\mu(\delta)^{\frac{1}{p}}\right)^p\right)^{p'} \leq C2^{\ell A}\sum_{\beta\geq\alpha}\left(\sum_{d(\gamma,\gamma\wedge\beta)\leq\ell} 2^{-nd(\beta,\gamma\wedge\beta)}\mu(\gamma)^{\frac{1}{p}}\right)^p, \tag{5.37}$$

for $\alpha \in \mathcal{T}_n$, and for all $\ell \geq 0$. However, the methods used above to prove that (5.34) follows from (5.28), also show that (5.37) follows from (5.28) with a constant A depending only on n and p. This completes the proof of Lemma 5.15.

5.2.2. *Multiplier approximations.* The next lemma constructs a holomorphic function that is close to 1 on the Carleson region associated to a point $w \in \mathbb{B}_n$, and decays appropriately away from the Carleson region. We follow Böe's proof in [**Boe**], which adapts a real-variable argument of Marshall and Sundberg in [**MaSu**] to produce a holomorphic multiplier approximation. Given $\beta < \rho < \alpha < 1$, we will use the cutoff function $c_{\rho,\alpha}$ defined by

$$c_{\rho,\alpha}(\gamma) = \begin{cases} 0 & \text{for} \quad \gamma < \rho \\ \frac{\gamma-\rho}{\alpha-\rho} & \text{for} \quad \rho \leq \gamma \leq \alpha \\ 1 & \text{for} \quad \alpha < \gamma \end{cases}. \tag{5.38}$$

Lemma 5.18 (analogue of Lemma 4.1 in [**Boe**]). *Suppose $s > -1$. There are ρ and α satisfying $\beta < \rho < \alpha < 1$ such that for every $w \in \mathbb{B}_n$, we can find a function g_w so that*

$$\varphi_w(z) = \Gamma_s g_w(z) = \int_{\mathbb{B}_n} \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}}d\zeta$$

satisfies

$$\left\{ \begin{array}{lcll} \varphi_w(w) & = & 1 & \\ \varphi_w(z) & = & c_{\rho,\alpha}(\gamma_w(z)) + O\left(\left(\log \frac{1}{1-|w|^2}\right)^{-1}\right), & z \in V_w \\ |\varphi_w(z)| & \leq & C\left(\log \frac{1}{1-|w|^2}\right)^{1-p}, & z \notin V_w \end{array} \right. , \tag{5.39}$$

where $\gamma_w(z)$ *is defined by*

$$|1 - \overline{z} \cdot Pw| = \left(1 - |w|^2\right)^{\gamma_w(z)}$$

and $c_{\rho,\alpha}$ *is as in (5.38). Furthermore we have the estimate*

$$\int_{\mathbb{B}_n} \left| \left(1-|\zeta|^2\right)^n g_w(\zeta) \right|^p d\lambda_n(\zeta)\, d\zeta \leq C \left(\log \frac{1}{1-|w|^2} \right)^{1-p}. \tag{5.40}$$

Remark 5.19. The proof of Lemma 5.18 shows that the third estimate in (5.39) can be vastly improved, and also holds for a larger range of z; namely there is $\beta < \beta_1 < \rho$ such that

$$|\varphi_w(z)| \leq C \left(\log \frac{1}{1-|w|^2} \right)^{-1} \left(1-|w|^2\right)^{(\rho-\beta_1)(1+s)}, \quad z \notin V_w^{\beta_1}.$$

This fact will be used in the proof of Lemma 5.21 below.

Proof. Define $g_w(\zeta)$ by

$$\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}} = K \left(\log \frac{1}{1-|w|^2} \right)^{-1} \left|1-\overline{\zeta}\cdot Pw\right|^{-n-1}, \tag{5.41}$$

when ζ lives in the annular sector $\mathcal{S}$ centred at Pw given as the intersection of the annulus

$$\mathcal{A} = \mathcal{A}_w = \left\{ \zeta \in \mathbb{B}_n : \left(1-|w|^2\right)^\alpha \leq \left|1-\overline{\zeta}\cdot Pw\right| \leq \left(1-|w|^2\right)^\rho \right\} \tag{5.42}$$

and the cone

$$\mathcal{C} = \mathcal{C}_w = \left\{ \zeta \in \mathbb{B}_n : \left|\mathrm{Im}\left(\zeta\cdot\overline{Pw}\right)\right| + \left|\zeta - \left(\zeta\cdot\overline{Pw}\right)Pw\right|^2 \leq c\left(1-\left|\overline{\zeta}\cdot Pw\right|\right) \right\},$$

where c is a suitably small constant. Define $g_w(\zeta) = 0$ otherwise. The following observation will be used repeatedly.

Remark 5.20. The cone $\mathcal{C}_w$ corresponds to the geodesic in the Bergman tree $\mathcal{T}_n$ joining the root to the "boundary point" Pw. To see this, consider the case $w = (t, 0, ..., 0)$ and $\zeta = \left(re^{i\theta}, \zeta'\right)$ with $re^{i\theta} = x + iy$, so that $\mathrm{Im}\left(\zeta\cdot\overline{Pw}\right) = y$, $\zeta - \left(\zeta\cdot\overline{Pw}\right)Pw = (0, \zeta')$ and $1 - \left|\overline{\zeta}\cdot Pw\right| = 1 - r$.

Now choose K so that $\varphi_w(w) = 1$, i.e.

$$K = \left(\log \frac{1}{1-|w|^2} \right) \left(\int_{\mathcal{S}} \left|1-\overline{\zeta}\cdot Pw\right|^{-n-1} d\zeta \right)^{-1},$$

which satisfies

$$K \approx K_{\alpha,\rho,n} = \frac{c_n}{\alpha - \rho} \tag{5.43}$$

since the annular sector

$$\mathcal{E}_a = \left\{ \zeta \in \mathbb{B}_n : a \leq \left| 1 - \overline{\zeta} \cdot Pw \right| \leq 2a \right\} \cap \mathcal{C}$$

is comparable to a Bergman ball of radius one, $\left|1 - \overline{\zeta} \cdot Pw\right|^{-n-1} d\zeta$ is comparable to invariant measure $d\lambda_n(\zeta)$ on $\mathcal{E}_a$, and $\mathcal{S} \approx \cup_{j=0}^{J} \mathcal{E}_{2^j \left(1-|w|^2\right)^\alpha}$ where

$$J = \log \frac{\left(1 - |w|^2\right)^\rho}{\left(1 - |w|^2\right)^\alpha} = (\rho - \alpha) \log \left(1 - |w|^2\right).$$

Note also that

$$\left|1 - \overline{\zeta} \cdot Pw\right| \approx \left|1 - \overline{\zeta} \cdot w\right| \approx 1 - |\zeta|^2, \qquad \zeta \in \mathcal{S}, \tag{5.44}$$

and so g_w satisfies the estimate

$$|g_w(\zeta)| \leq C \left(\log \frac{1}{1 - |w|^2} \right)^{-1} \left|1 - \overline{\zeta} \cdot Pw\right|^{-n}, \qquad \zeta \in \mathbb{B}_n. \tag{5.45}$$

Now fix $z \in V_w$ and set

$$E_1 = \left\{ \zeta \in \mathbb{B}_n : \left|1 - \overline{\zeta} \cdot Pw\right| \leq \left(1 - |w|^2\right)^{\gamma_w(z)} \right\},$$
$$E_2 = \mathbb{B}_n \setminus E_1 = \left\{ \zeta \in \mathbb{B}_n : \left|1 - \overline{\zeta} \cdot Pw\right| > \left(1 - |w|^2\right)^{\gamma_w(z)} \right\}.$$

Thus the common boundary of E_1 and E_2 passes through z. The main contribution to $\varphi_w(z)$ will come from integration over E_2. Thus we write

$$\varphi_w(z) = \int_{E_1} \frac{g_w(\zeta) \left(1 - |\zeta|^2\right)^s}{\left(1 - \overline{\zeta} \cdot z\right)^{1+s}} d\zeta + \int_{E_2} \frac{g_w(\zeta) \left(1 - |\zeta|^2\right)^s}{\left(1 - \overline{\zeta} \cdot z\right)^{1+s}} d\zeta$$
$$= I + II.$$

By (5.44), (5.45) and the definition of $\gamma_w(z)$, term I is dominated by a constant multiple of $\left(\log \frac{1}{1-|w|^2}\right)^{-1}$ times

$$\int_{\left\{\left(1-|w|^2\right)^\alpha \leq \left|1 - \overline{\zeta} \cdot Pw\right| \leq |1 - \overline{z} \cdot Pw|\right\} \cap \mathcal{C}} \left(\frac{1 - |\zeta|^2}{\left|1 - \overline{\zeta} \cdot z\right|} \right)^{1+s} d\lambda_n(\zeta),$$

which is at most a constant C since

$$\left|1 - \overline{\zeta} \cdot z\right| \approx |1 - \overline{z} \cdot Pw|, \qquad \zeta \in \mathcal{C} \cap E_1.$$

Thus we have

$$|I| \leq C \left(\log \frac{1}{1 - |w|^2} \right)^{-1}.$$

We now write

$$
\begin{aligned}
II &= \int_{E_2\cap\mathcal{S}} \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}} d\zeta \\
&= \int_{E_2\cap\mathcal{S}} \left\{ \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}} - \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}} \right\} d\zeta \\
&\qquad\qquad + \int_{E_2\cap\mathcal{S}} \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}} d\zeta \\
&= III + IV.
\end{aligned}
$$

Using (5.43), and that g_w is supported in $\mathcal{S}$, we calculate that term IV is

$$
\begin{aligned}
K\left(\log\frac{1}{1-|w|^2}\right)^{-1} \int_{\left\{\left(1-|w|^2\right)^{\gamma_w(z)}\leq\left|1-\overline{\zeta}\cdot Pw\right|\leq\left(1-|w|^2\right)^{\rho}\right\}\cap\mathcal{C}} &\left|1-\overline{\zeta}\cdot Pw\right|^{-n-1} d\zeta \\
&= \frac{\gamma_w(z)-\rho}{\alpha-\rho}\log\frac{1}{1-|w|^2}
\end{aligned}
$$

in the case $\rho < \gamma_w(z) < \alpha$. We also have $IV = 0$ in the case $\gamma_w(z) < \rho$, and $IV = 1$ in the case $\alpha < \gamma_w(z)$. This gives the estimate $IV = c_{\rho,\alpha}(\gamma_w(z))$, for $z \in V_w$. Using

$$
\left| \frac{1}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}} - \frac{1}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}} \right| \leq C \frac{|z-w|}{\left(1-|\zeta|^2\right)^{2+s}}
$$

together with (5.44) and (5.45), we obtain that

$$
\begin{aligned}
|III| &\leq C\int_{E_2\cap\mathcal{S}} |g_w(\zeta)|\left(1-|\zeta|^2\right)^{-2}|z-w|\,d\zeta \\
&\leq C|z-w|\left(\log\frac{1}{1-|w|^2}\right)^{-1}\int_{E_2\cap\mathcal{S}}\left(1-|\zeta|^2\right)^{-1} d\lambda_n(\zeta) \\
&\leq C\left(\log\frac{1}{1-|w|^2}\right)^{-1},
\end{aligned}
$$

as required. This completes the proof of the second estimate in (5.39).

We now turn to the third estimate in (5.39). For $z \notin V_w$, we have $\left|1-\overline{\zeta}\cdot z\right| \geq c\left(1-|w|^2\right)^{\beta}$ for $\zeta \in \mathcal{S}$, and thus

$$
\begin{aligned}
|\varphi_w(z)| &\leq \left(\log\frac{1}{1-|w|^2}\right)^{-1}\int_{\mathcal{S}}\left(\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}\cdot z\right|}\right)^{1+s} d\lambda_n(\zeta)\\
&\leq C\left(\log\frac{1}{1-|w|^2}\right)^{-1}\left(1-|w|^2\right)^{(\rho-\beta)(1+s)}\\
&\leq C_p\left(\log\frac{1}{1-|w|^2}\right)^{1-p}.
\end{aligned}
$$

Finally, the estimate (5.40) is a calculation using (5.44), (5.45) the definition of the support of g_w. Indeed, the left side of (5.40) is at most

$$
C\int_{\mathcal{S}}\left(\log\frac{1}{1-|w|^2}\right)^{-p} d\lambda_n(\zeta) \leq C\left(\log\frac{1}{1-|w|^2}\right)^{1-p}.
$$

The next lemma uses Lemma 5.15 to construct inductively a holomorphic function whose restriction to the sequence $\{z_j\}_{j=1}^{\infty}$ approximates an arbitrarily prescribed bounded sequence $\{\xi_j\}_{j=1}^{\infty}$.

LEMMA 5.21 (analogue of Lemma 4.2 in [**Boe**]). *Suppose $s>-1$, that $\{\xi_j\}_{j=1}^{\infty} \in \ell^{\infty}$ and let $0<\delta<1$. Let φ_j, g_j and γ_j correspond to z_j as in Lemma 5.18 and with the same s. Then there is $\{a_i\}_{i=1}^{\infty} \in \ell^{\infty}$ such that $\varphi=\sum_{i=1}^{\infty} a_i\varphi_i$ satisfies*

$$
\left\|\{\xi_j-\varphi(z_j)\}_{j=1}^{\infty}\right\|_{\ell^{\infty}} < \delta\left\|\{\xi_j\}_{j=1}^{\infty}\right\|_{\ell^{\infty}} \tag{5.46}
$$

and

$$
\left\|\{a_i\}_{i=1}^{\infty}\right\|_{\ell^{\infty}}, \|\varphi\|_{H^{\infty}(\mathbb{B}_n)} \leq C\left\|\{\xi_j\}_{j=1}^{\infty}\right\|_{\ell^{\infty}}. \tag{5.47}
$$

REMARK 5.22. The series $\sum_{i=1}^{\infty} a_i\varphi_i$ in Lemma 5.21 converges absolutely for each $z \in \mathbb{B}_n$. In fact, the proof below will show that (using $\#\mathcal{G}_{\ell} \leq C\beta(0,z_{\ell})$)

$$
\sum_{i=1}^{\infty}|\varphi_i(z)| \leq C\left(1+\log\frac{1}{1-|z|^2}\right), \qquad z \in \mathbb{B}_n.
$$

REMARK 5.23. The construction in the proof below shows that both the sequence $\{a_i\}_{i=1}^{\infty}$ and the function φ depend linearly on the data $\{\xi_j\}_{j=1}^{\infty}$.

PROOF. We follow the proof of Lemma 4.2 in [**Boe**]. Let $\left\|\{\xi_j\}_{j=1}^{\infty}\right\|_{\ell^{\infty}}=1$. We first choose J so large that

$$
\sup_{j\geq J}\left(\log\frac{1}{1-|z_j|}\right)^{-1}+\sum_{j=J}^{\infty}\left(\log\frac{1}{1-|z_j|}\right)^{1-p} < \varepsilon, \tag{5.48}
$$

where $\varepsilon>0$ will be determined later. Note that the series above converges by the Carleson embedding in (5.10). By standard arguments, we may discard the finitely many points $\{z_j\}_{j=1}^{J-1}$ and assume that $J=1$ in (5.48). Indeed, this is elementary in dimension $n=1$ using the structure of zeroes of holomorphic functions of one variable. In higher dimensions we must work a bit harder. Suppose that $Z = \{z_j\}_{j=1}^{\infty}$ is an interpolating sequence for B_p, i.e. that (5.3) holds. Suppose that $w \in \mathbb{B}_n \setminus Z$. In order to show that $Z \cup \{w\}$ satisfies (5.3), it suffices to exhibit a function $h \in B_p$ that vanishes on Z, but not at w. Using the Hahn-Banach theorem and the reflexivity of B_p, this is equivalent to showing that the point evaluation $e_w = k_w^{\alpha,p}$ is *not* in the closure $\mathcal{S}$ of the linear span of the set of point evaluations $\left\{e_{z_j}\right\}_{j=1}^{\infty}$ in $B_p' = B_{p'}$. So suppose, in order to derive a contradiction, that $e_w \in \mathcal{S}$. Above, we showed the equivalence of (5.3) for Z and the Riesz condition (5.4) for Z,

$$\left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}} \approx \left(\sum_{j=1}^{\infty} |a_j|^{p'} \right)^{\frac{1}{p'}},$$

and consequently we have

$$e_w = \sum_{j=1}^{\infty} a_j \omega_j e_{z_j}, \tag{5.49}$$

where $\omega_j = \frac{1}{\left\|e_{z_j}\right\|_{B_{p'}}}$ and

$$\sum_{j=1}^{\infty} |a_j|^{p'} < \infty.$$

Now let $f_\ell \in B_p$ satisfy $f_\ell(z_k) = \delta_k^\ell$. Then from (5.49) we have

$$f_\ell(w) = \sum_{j=1}^{\infty} a_j \omega_j f_\ell(z_j) = \omega_\ell a_\ell,$$

for all $\ell \geq 1$, and we'll have the desired contradiction, namely $e_w = 0$, if we can show $f_\ell(w) = 0$ for all $\ell \geq 1$. To see this, choose a linear function g_ℓ that vanishes at z_ℓ and takes the value 1 at w. Then $g_\ell \in M_{B_p}$ implies $f_\ell g_\ell \in B_p$, and (5.49) yields

$$f_\ell(w) = (f_\ell g_\ell)(w) = \sum_{j=1}^{\infty} a_j \omega_j (f_\ell g_\ell)(z_j) = a_\ell \omega_\ell g_\ell(z_\ell) = 0.$$

Now order the points $\{z_j\}_{j=1}^{\infty}$ so that $1 - |z_{j+1}| \leq 1 - |z_j|$ for $j \geq 1$. We define a "forest structure" on the index set $\mathbb{N}$ by declaring that j is a child of i (or that i is a parent of j) provided that

$$\begin{aligned} & i < j, \\ & V_{z_j} \subset V_{z_i}, \\ & V_{z_j} \subsetneq V_{z_k} \text{ for } i < k < j. \end{aligned} \tag{5.50}$$

Note that a child j chooses the "nearest" parent i if we have competing indices i and i' with $V_{z_j} \subset V_{z_i} \cap V_{z_{i'}}$. We define a partial order associated with this parent-child relationship by declaring that j is a successor of i (or that i is a predecessor of j) if there is a "chain" of indices $\{i = k_1, k_2, ..., k_m = j\} \subset \mathbb{N}$ such that $k_{\ell+1}$ is a child of k_ℓ for $1 \leq \ell < m$. Under this partial ordering, $\mathbb{N}$ decomposes into a disjoint union of trees. Thus associated to each index $\ell \in \mathbb{N}$, there is a unique tree containing ℓ and, unless ℓ is the root of the tree, a unique parent $P(\ell)$ of ℓ in that tree. Denote by $\mathcal{G}_\ell$ the unique geodesic joining the root of the tree to ℓ. We will now define the coefficients $\{a_i\}_{i=1}^{\infty}$ of $\varphi = \sum_{i=1}^{\infty} a_i \varphi_i$, where φ_i is the function φ_{z_i} in Lemma 5.18 with w there replaced by z_i, by considering separately the indices in each tree of the forest $\mathbb{N}$.

Let $\mathcal{Y}$ be a tree in the forest $\mathbb{N}$ with root k_0. For each $k \in \mathcal{Y} \setminus \{k_0\}$, define $\beta_k \in [0,1]$ by

$$\beta_k = c\left(\gamma_{P(k)}\left(z_k\right)\right),$$

where the functions $c = c_{\rho,\alpha}$ and $\gamma_j = \gamma_{z_j}$ are defined as in the statement of Lemma 5.18 with w there replaced by z_j. Note that by Lemma 5.18 with $w = z_{P(k)}$, we have the estimate

$$\begin{aligned} \varphi_{P(k)}\left(z_k\right) &= c\left(\gamma_{P(k)}\left(z_k\right)\right) + O\left(\left(\log \frac{1}{1-\left|z_{P(k)}\right|^2}\right)^{-1}\right) \\ &= \beta_k + O\left(\left(\log \frac{1}{1-\left|z_{P(k)}\right|^2}\right)^{-1}\right), \end{aligned}$$

which can serve as motivation for the definition of the coefficients given below in (5.52). Indeed, with gross oversimplification, what we want is

$$\begin{aligned} \xi_k = \varphi\left(z_k\right) &\approx a_k \varphi_k\left(z_k\right) + a_{P(k)} \varphi_{P(k)}\left(z_k\right) + ... \\ &\approx a_k + a_{P(k)} \beta_k + ..., \end{aligned}$$

which leads to (5.52).

We will now define numbers $\{a_k\}_{k\in\mathcal{Y}}$ by induction on the linear ordering in $\mathcal{Y}$ induced from the natural ordering of $\mathbb{N}$, so that

$$(5.51) \qquad \begin{cases} \left|a_k\right| & \leq \; 2 \\ \left|\sum_{i\in\mathcal{G}_k\setminus\{k_0\}} \beta_i a_{P(i)}\right| & \leq \; 1 \end{cases}$$

holds for all $k \in \mathcal{Y}$. First define $a_{k_0} = \xi_{k_0}$. Now fix $\ell \in \mathcal{Y} \setminus \{k_0\}$ and assume that a_k has been defined for all $k \in \mathcal{Y}$ for which $k < \ell$ so that (5.51) holds for all $k \in \mathcal{Y}$ for which $k < \ell$. We now define a_ℓ by

$$(5.52) \qquad a_\ell = \xi_\ell - \sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} \beta_i a_{P(i)}.$$

Of crucial importance is the observation that the geodesics $\mathcal{G}_\ell$ and $\mathcal{G}_{P(\ell)}$ are related by

$$\mathcal{G}_{P(\ell)} = \mathcal{G}_\ell \setminus \{\ell\},$$

i.e. if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$ with $k_m = \ell$, then $k_{m-1} = P(\ell)$ and $\mathcal{G}_{P(\ell)} = [k_0, k_1, ..., k_{m-1}]$. By the induction assumption and the fact that $P(\ell) \in \mathcal{Y}$ and $P(\ell) < \ell$, we have

$$\left|\sum_{i\in\mathcal{G}_{P(\ell)}\setminus\{k_0\}} \beta_i a_{P(i)}\right| \leq 1.$$

We have from (5.52) and the above that

$$\begin{aligned}\left|\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}}\beta_i a_{P(i)}\right| &= \left|\left(\sum_{i\in\mathcal{G}_{P(\ell)}\setminus\{k_0\}}\beta_i a_{P(i)}\right)+\beta_\ell\left(\xi_{P(\ell)}-\sum_{i\in\mathcal{G}_{P(\ell)}\setminus\{k_0\}}\beta_i a_{P(i)}\right)\right| \\ &= \left|\beta_\ell\xi_{P(\ell)}+(1-\beta_\ell)\sum_{i\in\mathcal{G}_{P(\ell)}\setminus\{k_0\}}\beta_i a_{P(i)}\right| \\ &\leq \beta_\ell\left|\xi_{P(\ell)}\right|+(1-\beta_\ell)\left|\sum_{i\in\mathcal{G}_{P(\ell)}\setminus\{k_0\}}\beta_i a_{P(i)}\right|\leq 1.\end{aligned}$$

From this and (5.52) once more it immediately follows that $|a_\ell|\leq 2$, which shows that (5.51) holds for $k=\ell$ as well. This completes the inductive definition of the sequence $\{a_k\}_{k\in\mathcal{Y}}$ satisfying (5.51) on the tree $\mathcal{Y}$, and hence defines the entire sequence $\{a_i\}_{i=1}^\infty$.

We now show that both (5.46) and (5.47) hold for the function $\varphi=\sum_{i=1}^\infty a_i\varphi_i$. Fix an index $\ell\in\mathbb{N}$, and with notation as above, let $\mathcal{F}_\ell=\mathbb{N}\setminus\mathcal{G}_\ell$ and write using $\varphi_\ell(z_\ell)=1$ and (5.52),

$$\begin{aligned}\varphi(z_\ell)-\xi_\ell &= \sum_{i=1}^\infty a_i\varphi_i(z_\ell)-\xi_\ell \\ &= \left(\sum_{i\in\mathcal{G}_{P(\ell)}} a_i\varphi_i(z_\ell)+a_\ell\varphi_\ell(z_\ell)+\sum_{i\in\mathcal{F}_\ell}a_i\varphi_i(z_\ell)\right) \\ &\qquad -\left(a_\ell+\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}}\beta_i a_{P(i)}\right) \\ &= \sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} a_{P(i)}\left(\varphi_{P(i)}(z_\ell)-\beta_i\right)+\sum_{i\in\mathcal{F}_\ell}a_i\varphi_i(z_\ell) \\ &= I+II.\end{aligned}$$

We now claim that

$$\begin{aligned}|I| &\leq C\sup_{i\geq 1}\left(\log\frac{1}{1-|z_i|}\right)^{-1}, \\ |II| &\leq C\sum_{i=1}^\infty\left(\log\frac{1}{1-|z_i|}\right)^{1-p}.\end{aligned}\tag{5.53}$$

With this done we obtain from (5.48) (recall that we assume $J=1$ there) that

$$\sup_{j\geq 1}|\varphi(z_j)-\xi_j|\leq C\varepsilon<\delta$$

provided we choose $\varepsilon>0$ small enough, and this proves (5.46). We note in passing that the proof below will show that the supremum in (5.53) need only be taken over indices i that are a root of a tree in the forest $\mathbb{N}$.

To estimate term I, we begin with

$$\begin{aligned}|I| &= \left|\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} a_{P(i)}\left(\varphi_{P(i)}\left(z_\ell\right)-\beta_i\right)\right| \\ &\leq \sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} \left|a_{P(i)}\right|\left|\varphi_{P(i)}\left(z_\ell\right)-c\left(\gamma_{P(i)}\left(z_i\right)\right)\right| \\ &\leq 2\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}}\left\{\left|c\left(\gamma_{P(i)}\left(z_\ell\right)\right)-c\left(\gamma_{P(i)}\left(z_i\right)\right)\right|+C\left(\log\frac{1}{1-\left|z_{P(i)}\right|^2}\right)^{-1}\right\},\end{aligned}$$

where the final inequality follows from Lemma 5.18 since $z_\ell \in V_{z_{P(i)}}$. Now if V_{z_i} has empty intersection with the annulus $\mathcal{A}_{P(i)}$ given in (5.42) with $w = z_{P(i)}$, then both $c\left(\gamma_{P(i)}\left(z_\ell\right)\right)$ and $c\left(\gamma_{P(i)}\left(z_i\right)\right)$ have the same value, either 0 or 1. Otherwise, since c is Lipschitz continuous with norm $\frac{1}{\alpha-\rho}$, we have

$$\begin{aligned}\left|c\left(\gamma_{P(i)}\left(z_\ell\right)\right)-c\left(\gamma_{P(i)}\left(z_i\right)\right)\right| &\leq C\left|\gamma_{P(i)}\left(z_\ell\right)-\gamma_{P(i)}\left(z_i\right)\right| \\ &= C\left|\frac{-\log\left|1-\overline{z_\ell}\cdot Pz_{P(i)}\right|}{-\log\left(1-\left|z_{P(i)}\right|^2\right)}-\frac{-\log\left|1-\overline{z_i}\cdot Pz_{P(i)}\right|}{-\log\left(1-\left|z_{P(i)}\right|^2\right)}\right| \\ &\leq C\left(\log\frac{1}{1-\left|z_{P(i)}\right|^2}\right)^{-1},\end{aligned}$$

since

$$C^{-1} \leq \frac{\left|1-\overline{z_\ell}\cdot Pz_{P(i)}\right|}{\left|1-\overline{z_i}\cdot Pz_{P(i)}\right|} \leq C$$

for $z_i, z_\ell \in \mathcal{A}_{P(i)}$. Now if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$, then by applying the separation condition repeatedly, we obtain

$$\left(1-|z_{k_i}|^2\right) \leq \left(1-|z_{k_0}|^2\right)^{\eta^i},$$

and so combining these estimates we have

$$\begin{aligned}|I| &\leq C\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}}\left(\log\frac{1}{1-\left|z_{P(i)}\right|^2}\right)^{-1} \\ &\leq C\left(\sum_{j=0}^{m-1}\eta^{-j}\right)\left(\log\frac{1}{1-|z_{k_0}|^2}\right)^{-1} \\ &\leq C_\eta\left(\log\frac{1}{1-|z_{k_0}|^2}\right)^{-1}\end{aligned}$$

since $\eta > 1$, which shows the first inequality in (5.53).

To estimate term $II = \sum_{i\in\mathcal{F}_\ell} a_i\varphi_i\left(z_\ell\right)$ in (5.53), we first note that if $z_\ell \notin V_{z_i}$, then

$$|\varphi_i\left(z_\ell\right)| \leq C\left(\log\frac{1}{1-|z_i|^2}\right)^{-(p-1)} \tag{5.54}$$

by Lemma 5.18. On the other hand, if $z_\ell \in V_{z_i}$, then $|z_i| < |z_\ell|$, and if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$, then either $|z_i| < |z_{k_0}|$ or there is j such that $\left|z_{k_{j-1}}\right| < |z_i| \leq \left|z_{k_j}\right|$. Note however that equality cannot hold here by Lemma 5.13, and so we actually have $\left|z_{k_{j-1}}\right| < |z_i| < \left|z_{k_j}\right|$. From (5.50) we obtain that no index $m \in (k_{j-1}, k_j)$ satisfies $V_{z_{k_j}} \subset V_{z_m}$. Since $i \notin \mathcal{G}_\ell$, we have $i \in (k_{j-1}, k_j)$ and thus we have both

$$V_{z_{k_j}} \subsetneq V_{z_i} \text{ and } \left|z_{k_j}\right| > |z_i| .$$

Now using Lemma 5.13 and $\beta\eta > 1$, we obtain

$$\left(1 - \left|z_{k_j}\right|^2\right)^\beta \leq \left(1 - |z_i|^2\right)^{\beta\eta} \ll \left(1 - |z_i|^2\right).$$

If we choose $w \in V_{z_{k_j}} \setminus V_{z_i}$, then $w, z_\ell \in V_{z_{k_j}}$ implies $|z_\ell - w| \leq C\left(1 - \left|z_{k_j}\right|^2\right)^\beta$ by definition, and $w \notin V_{z_i}$ implies $|1 - \overline{w} \cdot Pz_i| \geq c\left(1 - |z_i|^2\right)^\beta$. Together with the reverse triangle inequality we thus have

$$\begin{aligned}
|1 - \overline{z_\ell} \cdot Pz_i| &\geq |1 - \overline{w} \cdot Pz_i| - |\overline{z_\ell} \cdot Pz_i - \overline{w} \cdot Pz_i| \\
&\geq c\left(1 - |z_i|^2\right)^\beta - C\left(1 - |z_i|^2\right)^{\beta\eta} \\
&\geq (1 - |z_i|)^{\beta_1},
\end{aligned}$$

for some $\beta_1 \in (\beta, \rho)$ (again provided the $|z_i|$ are large enough). Thus estimate (5.54) now follows by Remark 5.19. With this done, we have completed the proof of (5.53), as well as the first estimate in (5.47).

To prove the second estimate $\|\varphi\|_{H^\infty(\mathbb{B}_n)} \leq C$ in (5.47), we fix $z \in \mathbb{B}_n$. If $z \in \cup_{k \in \mathbb{N}} V_{z_k}$, let ℓ be such that

$$z \in V_{z_\ell} \text{ and } z \notin V_{z_k} \text{ for any } k > \ell.$$

Then we have using (5.52) that

$$\begin{aligned}
\varphi(z) - \xi_\ell &= \sum_{i=1}^\infty a_i \varphi_i(z) - \xi_\ell \\
&= \left(\sum_{i \in \mathcal{G}_{P(\ell)}} a_i \varphi_i(z) + a_\ell \varphi_\ell(z) + \sum_{i \in \mathcal{F}_\ell} a_i \varphi_i(z)\right) \\
&\qquad - \left(a_\ell + \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \beta_i a_{P(i)}\right) \\
&= \left(\sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} a_{P(i)}\left(\varphi_{P(i)}(z) - \beta_i\right)\right) + a_\ell(\varphi_\ell(z) - 1) + \sum_{i \in \mathcal{F}_\ell} a_i \varphi_i(z),
\end{aligned}$$

and the estimates we proved above for z_ℓ show that with z_ℓ replaced by z, we also have

$$\begin{aligned}|\varphi(z)-\xi_\ell| &\leq \left|\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} a_{P(i)}(\varphi_{P(i)}(z)-\beta_i)\right| + |a_\ell(\varphi_\ell(z)-1)| + \left|\sum_{i\in\mathcal{F}_\ell} a_i\varphi_i(z)\right| \\ &\leq C_\eta\left(\log\frac{1}{1-|z_{k_0}|^2}\right)^{-1} + C + C\sum_{i\in\mathcal{F}_\ell}\left(\log\frac{1}{1-|z_i|^2}\right)^{-(p-1)},\end{aligned}$$

which yields $|\varphi(z)|\leq C$. Finally, if $z\notin\cup_{k\in\mathbb{N}}V_{z_k}$, then

$$|\varphi(z)| = \left|\sum_{i=1}^{\infty} a_i\varphi_i(z)\right| \leq C\sum_{i\in\mathbb{N}}\left(\log\frac{1}{1-|z_i|^2}\right)^{-(p-1)} \leq C$$

by Lemma 5.18. Easy modifications of these estimates also prove Remark 5.22. At this point we can use a normal families argument to prove that φ is holomorphic. This completes the proof of Lemma 5.21.

5.2.3. *The proof of multiplier interpolation.* Using Lemma 5.21, we first complete the proof that (5.8) implies (5.6) for $1<p<\infty$. Fix $s>-1$, $0<\delta<1$ and $\{\xi_j\}_{j=1}^\infty$ with $\left\|\{\xi_j\}_{j=1}^\infty\right\|_{\ell^\infty}=1$. Then by Lemma 5.21 there is $f_1=\sum_{i=1}^\infty a_i^1\varphi_i\in H^\infty(\mathbb{B}_n)$ such that $\left\|\{\xi_j-f_1(z_j)\}_{j=1}^\infty\right\|_{\ell^\infty}<\delta$ and $\left\|\{a_i^1\}_{i=1}^\infty\right\|_{\ell^\infty}$, $\|f_1\|_{H^\infty(\mathbb{B}_n)}\leq C$ where C is as in (5.47). Now apply Lemma 5.21 to the sequence $\{\xi_j-f_1(z_j)\}_{j=1}^\infty$ to obtain the existence of $f_2=\sum_{i=1}^\infty a_i^2\varphi_i\in H^\infty(\mathbb{B}_n)$ such that $\left\|\{\xi_j-f_1(z_j)-f_2(z_j)\}_{j=1}^\infty\right\|_{\ell^\infty}<\delta^2$ and $\left\|\{a_i^2\}_{i=1}^\infty\right\|_{\ell^\infty}$, $\|f_2\|_{H^\infty(\mathbb{B}_n)}\leq C\delta$ where C is as in (5.47). Continuing inductively, we obtain $f_m=\sum_{i=1}^\infty a_i^m\varphi_i\in H^\infty(\mathbb{B}_n)$ such that

$$\begin{aligned}\left\|\left\{\xi_j-\sum_{i=1}^m f_i(z_j)\right\}_{j=1}^\infty\right\|_{\ell^\infty} &< \delta^m, \\ \|\{a_i^m\}_{i=1}^\infty\|_{\ell^\infty}, \|f_m\|_{H^\infty(\mathbb{B}_n)} &\leq C\delta^{m-1}.\end{aligned}$$

If we now take $\varphi=\sum_{m=1}^\infty f_m$, we have

$$\begin{aligned}\xi_j &= \varphi(z_j), \quad 1\leq j<\infty, \\ \|\varphi\|_{H^\infty(\mathbb{B}_n)} &\leq C_\delta,\end{aligned} \tag{5.55}$$

as well as $\varphi=\sum_{i=1}^\infty a_i\varphi_i$ with $\|\{a_i\}_{i=1}^\infty\|_{\ell^\infty}\leq C_\delta$. Recall that the series $\varphi=\sum_{i=1}^\infty a_i\varphi_i$ converges absolutely by Remark 5.22, and depends linearly on the data $\{\xi_j\}_{j=1}^\infty$ by Remark 5.23 and the linear construction in this paragraph. Thus $\varphi\in H^\infty(\mathbb{B}_n)$ linearly interpolates the values $\{\xi_j\}_{j=1}^\infty$ on the sequence $\{z_j\}_{j=1}^\infty$, and it remains to prove that $\varphi\in M_{B_p}$. Recall that our function φ depends on our choice of $s>-1$.

By Theorem 4.2, $\varphi\in M_{B_p}$ will follow if we show that

$$\left\|\left|\left(1-|z|^2\right)^n\nabla^n\varphi(z)\right|^p d\lambda_n(z)\right\|_{Carleson}\leq C. \tag{5.56}$$

Since

$$\varphi=\sum_{i=1}^{\infty}a_i\varphi_i=\sum_{i=1}^{\infty}a_i\Gamma_s g_i=\Gamma_s g$$

where $g=\sum_{i=1}^{\infty}a_i g_i$ with $\sup_{i\geq 1}|a_i|\leq C_\delta$, (5.56) will follow from Theorem 3.1 and Lemma 5.15 for s sufficiently large provided we show that (5.22) holds and that

$$\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

satisfies the tree condition (3.2). From the definition of g_i in (5.41), and the fact that the supports of the g_i are pairwise disjoint by the separation condition, we may assume that the reverse Hölder condition on Bergman balls in (5.22) holds. The tree condition estimate will follow from the next lemma.

Lemma 5.24. *With $s>n-\frac{1}{p'}$ and $g=\sum_{i=1}^{\infty}a_i g_i$ as above, we have*

$$\left\|\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)\right\|_{tree\ condition}\leq C. \tag{5.57}$$

Proof. Inequality (5.57) follows from the estimate (5.40) as follows. If we discretize (5.40), we obtain with $w=z_i$ and $S(\alpha)\approx V_{z_i}$,

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(1-|c_\beta|^2\right)^{np}g_{z_i}(\beta)^p\leq C\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}. \tag{5.58}$$

Denote the Carleson tent at c_β by $S(\beta)=T_\beta=\cup_{\gamma\geq\beta}K_\gamma$. We are assuming that (3.2) holds for the measure $\nu=\sum_{j=1}^{\infty}\left(\log\frac{1}{1-|z_j|^2}\right)^{1-p}\delta_{z_j}$,

(5.59)

$$\begin{aligned}\sum_{\substack{\beta\in\mathcal{T}_n\\ \beta\geq\alpha}}\left(\sum_{z_j\in T_\beta}\left(\log\frac{1}{1-|z_j|^2}\right)^{1-p}\right)^{p'} &= \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}I^*\nu(\beta)^{p'}\\ &\leq C^{p'}I^*\nu(\alpha)\\ &= C^{p'}\sum_{z_j\in T_\alpha}\left(\log\frac{1}{1-|z_j|^2}\right)^{1-p}<\infty,\qquad \alpha\in\mathcal{T}_n.\end{aligned}$$

If we now also discretize (5.57), we see that we must prove

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}I^*\mu(\beta)^{p'}\leq C^{p'}I^*\mu(\alpha)<\infty,\qquad \alpha\in\mathcal{T}_n,$$

where

$$\begin{aligned}\mu(\beta)&=\left|\left(1-|c_\beta|^2\right)^n g(\beta)\right|^p,\\ g(\beta)&=\int_{K_\alpha}|g|\,d\lambda_n,\\ I^*\mu(\alpha)&=\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\mu(\beta),\end{aligned}$$

for all $g=\sum_{i=1}^{\infty}a_ig_i$ as above. Since $\nu=\sum_{j=1}^{\infty}\left(\log\frac{1}{1-|z_j|^2}\right)^{1-p}\delta_{z_j}$ satisfies the tree condition (3.2), it suffices to prove

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}I^*\mu\left(\beta\right)^{p'}\leq C^{p'}I^*\nu\left(\alpha\right),\qquad\alpha\in\mathcal{T}_n. \tag{5.60}$$

Indeed, (5.60) shows that $\mu+\nu$ satisfies (3.2), and now the equivalence of 2 and 3 in Theorem 3.1 shows that μ satisfies (3.2) as well (since the Carleson embedding is preserved for smaller measures). Alternatively, we can obtain this fact directly from Lemma 10.1 in the appendix at the end of the paper.

Now $g=\sum_{i=1}^{\infty}a_ig_i$ where the supports of the $g_i=g_{z_i}$ are pairwise disjoint by the separation condition on $\{z_i\}_{i=1}^{\infty}$. Fix $\alpha\in\mathcal{T}_n$. Then we have

$$\begin{aligned}\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}I^*\mu\left(\beta\right)^{p'}&=\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\mu\left(\gamma\right)\right)^{p'}\\&=\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\left|\left(1-|c_\gamma|^2\right)^n\left(\sum_{i=1}^{\infty}a_ig_i\right)(\gamma)\right|^p\right)^{p'}\\&=\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{i:z_i\in T_\beta}|a_i|^p\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\left|\left(1-|c_\gamma|^2\right)^n g_i\left(\gamma\right)\right|^p\right)^{p'}.\end{aligned}$$

Now we use (5.58) to dominate the last sum above by

$$\begin{aligned}&C\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{i:z_i\in T_\beta}|a_i|^p\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}\right)^{p'}\\&\leq C\left\|\{a_i\}_{i=1}^{\infty}\right\|_{\ell^\infty}^{pp'}\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{i:z_i\in T_\beta}\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}\right)^{p'}\\&\leq C\sum_{j:z_j\in T_\alpha}\left(\log\frac{1}{1-|z_j|^2}\right)^{1-p}=CI^*\nu\left(\alpha\right),\end{aligned}$$

where the final inequality follows from (5.59). This establishes (5.60), and completes the proof of Lemma 5.24.

With this done, we have completed the proof that (5.8) implies (5.6) for $1<p<\infty$.

We now prove that (5.5) implies (5.6) for $p>2n$. For this we will argue as above but with Lemma 5.15 replaced by Lemma 5.16, and with Lemma 5.24 replaced by the following analogue.

LEMMA 5.25. *Suppose (5.5) holds. With $p>2n$, $s>n-\frac{1}{p'}$ and $g=\sum_{i=1}^{\infty}a_ig_i$ as above, we have*

$$\left\|\left|\left(1-|z|^2\right)^n g\left(z\right)\right|^p d\lambda_n\left(z\right)\right\|_{Carleson}\leq C. \tag{5.61}$$

Proof. Inequality (5.61) follows from estimate (5.40),

$$\int_{\mathbb{B}_n} \left|\left(1-|\zeta|^2\right)^n g_{z_i}(\zeta)\right|^p d\lambda_n(\zeta)\, d\zeta \leq C\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}, \tag{5.62}$$

as follows. Fix an index i. From Remark 5.20 we see that the support of g_{z_i} is essentially the union of a geodesic segment of Bergman kubes $K_1^i, K_2^i, ..., K_{M_i}^i$ where

$$M_i \approx (\alpha-\rho)\log\frac{1}{1-|z_i|^2}.$$

Indeed, recall that the support g_{z_i} is contained in the intersection of the cone $\mathcal{C}_{z_i}$ and the annulus $\mathcal{A}_{z_i}$. Now for ζ in the cone $\mathcal{C}_{z_i}$, we have $\left|1-\overline{\zeta}\cdot Pz_i\right| \approx 1-|\zeta|^2$, and thus for ζ in the annulus $\mathcal{A}_{z_i}$ as well, we have approximately

$$\log\frac{1}{1-|\zeta|^2} \in \left(\rho\log\frac{1}{1-|z_i|^2}, \alpha\log\frac{1}{1-|z_i|^2}\right).$$

Thus $\zeta \in supp\, g_{z_i}$ lies in the union of those kubes in $\mathcal{T}_n$ along the geodesic joining the root to the "boundary point" Pz_i, and having tree distance from the root lying roughly between $\rho\beta(0,z_i)$ and $\alpha\beta(0,z_i)$. Moreover, this segment can be continued to a longer sequence of adjacent Bergman kubes $K_1^i, K_2^i, ..., K_{M_i}^i, ...K_{J_i}^i = K_{z_i}$ connecting the support of g_{z_i} to the kube K_{z_i} containing z_i, and where

$$J_i \approx \log\frac{1}{1-|z_i|^2}. \tag{5.63}$$

Choose $w_j \in K_j^i$ for $1 \leq j < J_i$. Then we have for $z \in K_m^i$, $1 \leq m \leq M_i$, and $f \in B_p(\mathbb{B}_n)$,

$$\begin{aligned}
|f(z)|^p &= f(z) - f(w_m) \\
&\qquad + \sum_{j=m}^{J_i-1}\left[f(w_j)-f(w_{j+1})\right] + \left[f(w_{J_i})-f(z_i)\right] + f(z_i)|^p \\
&\lesssim |f(z)-f(w_m)|^p + (J_i)^{p-1}\sum_{j=1}^{J_i-1}|f(w_j)-f(w_{j+1})|^p \\
&\qquad + |f(w_{J_i})-f(z_i)|^p + |f(z_i)|^p \\
&\leq C(J_i)^{p-1}\sum_{j=1}^{J_i}\left(\max_{z_1,z_2\in\left(K_j^i\right)^*}|f(z_1)-f(z_2)|\right)^p + C|f(z_i)|^p.
\end{aligned}$$

Using this together with (5.62), (5.63) and the fact that the supports of the g_{z_i} are pairwise disjoint, we obtain

$$\begin{aligned}
\int_{\mathbb{B}_n}|f(z)|^p\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z) &\leq C\sum_i\sum_{j=1}^{J_i}\left(\max_{z_1,z_2\in\left(K_j^i\right)^*}|f(z_1)-f(z_2)|\right)^p \\
&\qquad + C\sum_i|f(z_i)|^p\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}.
\end{aligned}$$

Since the kubes $\{K_j^i\}$ are pairwise disjoint by Lemma 5.13, the first term on the right is dominated by

$$\sum_{\alpha\in\mathcal{T}_n}\left(\max_{z_1,z_2\in K_\alpha}|f(z_1)-f(z_2)|\right)^p\le C\|f\|^p_{B_p(\mathbb{B}_n)}$$

by Theorem 6.30 of [**Zhu**] (or see Lemma 7.8 below). The second term is dominated by $C\|f\|^p_{B_p(\mathbb{B}_n)}$ since we are assuming in (5.5) that $\sum_i\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}\delta_{z_i}$ is a $B_p(\mathbb{B}_n)$-Carleson measure. This completes the proof of Lemma 5.25.

Arguing as above, the proof that (5.5) implies (5.6) will follow from Theorem 3.1 and Lemma 5.16 provided we show that (5.23) holds and that

$$\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

is a $B_p(\mathbb{B}_n)$-Carleson measure. From the definition of g_i in (5.41), and the fact that the supports of the g_i are pairwise disjoint by the separation condition, we have that (5.23) holds. Lemma 5.25 above shows that $\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$ is a $B_p(\mathbb{B}_n)$-Carleson measure for $p>2n$, and this completes the proof that (5.5) implies (5.6) for $p>2n$.

5.3. Besov space interpolation. Here we adapt the above arguments to prove that (5.10), or equivalently (5.5), implies (5.3) for $1<p<\infty$, and with linear interpolation. We note in passing that we already have that (5.10) implies (5.3) for $1<p\le 2$. Indeed, the Carleson embedding in (5.10) implies that the map

$$Tf=\left\{\frac{f(z_j)}{\left\|k^{\alpha,p}_{z_j}\right\|_{B_{p'}}}\right\}_{j=1}^\infty$$

in (5.3) maps B_p *into* ℓ^p. On the other hand, (5.10) implies (5.6) implies (5.7) implies (this is where we use $p\le 2$) the inequality $\gtrsim$ of (5.4), which in turn implies that the map T in (5.3) maps B_p *onto* ℓ^p.

Fix $\alpha,s>-1$ and $\{\xi_j\}_{j=1}^\infty$ with

$$\left\|\left\{\frac{\xi_j}{\left\|k^{\alpha,p}_{z_j}\right\|_{B_{p'}}}\right\}_{j=1}^\infty\right\|_{\ell^p}=1.$$

Recall that by (5.9), $\left\|k^{\alpha,p}_{z_j}\right\|_{B_{p'}}\approx\left(\log\frac{1}{1-|z_j|^2}\right)^{\frac{1}{p'}}$. We may discard finitely many points and reorder them so that with $d\mu=\sum_{j=1}^\infty\left\|k^{\alpha,p}_{z_j}\right\|^{-p}_{B_{p'}}\delta_{z_j}$, we have

$$\|\mu\|\approx\sum_{j=1}^\infty\left(\log\frac{1}{1-|z_j|}\right)^{1-p}<\varepsilon, \tag{5.64}$$

where $\varepsilon>0$ will be determined later. Moreover, we may suppose that the sequence $\{z_j\}_{j=1}^J$ is finite, and obtain an appropriate estimate independent of $J\ge 1$ (see (5.66) below). Since the sequence $\{\xi_j\}_{j=1}^J$ is bounded (we are not concerned that this bound blows up as $J \to \infty$), the construction in the previous section leading to (5.55) yields $\varphi = \sum_{m=1}^{J} \varphi_m$ satisfying

$$\xi_j = \varphi(z_j), \quad 1 \leq j < J,$$
$$\|\varphi\|_{H^\infty(\mathbb{B}_n)} \leq C \left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^\infty},$$

as well as $\varphi = \sum_{i=1}^{J} a_i \varphi_i$ with

$$\left\| \{a_i\}_{i=1}^J \right\|_{\ell^\infty} \leq C \left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^\infty}.$$

We need additional weighted ℓ^p control on the coefficients $\{a_i\}_{i=1}^J$. View $d\mu$ as the measure assigning mass $\left(\log \frac{1}{1-|z_j|^2}\right)^{1-p}$ to the point $j \in \{0, 1, 2, ..., J\}$. By (5.9), we have

$$\left\| \{b_j\}_{j=1}^J \right\|_{\ell^p(d\mu)} \approx \left\| \left\{ \frac{b_j}{\left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}} \right\}_{j=1}^J \right\|_{\ell^p},$$

for any sequence $\{b_j\}_{j=1}^J$.

Lemma 5.26. *The sequence $\{a_i\}_{i=1}^J$ constructed in (5.55) using Lemma 5.21 satisfies*

$$\left\| \{a_j\}_{j=1}^J \right\|_{\ell^p(d\mu)} \leq C \left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^p(d\mu)} \approx C. \tag{5.65}$$

With Lemma 5.26 established, we can easily complete the proof. We have that $\varphi \in H^\infty(\mathbb{B}_n)$ interpolates the values $\{\xi_j\}_{j=1}^J$ on the sequence $\{z_j\}_{j=1}^J$, and it remains only to prove that $\varphi \in B_p$ with $\|\varphi\|_{B_p} \leq C$ whenever $\left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^p(d\mu)} = 1$, independent of $J \geq 1$. Thus we must show that

$$\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^n \nabla^n \varphi(z) \right|^p d\lambda_n(z) \leq C,$$

independent of $J \geq 1$. Now

$$\varphi = \sum_{i=1}^{J} a_i \varphi_i = \sum_{i=1}^{J} a_i \Gamma_s g_i = \Gamma_s g$$

where $g = \sum_{i=1}^{J} a_i g_i$ with $\left\| \{a_i\}_{i=1}^J \right\|_{\ell^\infty} \leq C_\delta \left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^\infty}$. Moreover,

$$|\nabla^n \Gamma_s g(z)| \leq C'_{s,n} \widehat{T_s} |g|(z)$$

by (5.26) where the operator $\widehat{T_s}$ is given by

$$\widehat{T_s} f(z) = c_{s,n} \int_{\mathbb{B}_n} \frac{f(w) \left(1 - |w|^2\right)^s}{|1 - \overline{w} \cdot z|^{n+1+s}} dw.$$

Thus we must estimate

$$\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^n \widehat{T_s} |g|(z) \right|^p d\lambda_n(z) = \int_{\mathbb{B}_n} \left| \widehat{T_s} |g|(z) \right|^p d\nu_t(z)$$

where $t = pn - n - 1$. Now by Theorem 2.10 in [**Zhu**], $\widehat{T_s}$ is bounded on $L^p(d\nu_t)$ if and only if $0 < t + 1 < p(s+1)$, i.e.

$$p > 1 \text{ and } s > \frac{n}{p'} - 1.$$

Thus choosing $s > \frac{n}{p'} - 1$, we have

$$\begin{aligned}\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^n \nabla^n \varphi(z)\right|^p d\lambda_n(z) &\le C \int_{\mathbb{B}_n} |g(z)|^p d\nu_t(z) \\ &= C \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z).\end{aligned}$$

Since the supports of the g_i are pairwise disjoint by the separation condition, we obtain from (5.40) and then (5.65) that $g = \sum_{i=1}^J a_i g_i$ satisfies

$$\begin{aligned}\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z) &= \sum_{j=1}^J |a_i|^p \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^n g_i(z)\right|^p d\lambda_n(z) \\ &\le C \sum_{j=1}^J |a_i|^p \left(\log \frac{1}{1-|z_j|^2}\right)^{1-p} \\ &= C \left\|\{a_j\}_{j=1}^J\right\|_{\ell^p(d\mu)} \\ &\le C \left\|\{\xi_j\}_{j=1}^J\right\|_{\ell^p(d\mu)} = C.\end{aligned}$$

Altogether we have proved that $\varphi \in B_p$ interpolates the values $\{\xi_j\}_{j=1}^J$ on the sequence $\{z_j\}_{j=1}^J$ with the norm estimate

$$\|\varphi\|_{B_p} \le C \left\|\{\xi_j\}_{j=1}^J\right\|_{\ell^p(d\mu)}, \tag{5.66}$$

where C is independent of $J \ge 1$ as required. A limiting argument now finishes the proof that (5.10) implies (5.3).

It remains to prove Lemma 5.26 above. Let $\left\|\{\xi_j\}_{j=1}^J\right\|_{\ell^2(\mu)} = 1$. Recall from the proof of Lemma 5.21 that the approximating sequence $\left\{a_j^m\right\}_{j=1}^J$ at the m^{th} step of the proof there is given in terms of the data $\left\{\xi_j^m\right\}_{j=1}^J$ at the m^{th} step by (5.52), i.e.

$$a_\ell^m = \xi_\ell^m - \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \beta_i a_{A(i)}^m, \qquad 1 \le \ell \le J, \tag{5.67}$$

where for any given ℓ, the numbers β_i lie in $[0,1]$, and the geodesics $\mathcal{G}_\ell$ and $\mathcal{G}_{A(\ell)}$ lie in a fixed tree $\mathcal{Y}$ of the forest $\{j\}_{j=1}^J$, and are related by

$$\mathcal{G}_{A(\ell)} = \mathcal{G}_\ell \setminus \{\ell\},$$

i.e. if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$ with $k_m = \ell$, then $k_{m-1} = A(\ell)$ and $\mathcal{G}_{A(\ell)} = [k_0, k_1, ..., k_{m-1}]$. We will first prove the estimate (5.65) for the sequence $\left\{a_j^m\right\}_{j=1}^J$ given in terms of the data $\left\{\xi_j^m\right\}_{j=1}^J$, i.e.

$$\left\|\left\{a_j^m\right\}_{j=1}^J\right\|_{\ell^p(d\mu)} \le C \left\|\left\{\xi_j^m\right\}_{j=1}^J\right\|_{\ell^p(d\mu)}. \tag{5.68}$$

Without loss of generality, we may assume that the forest of indices $\{j\}_{j=1}^{J}$ is actually a single tree $\mathcal{Y}$. For convenience we will drop the superscript m from both a_ℓ^m and ξ_ℓ^m and write simply a_ℓ and ξ_ℓ.

Now fix ℓ. At this point it will be convenient for notation to momentarily relabel the points $\{z_j\}_{j\in\mathcal{G}_\ell} = \{z_{k_0}, z_{k_1}, ..., z_{k_m}\}$ as $\{z_0, z_1, ..., z_m\}$, and similarly relabel $\{a_0, a_1, ..., a_m\}$, $\{\xi_0, \xi_1, ..., \xi_m\}$ and $\{\beta_0, \beta_1, ..., \beta_m\}$ so that

$$a_k = \xi_k - \sum_{i=1}^{k} \beta_i a_{i-1}, \qquad 0 \le k \le \ell.$$

We also have $d\mu(j) = \left(\log \frac{1}{1-|z_j|^2}\right)^{1-p}$ where z_j now denotes the point z_{k_j} in the ball corresponding to k_j before the relabelling. In other words, we are restricting attention to the geodesic $\mathcal{G}_\ell$ and relabeling sequences so as to conform to the ordering in the geodesic. Now let

$$\omega_k = \sum_{i=1}^{k} \beta_i a_{i-1}$$

for $1 \le k \le \ell$ so that

$$a_k = \xi_k - \omega_k, \qquad 0 \le k \le \ell.$$

We now claim that

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_k \end{pmatrix} = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{bmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{k-1} \end{pmatrix}, \qquad 1 \le k \le \ell, \tag{5.69}$$

where $b_{ij} \in [0,1]$ and $b_{ij} = 0$ if $i < j$. We prove this by induction on k. Indeed, it is evident for $k = 1$ since then $\omega_1 = \beta_1 a_0 = \beta_1 \xi_0$. Assuming its truth for k, we obtain from

$$\omega_{k+1} = \beta_{k+1} a_k + \sum_{i=1}^{k} \beta_i a_{i-1} = \beta_{k+1}(\xi_k - \omega_k) + \omega_k = (1 - \beta_{k+1})\,\omega_k + \beta_{k+1}\xi_k,$$

that the vector $\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_k \\ \omega_{k+1} \end{pmatrix}$ is given by

$$\begin{bmatrix} b_{11} & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} & 0 \\ (1-\beta_{k+1})\,b_{k1} & (1-\beta_{k+1})\,b_{k2} & \cdots & (1-\beta_{k+1})\,b_{kk} & \beta_{k+1} \end{bmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{k-1} \\ \xi_k \end{pmatrix},$$

which has the desired form, thus proving (5.69). The consequence we need from (5.69) is

$$|a_k| = |\xi_k - \omega_k| \le |\xi_k| + |\omega_k| \le \sum_{j=0}^{k} |\xi_j|, \qquad 0 \le k \le \ell. \tag{5.70}$$

We now return our attention to the tree $\mathcal{Y}$. For each $\alpha \in \mathcal{Y}$, with corresponding index $j \in \{j\}_{j=1}^{J}$, there are values $a(\alpha) = a_j$ and $\xi(\alpha) = \xi_j$. Define functions $f(\alpha) = |a(\alpha)|$ and $g(\alpha) = |\xi(\alpha)|$ on the tree $\mathcal{Y}$. Note that we are simply relabelling the indices $\{j\}_{j=1}^{J}$ as $\alpha \in \mathcal{Y}$ to emphasize the tree structure of $\mathcal{Y}$ when convenient. The inequality (5.70) says that

$$f(\alpha) \le Ig(\alpha), \qquad \alpha \in \mathcal{Y}. \tag{5.71}$$

Recall also that we are assuming that the measure $d\mu = \sum_{\alpha \in \mathcal{Y}} \left(\log \frac{1}{1-|z_\alpha|^2}\right)^{1-p}$ is a B_p Carleson measure on the ball $\mathbb{B}_n$, where $z_\alpha = z_j \in \mathbb{B}_n$ if α corresponds to j. The only consequence of this that we need here is the simple condition,

$$\beta(0,\alpha)^{p-1} \sum_{j: z_j \in S(\alpha)} \mu(j) \le C, \qquad \alpha \in \mathcal{T}_n. \tag{5.72}$$

Note that this last inequality refers to the tree $\mathcal{T}_n$ rather than to $\mathcal{Y}$. Using the fact that $\beta(0,\alpha) \approx \log \frac{1}{1-|z_\alpha|^2}$, we obtain from this simple condition that if $S(\alpha) \approx V_{z_k}$, i.e. $\alpha \approx \left[1 - \left(1 - |z_k|^\beta\right)\right] z_k$, then

$$\begin{aligned}
\sum_{\beta: \beta \ge \alpha} \mu(\beta) = \sum_{j: z_j \in S(\alpha)} \mu(j) &\le C\beta(0,\alpha)^{1-p} \\
&\approx C\left(\log \frac{1}{\left(1-|z_k|^2\right)^\beta}\right)^{1-p} \\
&\approx C\left(\log \frac{1}{1-|z_k|^2}\right)^{1-p} = C\mu(\alpha),
\end{aligned}$$

by the definition of the region V_{z_k}. Thus on the tree $\mathcal{Y}$, we have

$$I^*\mu(\alpha) \le C\mu(\alpha), \qquad \alpha \in \mathcal{Y}. \tag{5.73}$$

We now claim that (5.73) implies the inequality

$$\sum_{\alpha \in \mathcal{Y}} Ig(\alpha)^p \mu(\alpha) \le C \sum_{\alpha \in \mathcal{Y}} g(\alpha)^p \mu(\alpha), \tag{5.74}$$

which together with (5.71) yields (5.68). Note that (5.73) is obviously necessary for (5.74).

To see (5.74), we use our more general tree theorem for the tree $\mathcal{Y}$. Recall from Subsection 1.3 on the tree theorem that

$$\sum_{\alpha \in \mathcal{Y}} Ig(\alpha)^p w(\alpha) \le C \sum_{\alpha \in \mathcal{Y}} g(\alpha)^p v(\alpha), \qquad g \ge 0,$$

if and only if

$$\sum_{\beta \ge \alpha} I^*w(\beta)^{p'} v(\beta)^{1-p'} \le CI^*w(\alpha) < \infty, \qquad \alpha \in \mathcal{Y}. \tag{5.75}$$

With $w = v = \mu$, (5.73) yields condition (5.75) as follows:

$$\sum_{\beta \ge \alpha} I^*\mu(\beta)^{p'} \mu(\beta)^{1-p'} \le C \sum_{\beta \ge \alpha} \mu(\beta)^{p'} \mu(\beta)^{1-p'} = C \sum_{\beta \ge \alpha} \mu(\beta) = CI^*\mu(\alpha),$$

and this completes the proof of (5.68) for the sequence $\left\{a_j^m\right\}_{j=1}^J$ given in terms of the data $\left\{\xi_j^m\right\}_{j=1}^J$ by (5.67).

Next, we prove an estimate relating the data $\left\{\xi_j^{m+1}\right\}_{j=1}^J$ at the $(m+1)^{st}$ stage to the coefficients $\left\{a_j^m\right\}_{j=1}^J$ at the m^{th} stage:

$$\left\|\left\{\xi_j^{m+1}\right\}_{j=1}^J\right\|_{\ell^p(d\mu)} \leq C\left\|\mu\right\|^{\min\left\{p'-1,1\right\}}\left\|\left\{a_j^m\right\}_{j=1}^J\right\|_{\ell^p(d\mu)}. \tag{5.76}$$

To see (5.76) we recall from the previous subsubsection that

$$\xi_j^{m+1} = f_m\left(z_j\right) - \xi_j^m = \sum_{i=1}^{\infty} a_i^m \varphi_i\left(z_j\right) - \xi_j^m.$$

We will again drop the superscript m and replace f_m by φ in accordance with the notation used in the proof of Lemma 5.21. We then have from the argument given there that

$$\begin{aligned}
\varphi\left(z_\ell\right) - \xi_\ell &= \sum_{i=1}^{\infty} a_i \varphi_i\left(z_\ell\right) - \xi_\ell \\
&= \sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} a_{P(i)}\left(\varphi_{P(i)}\left(z_\ell\right) - \beta_i\right) + \sum_{i\in\mathcal{F}_\ell} a_i\varphi_i\left(z_\ell\right) \\
&= I_\ell + II_\ell.
\end{aligned}$$

We now claim that

$$\begin{aligned}
\left|I_\ell\right| &\leq C\left\|\mu\right\|^{\frac{1}{p(p-1)}}\left\|\left\{a_j\right\}_{j=1}^J\right\|_{\ell^p(\mu)} \\
\left|II_\ell\right| &\leq C\left\|\mu\right\|^{\frac{1}{p'}}\left\|\left\{a_j\right\}_{j=1}^J\right\|_{\ell^p(\mu)}.
\end{aligned} \tag{5.77}$$

For the term I_ℓ we have from previous arguments

$$\begin{aligned}
\left|I_\ell\right| &= \left|\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} a_{P(i)}\left(\varphi_{P(i)}\left(z_\ell\right) - \beta_i\right)\right| \\
&\leq \sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} \left|a_{P(i)}\right|\left|\varphi_{P(i)}\left(z_\ell\right) - c\left(\gamma_{P(i)}\left(z_i\right)\right)\right| \\
&\leq \sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} \left|a_{P(i)}\right| \\
&\qquad\times\left\{\left|c\left(\gamma_{P(i)}\left(z_\ell\right)\right) - c\left(\gamma_{P(i)}\left(z_i\right)\right)\right| + C\left(\log\frac{1}{1-\left|z_{P(i)}\right|^2}\right)^{-1}\right\} \\
&\leq C\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}} \left|a_{P(i)}\right|\left(\log\frac{1}{1-\left|z_{P(i)}\right|^2}\right)^{-1}
\end{aligned}$$

$$\leq C \left\{ \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \left(\log \frac{1}{1 - \left| z_{P(i)} \right|^2} \right)^{1-p'} \right\}^{\frac{1}{p'}}$$

$$\times \left\{ \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \left| a_{P(i)} \right|^p \left(\log \frac{1}{1 - \left| z_{P(i)} \right|^2} \right)^{1-p} \right\}^{\frac{1}{p}}$$

$$\leq C_\eta \left(\log \frac{1}{1 - |z_{k_0}|^2} \right)^{-\frac{1}{p}} \left\{ \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \left| a_{P(i)} \right|^p \mu \left(z_{P(i)} \right) \right\}^{\frac{1}{p}}$$

$$\leq C \left\| \mu \right\|^{\frac{1}{p(p-1)}} \left\{ \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \left| a_{P(i)} \right|^p \mu \left(z_{P(i)} \right) \right\}^{\frac{1}{p}},$$

which shows the first inequality in (5.77).

To estimate term $II_\ell = \sum_{i \in \mathcal{F}_\ell} a_i \varphi_i (z_\ell)$ in (5.77), we first note that if $z_\ell \notin V_{z_i}$, then

$$|\varphi_i (z_\ell)| \leq C \left(1 - |z_i|^2 \right)^\sigma, \qquad \sigma > 0, \tag{5.78}$$

by Remark 5.19. On the other hand, if $z_\ell \in V_{z_i}$, then the corresponding argument in the previous subsubsection shows that Remark 5.19 again yields (5.78). From the obvious inequality

$$\left(1 - |z_i|^2 \right)^\sigma \leq C_{\sigma,p} \left(\log \frac{1}{1 - |z_i|^2} \right)^{1-p} = C \mu (z_i),$$

we thus obtain

$$|II_\ell| = \left| \sum_{i \in \mathcal{F}_\ell} a_i \varphi_i (z_\ell) \right| \leq \sum_{i \in \mathcal{F}_\ell} |a_i| \mu (z_i) \leq C \left\| \mu \right\|^{\frac{1}{p'}} \left\{ \sum_{i \in \mathcal{F}_\ell} |a_i|^p \mu (z_i) \right\}^{\frac{1}{p}},$$

which shows the second inequality in (5.77). Using (5.77) and (5.68) we then have

$$\begin{aligned} \left\| \left\{ \xi_j^{m+1} \right\}_{j=1}^J \right\|_{\ell^p(d\mu)} &\leq C \left\| \left\{ \left| I_j^m \right| + \left| II_j^m \right| \right\}_{j=1}^J \right\|_{\ell^p(d\mu)} \\ &= C \left\{ \sum_{j=1}^J \left(\left| I_j^m \right| + \left| II_j^m \right| \right)^p \mu (z_i) \right\}^{\frac{1}{p}} \\ &\leq C \left(\left\| \mu \right\|^{\frac{1}{p(p-1)}} + \left\| \mu \right\|^{\frac{1}{p'}} \right) \left\| \{a_j\}_{j=1}^J \right\|_{\ell^p(\mu)} \left\| \mu \right\|^{\frac{1}{p}}, \end{aligned}$$

which is (5.76).

Finally, we alternately iterate (5.76) and (5.68) to obtain

$$\left\| \left\{ \xi_j^m \right\}_{j=1}^J \right\|_{\ell^p(d\mu)} \leq \left(C \left\| \mu \right\|^{\min\left\{ p'-1, 1 \right\}} \right)^m \left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^p(d\mu)},$$

and then using $\{a_j\}_{j=1}^J = \sum_{m=1}^\infty \left\{a_j^m\right\}_{j=1}^J$ with Minkowski's inequality and (5.68) once more, we have

$$\begin{aligned}\left\|\{a_j\}_{j=1}^J\right\|_{\ell^p(d\mu)} &\leq \sum_{m=1}^\infty \left\|\{a_j^m\}_{j=1}^J\right\|_{\ell^p(d\mu)} \\ &\leq \sum_{m=1}^\infty C \sum_{m=1}^\infty \left\|\{\xi_j^m\}_{j=1}^J\right\|_{\ell^p(d\mu)} \\ &\leq C\left(\sum_{m=1}^\infty \left(C\,\|\mu\|^{\min\{p'-1,1\}}\right)^m\right)\left\|\{\xi_j\}_{j=1}^J\right\|_{\ell^p(d\mu)}.\end{aligned}$$

If $\varepsilon > 0$ is chosen small enough in (5.64), the series above converges and yields (5.65). This completes the proof of Lemma 5.26, and thus also finishes the proof that (5.10) implies (5.3) for all $1 < p < \infty$.

REMARK 5.27. Note that the above proof shows that the map T in (5.3) is *onto*, provided both (5.72) and the separation condition in (5.5) hold. While the separation condition is necessary for the map T in (5.3) to be *onto*, we do not know a characterization of when the map T in (5.3) is *onto*.

5.4. The plan for completing the proof. At this point Theorem 5.1 has been completely proved, and the only assertion remaining to be proved in Theorem 5.2 is that (5.6) implies (5.8) in the range $1 + \frac{1}{n-1} \leq p < 2$. This latter will be accomplished by introducing analogues $HB_p(\mathcal{T}_n)$ and $M_{HB_p(\mathcal{T}_n)}$ on the Bergman tree $\mathcal{T}_n$ of the Besov and multiplier spaces $B_p(\mathbb{B}_n)$ and $M_{B_p(\mathbb{B}_n)}$ on the ball, that behave in some ways similar to martingales. This will require the introduction of an elaborate complex structure on $\mathcal{T}_n$ so that three important properties hold for the new spaces:

1. The restriction map is bounded from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_p(\mathcal{T}_n)}$.
2. The reproducing kernels for $HB_p(\mathcal{T}_n)$ satisfy a positivity property analogous to $\operatorname{Re}\frac{1-|w|^2}{1-\overline{z}\cdot w} > 0$ for $z, w \in \mathbb{B}_n$.
3. Carleson measures for $HB_p(\mathcal{T}_n)$ are characterized by the tree condition (3.2).

In order to define spaces $HB_p(\mathcal{T}_n)$ with these properties, we first introduce a more complex-geometric norm on $B_p(\mathbb{B}_n)$ that will serve as a model for that of $HB_p(\mathcal{T}_n)$, and permit natural localized norm estimates to be made between $HB_p(\mathcal{T}_n)$ and $B_p(\mathbb{B}_n)$. This is accomplished in the next section.

6. An almost invariant holomorphic derivative

We now construct a differential operator D_a on each Bergman kube K_α that is close there to the invariant gradient $\widetilde{\nabla}$, and which has the additional property that $D_a^m f(z)$ is holomorphic for $m \geq 1$ and $z \in K_\alpha$ when f is. For our purposes the powers $D_a^m f$, $m \geq 1$, are easier to work with than the corresponding powers $\widetilde{\nabla}^m f$, which fail to be holomorphic. We will show that D_a^m can be used to define an equivalent norm on the Besov space $B_p(\mathbb{B}_n)$, and then establish an oscillation inequality for $B_p(\mathbb{B}_n)$. We then use this oscillation inequality in the subsequent sections to show that the restriction map

$$Tf = \{f(\alpha)\}_{\alpha\in\mathcal{T}_n}$$

is bounded from $B_p(\mathbb{B}_n)$ to $HB_p(\mathcal{T}_n)$, and, provided $p < 2 + \frac{1}{n-1}$, from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_p(\mathcal{T}_n)}$ as well (see Section 8 for $HB_p(\mathcal{T}_n)$). This latter result will require a choice of the structural constant θ sufficiently large in the construction of the Bergman tree $\mathcal{T}_n$. We will also establish a positivity property for reproducing kernels of $HB_p(\mathcal{T}_n)$ which will further require a choice of λ sufficiently small in the construction of $\mathcal{T}_n$. This restriction theorem and positivity property will enable us to complete the proof of Theorem 5.2 by showing that (5.6) implies (5.8).

Fix $\alpha \in \mathcal{T}_n$ and let $a = c_\alpha$. Recall that the gradient with invariant length given by

$$\begin{aligned}\widetilde{\nabla} f(a) &= (f \circ \varphi_a)'(0) = f'(a)\varphi_a'(0) \\ &= -f'(a)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\}\end{aligned}$$

fails to be holomorphic in a. To rectify this, we define

$$\begin{aligned}D_a f(z) &= f'(z)\varphi_a'(0) \\ &= -f'(z)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\},\end{aligned} \tag{6.1}$$

for $z \in \mathbb{B}_n$. Note that $\nabla_z(\overline{a}\cdot z) = \overline{a}^t$ when we view $w \in \mathbb{B}_n$ as an $n \times 1$ complex matrix, and denote by w^t the $1 \times n$ transpose of w. With this interpretation, we observe that $P_a z = \frac{\overline{a}\cdot z}{|a|^2} a$ has derivative $P_a = P_a' z = \frac{a\overline{a}^t}{|a|^2} = |a|^{-2}\left[a_i\overline{a_j}\right]_{1\le i,j\le n}$.

Example 6.1. *For $n = 2$ we can calculate the differential operator D_a for $a \neq 0$ more explicitly using the basis $\left\{a, a^\perp\right\}$ of $\mathbb{C}^2$ where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $a^\perp = \begin{pmatrix} -\overline{a_2} \\ \overline{a_1} \end{pmatrix}$. For $w = \mu a + \nu a^\perp$, we compute the action of the linear operator $D_a f(z)$ on w as*

$$\begin{aligned}-D_a f(z) w &= f'(z)\left\{\left(1-|a|^2\right)\mu a + \left(1-|a|^2\right)^{\frac{1}{2}}\nu a^\perp\right\} \\ &= \mu\left(1-|a|^2\right)\left(a_1\frac{\partial}{\partial z_1}f(z) + a_2\frac{\partial}{\partial z_2}f(z)\right) \\ &\qquad + \nu\left(1-|a|^2\right)^{\frac{1}{2}}\left(-\overline{a_2}\frac{\partial}{\partial z_1}f(z) + \overline{a_1}\frac{\partial}{\partial z_2}f(z)\right).\end{aligned}$$

Thus in the basis $\left\{a, a^\perp\right\}$, we have

$$-D_a = \left(\left(1-|a|^2\right)\left(a_1\frac{\partial}{\partial z_1} + a_2\frac{\partial}{\partial z_2}\right), \left(1-|a|^2\right)^{\frac{1}{2}}\left(-\overline{a_2}\frac{\partial}{\partial z_1} + \overline{a_1}\frac{\partial}{\partial z_2}\right)\right),$$

where at the point a, $\left(a_1\frac{\partial}{\partial z_1} + a_2\frac{\partial}{\partial z_2}\right)f(a) = f'(a)a$ is the complex radial derivative of f at a, and $\left(-\overline{a_2}\frac{\partial}{\partial z_1} + \overline{a_1}\frac{\partial}{\partial z_2}\right)f(a) = f'(a)a^\perp$ is the complex tangential derivative of f at a.

The next lemma shows that D_a^m and D_b^m are comparable when a and b are close in the Bergman metric.

LEMMA 6.2. *Let $a, b \in \mathbb{B}_n$ satisfy $\beta(a,b) \le C$. There is a positive constant C_m depending only on C and m such that*

$$C_m^{-1} \left| D_b^m f(z) \right| \le \left| D_a^m f(z) \right| \le C_m \left| D_b^m f(z) \right|,$$

for all $f \in H(\mathbb{B}_n)$.

PROOF. The equivalence is easy if $a = 0$, so we may assume that $|a|, |b| \ge \frac{1}{2}$. We first note that $b \in B_d(a, C)$ implies the estimates

$$\begin{aligned} \left| P_a(a-b) \right| + \left| Q_a(a-b) \right|^2 &\le C\left(1 - |a|^2\right), \\ \left(1 - |a|^2\right) &\approx \left(1 - |b|^2\right), \end{aligned}$$

and since $Q_a a = 0$, this yields

$$|Q_a b| = |Q_a(a-b)| \le C\left(1 - |a|^2\right)^{\frac{1}{2}},$$

and by symmetry,

$$|Q_b a| \le C\left(1 - |b|^2\right)^{\frac{1}{2}} \approx \left(1 - |a|^2\right)^{\frac{1}{2}}.$$

We can now estimate the modulus of $\varphi_b'(0)\, a$ by

$$\begin{aligned} \left| \varphi_b'(0)\, a \right| &= \left| \left(1 - |b|^2\right) P_b a + \left(1 - |b|^2\right)^{\frac{1}{2}} Q_b a \right| \\ &\le C\left(1 - |a|^2\right) + C\left(1 - |a|^2\right)^{\frac{1}{2}} |Q_b a| \\ &\le C\left(1 - |a|^2\right). \end{aligned}$$

Thus we may write

$$\varphi_b'(0)\, a = \left(1 - |a|^2\right)\left(\sigma a + \tau v^{\perp}\right), \qquad |\sigma|, |\tau| \le C, \tag{6.2}$$

where $v^{\perp}$ is a unit vector in the orthogonal complement $(\mathbb{C}a)^{\perp}$ of $\mathbb{C}a$.

Next we show that if $a^{\perp}$ is any unit vector in $(\mathbb{C}a)^{\perp}$, then

$$\begin{aligned} \varphi_b'(0)\, a^{\perp} &= \left(1 - |b|^2\right) P_b a^{\perp} + \left(1 - |b|^2\right)^{\frac{1}{2}} Q_b a^{\perp} \\ &= \left(1 - |a|^2\right) \mu a + \left(1 - |a|^2\right)^{\frac{1}{2}} \nu w^{\perp}, \qquad |\mu|, |\nu| \le C, \end{aligned} \tag{6.3}$$

where $w^{\perp}$ is another unit vector in $(\mathbb{C}a)^{\perp}$. To see this we use

$$P_b a^{\perp} = P_a P_b a^{\perp} + Q_a P_b a^{\perp}$$

and

$$\begin{aligned} Q_b a^{\perp} &= a^{\perp} - P_b a^{\perp} = a^{\perp} - \frac{\overline{b} \cdot a^{\perp}}{|b|^2} b \\ &= a^{\perp} - \frac{\overline{b} \cdot a^{\perp}}{|b|^2} \left(P_a b + Q_a b\right) \end{aligned}$$

to write

$$\varphi_b'(0)\,a^{\perp} = \left(1-|b|^2\right)\left(P_aP_ba^{\perp}+Q_aP_ba^{\perp}\right) + \left(1-|b|^2\right)^{\frac{1}{2}}\left(a^{\perp}-\frac{\overline{b}\cdot a^{\perp}}{|b|^2}P_ab-\frac{\overline{b}\cdot a^{\perp}}{|b|^2}Q_ab\right),$$

so that for (6.3) to hold we must have

$$\left(1-|a|^2\right)\mu a = \left(1-|b|^2\right)P_aP_ba^{\perp} - \left(1-|b|^2\right)^{\frac{1}{2}}\frac{\overline{b}\cdot a^{\perp}}{|b|^2}P_ab$$

and

$$\left(1-|a|^2\right)^{\frac{1}{2}}\nu w^{\perp} = \left(1-|b|^2\right)Q_aP_ba^{\perp} + \left(1-|b|^2\right)^{\frac{1}{2}}\left(a^{\perp}-\frac{\overline{b}\cdot a^{\perp}}{|b|^2}Q_ab\right).$$

Since P_a and Q_a are orthogonal projections onto $\mathbb{C}a$ and $(\mathbb{C}a)^{\perp}$ respectively, we see that the representation (6.3) holds with the stated bounds upon using

$$\left|\overline{b}\cdot a^{\perp}\right| = \left|\overline{(b-a)}\cdot a^{\perp}\right| \leq C\,|b-a| \leq C\left(1-|a|^2\right)^{\frac{1}{2}}.$$

From (6.2) and (6.3) above we obtain in particular that for every unit vector $v\in\mathbb{C}^n$, there is a unit vector $v^{\perp}\in(\mathbb{C}a)^{\perp}$ and bounded scalars μ,ν such that

$$\varphi_b'(0)\,v = \left(1-|a|^2\right)\mu a + \left(1-|a|^2\right)^{\frac{1}{2}}\nu v^{\perp} \tag{6.4}$$

(simply write v as a linear combination of a and a vector perpendicular to a and use (6.2) and (6.3)).

Now suppose that $m=1$. Then for any unit vector $v\in\mathbb{C}^n$, we have from (6.4) that

$$\begin{aligned}|D_bf(z)\,v| &= |f'(z)\,\varphi_b'(0)\,v|\\ &= \left|\left(1-|a|^2\right)\mu f'(z)\,a + \left(1-|a|^2\right)^{\frac{1}{2}}\nu f'(z)\,v^{\perp}\right|\\ &\leq C\left(1-|a|^2\right)|f'(z)\,a| + C\left(1-|a|^2\right)^{\frac{1}{2}}\left|f'(z)\,v^{\perp}\right|\\ &\leq C\,|D_af(z)|,\end{aligned}$$

since

$$\begin{aligned}|D_af(z)| &= \left|f'(z)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}}Q_a\right\}\right|\\ &\approx \sup_{|v|\leq 1}\left|f'(z)\left\{\left(1-|a|^2\right)P_av + \left(1-|a|^2\right)^{\frac{1}{2}}Q_av\right\}\right|\\ &\approx \left(1-|a|^2\right)|f'(z)\,P_a| + \left(1-|a|^2\right)^{\frac{1}{2}}|f'(z)\,Q_a|.\end{aligned}$$

Now suppose that $m=2$. Then viewing $D_a^2 f(z)$ as a symmetric bilinear form, we have that

$$\left|D_a^2 f(z)\right| \approx \left(1-|a|^2\right)^2 |P_a f''(z) P_a| + \left(1-|a|^2\right)^{\frac{3}{2}} \left(|P_a f''(z) Q_a| + |Q_a f''(z) P_a|\right) + \left(1-|a|^2\right) |Q_a f''(z) Q_a| . \tag{6.5}$$

Indeed, write $\widetilde{P_a} = \left(1-|a|^2\right) P_a$ and $\widetilde{Q_a} = \left(1-|a|^2\right)^{\frac{1}{2}} Q_a$, and note that the set of vectors

$$\mathcal{V} = \{\sigma P_a v + \tau Q_a w : \sigma, \tau \in \mathbb{C}; v, w \in \mathbb{C}^n; |\sigma|, |\tau|, |v|, |w| \leq 1\}$$

satisfies

$$\{V \in \mathbb{C}^n : |V| \leq 1\} \subset \mathcal{V} \subset \{V \in \mathbb{C}^n : |V| \leq 2\}.$$

Then, with $V_i = \sigma_i P_a v_i + \tau_i Q_a w_i$ for $i = 1, 2$, we have

$$\widetilde{P_a} V_i + \widetilde{Q_a} V_i = \sigma_i \widetilde{P_a} v_i + \tau_i \widetilde{Q_a} w_i, \qquad i = 1, 2$$

and so $\left|D_a^2 f(z)\right|$ is approximately

$$\begin{aligned}
&\sup_{V_1, V_2 \in \mathcal{V}} \left|\left\{\widetilde{P_a} V_1 + \widetilde{Q_a} V_1\right\} f''(z) \left\{\widetilde{P_a} V_2 + \widetilde{Q_a} V_2\right\}\right| \\
&= \sup_{\substack{|\sigma_i|, |\tau_i| \leq 1 \\ |v_i|, |w_i| \leq 1}} \left|\left\{\sigma_1 \widetilde{P_a} v_1 + \tau_1 \widetilde{Q_a} w_1\right\} f''(z) \left\{\sigma_2 \widetilde{P_a} v_2 + \tau_2 \widetilde{Q_a} w_2\right\}\right| \\
&= \sup_{\substack{|\sigma_i|, |\tau_i| \leq 1 \\ |v_i|, |w_i| \leq 1}} \left|(\sigma_1, \tau_1) \begin{bmatrix} \widetilde{P_a} v_1 f''(z) \widetilde{P_a} v_2 & \widetilde{P_a} v_1 f''(z) \widetilde{Q_a} w_2 \\ \widetilde{Q_a} w_1 f''(z) \widetilde{P_a} v_2 & \widetilde{Q_a} w_1 f''(z) \widetilde{Q_a} w_2 \end{bmatrix} \begin{pmatrix} \sigma_2 \\ \tau_2 \end{pmatrix}\right| \\
&\approx \sup_{|v_i|, |w_i| \leq 1} \left\{ \begin{array}{c} \left|\widetilde{P_a} v_1 f''(z) \widetilde{P_a} v_2\right| + \left|\widetilde{P_a} v_1 f''(z) \widetilde{Q_a} w_2\right| \\ + \left|\widetilde{Q_a} w_1 f''(z) \widetilde{P_a} v_2\right| + \left|\widetilde{Q_a} w_1 f''(z) \widetilde{Q_a} w_2\right| \end{array} \right\},
\end{aligned}$$

which yields (6.5). Since $v^t D_b^2 f(z) w = (\varphi_b'(0) v)^t f''(z) \varphi_b'(0) w$, we must now show that the expression

$$\left|\left[\left(1-|a|^2\right) \sigma a + \left(1-|a|^2\right)^{\frac{1}{2}} \tau v^\perp\right]^t f''(z) \left[\left(1-|a|^2\right) \mu a + \left(1-|a|^2\right)^{\frac{1}{2}} \nu w^\perp\right]\right| \tag{6.6}$$

is at most $\left|D_a^2 f(z)\right|$. However, the expression (6.6) is dominated by

$$C \left(1-|a|^2\right)^2 |P_a f''(z) P_a| + C \left(1-|a|^2\right)^{\frac{3}{2}} |P_a f''(z) Q_a| + C \left(1-|a|^2\right)^{\frac{3}{2}} |Q_a f''(z) P_a| + C \left(1-|a|^2\right) |Q_a f''(z) Q_a|,$$

which is at most $\left|D_a^2 f(z)\right|$ by (6.5). This completes the proof of Lemma 6.2 in cases $m = 1, 2$.

The case $m > 2$ is handled similarly viewing $D_b^m f(z)$ as a symmetric m-linear form on $\mathbb{C}^n \times ... \times \mathbb{C}^n$ (m times). Briefly, the argument is this:

$$\begin{aligned}\left|D_b^m f(z)\{v_j\}_{j=1}^m\right| &\leq \left|f^{(m)}(z)\{\varphi_b'(0)v_j\}_{j=1}^m\right| \\ &\leq \left|f^{(m)}(z)\left\{\left(1-|a|^2\right)\mu_j a + \left(1-|a|^2\right)^{\frac{1}{2}}\nu_j w_j^{\perp}\right\}_{j=1}^m\right| \\ &\leq C\sum_{\ell=0}^m \left(1-|a|^2\right)^{m-\ell}\left(1-|a|^2\right)^{\frac{\ell}{2}}\left|f^{(m)}(z)\{u_j\}_{j=1}^m\right|,\end{aligned}$$

where in the summand corresponding to the index ℓ, $m-\ell$ of the u_j are equal to a and the remaining u_j are equal to $w_j^{\perp}$. Each summand is dominated by $|D_a^m f(z)|$, thereby completing the proof of Lemma 6.2.

Definition 6.3. *Suppose $1 < p < \infty$ and $m \geq 1$. We define a "tree semi-norm" $\|\cdot\|_{B_{p,m}}^*$ by*

$$\|f\|_{B_{p,m}}^* = \left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_d(c_\alpha,C_2)}\left|D_{c_\alpha}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}. \tag{6.7}$$

6.1. Equivalence of semi-norms. The semi-norms $\|\cdot\|_{B_{p,m}}^*$ turn out to be independent of $m > \frac{2n}{p}$. We will obtain this fact as a corollary of the equivalence of the norm in (2.9) with the corresponding "radial" derivative norm $\mathcal{R}_m$ as given in Proposition 2.1. Note that the restriction $m > \frac{2n}{p}$ is dictated by the fact that $\left|D_{c_\alpha}^m f(z)\right|$ involves the factor $\left(1-|z|^2\right)^{\frac{m}{2}}$ times m^{th} order tangential derivatives of f, and so we must have that $\left(1-|z|^2\right)^{\frac{m}{2}p} d\lambda_n(z)$ is a finite measure, i.e. $\frac{m}{2}p - n - 1 > -1$.

Lemma 6.4. *Let $1 < p < \infty$ and $m > \frac{2n}{p}$. Then*

(6.8)

$$\begin{aligned}\|f\|_{B_{p,m}}^* + \sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| &\equiv \left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_d(c_\alpha,C_2)}\left|D_{c_\alpha}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} + \sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\ &\approx \left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m \mathcal{R}_m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}.\end{aligned}$$

Proof. We have

$$|D_a f(z)| = \left|f'(z)\left\{\left(1-\left|a^2\right|\right)P_a + \left(1-\left|a^2\right|\right)^{\frac{1}{2}}Q_a\right\}\right| \geq \left|\left(1-\left|a^2\right|\right)f'(z)\right|, \tag{6.9}$$

and iterating with f replaced by (the components of) $D_a f$ in (6.9), we obtain

$$\left|D_a^2 f(z)\right| \geq \left|\left(1-\left|a^2\right|\right)(D_a f)'(z)\right|.$$

Applying (6.9) once more with f replaced by (the components of) f', we get

$$\left|\left(1-\left|a^2\right|\right)(D_a f)'(z)\right| = \left|\left(1-\left|a^2\right|\right)D_a(f')(z)\right| \geq \left|\left(1-\left|a^2\right|\right)^2 f''(z)\right|,$$

which when combined with the previous inequality yields

$$\left|D_a^2 f(z)\right| \geq \left|\left(1-\left|a^2\right|\right)^2 f''(z)\right|.$$

Continuing by induction we have

$$|D_a^m f(z)| \geq \left|\left(1-\left|a^2\right|\right)^m f^{(m)}(z)\right|, \quad m \geq 1. \tag{6.10}$$

Proposition 2.1 and (6.10) now show that

$$\begin{aligned}
&\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m \mathcal{R}_m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \\
&\leq C\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m f^{(m)}(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_d(c_\alpha,C_2)}\left|\left(1-|z|^2\right)^m f^{(m)}(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_d(c_\alpha,C_2)}\left|\left(1-|c_\alpha|^2\right)^m f^{(m)}(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_d(c_\alpha,C_2)}\left|D_a^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&= C\,\|f\|_{B_{p,m}}^{*}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right|.
\end{aligned}$$

For the opposite inequality, we employ some of the ideas in the proofs of Theorem 6.11 and Lemma 3.3 in [**Zhu**], where the case $m=1>\frac{2n}{p}$ is proved. Suppose $f\in H(\Omega)$ and that the right side of (6.8) is finite. By Proposition 2.1 and Theorem 6.7 of [**Zhu**] we have

$$f(z)=\frac{n!}{\pi^n}\int_{\mathbb{B}_n}\frac{g(w)}{(1-\overline{w}\cdot z)^{n+1}}dw, \quad z\in\mathbb{B}_n,$$

for some $g\in L^p(d\lambda_n)$ where

$$\|g\|_{L^p(d\lambda_n)} \approx \sum_{j=0}^{m-1}\left|\nabla^j f(0)\right|+\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m \mathcal{R}_m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}. \tag{6.11}$$

Fix $\alpha\in\mathcal{T}_n$ and let $a=c_\alpha\in\mathbb{B}_n$. We claim that
(6.12)

$$|D_a^m f(z)| \leq C_m\left(1-|a|^2\right)^{\frac{m}{2}}\int_{\mathbb{B}_n}\frac{|g(w)|}{|1-\overline{w}\cdot z|^{n+1+\frac{m}{2}}}dw, \quad m\geq 1, z\in B_d(a,C).$$

We now compute $D_a^m f(z)$ for $z\in B_d(a,C)$, beginning with the case $m=1$. Since

$$\begin{aligned}
D_a(\overline{w}\cdot z) &= (\overline{w}\cdot z)'\,\varphi_a'(0) = -\overline{w}^t\left\{\left(1-\left|a^2\right|\right)P_a+\left(1-\left|a^2\right|\right)^{\frac{1}{2}}Q_a\right\} \\
&= -\overline{\left\{(1-|a^2|)\,P_a w+(1-|a^2|)^{\frac{1}{2}}\,Q_a w\right\}}^{t},
\end{aligned}$$

we have

(6.13)

$$\begin{aligned}
&D_a f(z) \\
&= \frac{n!}{\pi^n} \int_{\mathbb{B}_n} D_a (1 - \overline{w} \cdot z)^{-(n+1)} g(w)\, dw \\
&= \frac{(n+1)!}{\pi^n} \int_{\mathbb{B}_n} (1 - \overline{w} \cdot z)^{-(n+2)} D_a (\overline{w} \cdot z) g(w)\, dw \\
&= -\frac{(n+1)!}{\pi^n} \int_{\mathbb{B}_n} (1 - \overline{w} \cdot z)^{-(n+2)} \overline{\left\{ (1 - |a^2|) P_a w + (1 - |a^2|)^{\frac{1}{2}} Q_a w \right\}}^t g(w)\, dw.
\end{aligned}$$

Taking absolute values inside, we obtain

$$|D_a f(z)| \leq C \left(1 - \left|a^2\right|\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{\left(1 - \left|a^2\right|\right)^{\frac{1}{2}} |P_a w| + |Q_a w|}{|1 - \overline{w} \cdot z|^{n+2}} |g(w)|\, dw. \tag{6.14}$$

From the following elementary inequalities

$$\begin{aligned}
|Q_a w| &= \left| w - \frac{w \cdot \overline{a}}{|a|^2} a \right| \leq |w - a| + |a|^{-1} |1 - w \cdot \overline{a}|, \\
|w - a|^2 &= |w|^2 + |a|^2 - 2 \operatorname{Re}(w \cdot \overline{a}) \\
&\leq \left(1 - |w|^2\right) + \left(1 - |a|^2\right) + |1 - w \cdot \overline{a}| \\
&\leq C |1 - w \cdot \overline{a}|,
\end{aligned}$$

we obtain that $|Q_a w| \leq C |1 - \overline{w} \cdot a|^{\frac{1}{2}}$. Using $\left(1 - \left|a^2\right|\right) + |1 - \overline{w} \cdot a| \leq C |1 - \overline{w} \cdot z|$ for $z \in B_d(a, C)$, we now see that

$$\frac{\left(1 - \left|a^2\right|\right)^{\frac{1}{2}} |P_a w| + |Q_a w|}{|1 - \overline{w} \cdot z|^{n+2}} \leq \frac{C}{|1 - \overline{w} \cdot z|^{n+\frac{3}{2}}}, \qquad z \in B_d(a, C).$$

Plugging this estimate into (6.14) yields

$$|D_a f(z)| \leq C \left(1 - \left|a^2\right|\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{|g(w)|}{|1 - \overline{w} \cdot z|^{n+\frac{3}{2}}} dw,$$

which is the case $m = 1$ of (6.12).

To obtain the case $m = 2$ of (6.12), we differentiate (6.13) again to get

$$D_a^2 f(z) = -\frac{(n+2)!}{\pi^n} \int_{\mathbb{B}_n} (1 - \overline{w} \cdot z)^{-(n+3)} W \overline{W}^t g(w)\, dw.$$

where we have written $W = \left\{ \left(1 - \left|a^2\right|\right) P_a w + \left(1 - \left|a^2\right|\right)^{\frac{1}{2}} Q_a w \right\}$ for convenience. Again taking absolute values inside, we obtain

$$\left|D_a^2 f(z)\right| \leq C \left(1 - \left|a^2\right|\right) \int_{\mathbb{B}_n} \frac{\left(\left(1 - \left|a^2\right|\right)^{\frac{1}{2}} |P_a w| + |Q_a w|\right)^2}{|1 - \overline{w} \cdot z|^{n+3}} |g(w)|\, dw.$$

Once again, using $|Q_a w| \leq C |1-\overline{w}\cdot a|^{\frac{1}{2}}$ and $\left(1-\left|a^2\right|\right)+|1-\overline{w}\cdot a| \leq C|1-\overline{w}\cdot z|$ for $z\in B_d(a,C)$, we see that

$$\frac{\left(\left(1-\left|a^2\right|\right)^{\frac{1}{2}}|P_a w|+|Q_a w|\right)^2}{|1-\overline{w}\cdot z|^{n+3}} \leq \frac{C}{|1-\overline{w}\cdot z|^{n+2}}, \qquad z\in B_d(a,C),$$

which yields the case $m=2$ of (6.12). The general case of (6.12) follows by induction on m.

Now with $a=\frac{m}{2}$, $b=0$ and $t=-n-1$, our assumption that $m>\frac{2n}{p}$ yields

$$-pa<t+1<p(b+1).$$

The inequality (6.12) shows that $|D_a^m f(z)| \leq C_m S|g|(z)$ for $z\in B_d(a,C)$, where

$$Sg(z)=\int_{\mathbb{B}_n}\frac{\left(1-|a|^2\right)^{\frac{m}{2}}}{|1-\overline{w}\cdot z|^{n+1+\frac{m}{2}}}g(w)\,dw$$

is the operator in Theorem 2.10 of [**Zhu**] with parameters $a=\frac{m}{2}$ and $b=0$. Thus the finite overlap property of the balls $B_d(c_\alpha,C_2)$ together with Theorem 2.10 of [**Zhu**] yield

$$\begin{aligned}\|f\|_{B_{p,m}}^{*} &= \left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_d(c_\alpha,C_2)}\left|D_{c_\alpha}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\ &\leq C_m\left(\int_{\mathbb{B}_n}|Sg(z)|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\ &\leq C_m'\left(\int_{\mathbb{B}_n}|g(z)|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\ &\leq C_m''\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^m\mathcal{R}_m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\end{aligned}$$

by (6.11). This completes the proof of Lemma 6.4.

6.2. The oscillation inequality. We will obtain an oscillation inequality for Besov spaces by transporting Taylor polynomials at the origin to the kubes K_α using the affine map $\widetilde{\varphi_a}(z)=a+\varphi_a'(0)z$, and its inverse $\widetilde{\varphi_a}^{-1}(z)=\varphi_a'(a)(z-a)$ with $a=c_\alpha$. Inequality (6.15) below is due to Peloso [**Pel**].

PROPOSITION 6.5. *For $1<p<\infty$ and $m\geq 1$, we have the Taylor polynomial oscillation inequality*

$$\left\{\sum_{\alpha\in\mathcal{T}_n}\left(\sup_{z\in K_\alpha}\left|f(z)-\sum_{k=0}^{m-1}\frac{((z-a_\alpha)\cdot')^k f(a_\alpha)}{k!}\right|\right)^p\right\}^{\frac{1}{p}} \leq C\|f\|_{B_{p,m}^{*}}, \tag{6.15}$$

for every sequence of points $\{a_\alpha\}_{\alpha\in\mathcal{T}_n}$ with $a_\alpha\in K_\alpha$, $\alpha\in\mathcal{T}_n$. We also have the local estimate

$$\sup_{\xi\in K_\alpha}\left|f(\xi)-\sum_{k=0}^{m-1}\frac{((\xi-a_\alpha)\cdot')^k f(a_\alpha)}{k!}\right| \leq C\left(\int_{K_\alpha^*}\left|D_{c_\alpha}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}. \tag{6.16}$$

Proof. Denote by K_α^* the union of the Carleson box K_α and its neighbours at most M boxes away. From part 1 of Lemma 2.8, and provided we choose M sufficiently large, we obtain that there are constants A_1, A_2 and A_2^* in $[0,1)$ with $1-A_2^*=\frac{1}{2}(1-A_2)$ depending only on n such that

$$\begin{aligned} &B(c_\alpha,A_1)\subset K_\alpha\subset B(c_\alpha,A_2),\\ &B(c_\alpha,A_2^*)\subset K_\alpha^*, \end{aligned}$$

for $\alpha\in\mathcal{T}_n$. We first establish the estimate

$$\sup_{z\in B(0,A_2)}|f(z)|\le C_{m,n,p}\left(\int_{K_0^*}|D_0^m f(z)|^p\,d\lambda_n(z)\right)^{\frac{1}{p}},\tag{6.17}$$

for all $f\in B_p$ satisfying the initial conditions

$$\nabla^k f(0)=0,\qquad 0\le k<m.\tag{6.18}$$

We have by standard methods, for $|z|<A_2$ and $f\in B_p$ satisfying (6.18),

$$\begin{aligned} |f(z)| &\le C\int_{B\left(0,A_2^*\right)}|\nabla^m f(w)|\,dw\\ &\le C\int_{K_0^*}|\nabla^m f(w)|\,dw\\ &\le C\left(\int_{K_0^*}|\nabla^m f(\xi)|^p\,d\lambda_n(\xi)\right)^{\frac{1}{p}}, \end{aligned}$$

which establishes (6.17) since $D_0^m=\nabla^m$.

Using the affine maps $\widetilde{\varphi_a}$, we now show that (6.17) implies the estimate (6.16). Indeed, let $a=c_\alpha$. Since $\widetilde{\varphi_a}(B(0,A_2))\supset K_\alpha$ for A_2 large enough, we will apply (6.17) to the function

$$g(z)=(f\circ\widetilde{\varphi_a})(z)-\sum_{k=0}^{m-1}(z\cdot\nabla)^k(f\circ\widetilde{\varphi_a})(0),$$

which satisfies (6.18) with g in place of f. We also restrict the supremum over z in (6.17) to $\widetilde{\varphi_a}^{-1}(K_\alpha)$. We compute that

$$(w\cdot\nabla)(f\circ\widetilde{\varphi_a})(0)=f'(a)\,\varphi_a'(0)\,w=D_a f(a)\,w\equiv(w\cdot D_a)\,f(a),$$

and more generally

$$(w\cdot\nabla)^k(f\circ\widetilde{\varphi_a})(0)=(w\cdot D_a)^k f(a),\qquad 0\le k\le m-1, w\in\mathbb{C}^n,$$

so that

$$g(z)=f(\widetilde{\varphi_a}(z))-\sum_{k=0}^{m-1}\frac{(z\cdot D_a)^k f(a)}{k!}.$$

Now using $\widetilde{\varphi_a}^{-1}(\xi)=\varphi_a'(a)(\xi-a)$ and

$$\begin{aligned} (\varphi_a'(a)(\xi-a)\cdot D_a)\,f(a) &= f'(a)\,\varphi_a'(0)\,\varphi_a'(a)(\xi-a)\\ &= f'(a)(\xi-a)\\ &= ((\xi-a)\cdot{}')\,f(a), \end{aligned}$$

we get from (6.17), with g in place of f, that

$$\begin{aligned}
&\max_{\xi\in K_\alpha}\left|f(\xi)-\sum_{k=0}^{m-1}\frac{((\xi-a)\cdot')^k f(a)}{k!}\right|^p\\
&=\max_{z\in K_0}|g(z)|^p\\
&\leq C\int_{K_0^*}|D_0^m g(z)|^p\,d\lambda_n(z).
\end{aligned}$$

Noting that $D_0^m g = D_0^m(f\circ\widetilde{\varphi_a}) = (D_a^m f)\circ\widetilde{\varphi_a}$ and $d\lambda_n(\xi)\approx d\lambda_n(z)$, we have

$$\int_{K_0^*}|D_0^m(f\circ\widetilde{\varphi_a})(\xi)|^p\,d\lambda_n(\xi)\approx\int_{K_\alpha^*}|D_a^m f(z)|^p\,d\lambda_n(z),$$

which yields the first inequality (6.16) since $a=c_\alpha$.

Now we apply (6.16) to obtain the case $a_\alpha=c_\alpha$ of (6.15) as follows:

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{T}_n}\left(\max_{z\in K_\alpha}\left|f(z)-\sum_{k=0}^{m-1}\frac{((z-c_\alpha)\cdot')^k f(a_\alpha)}{k!}\right|\right)^p\\
&\qquad\leq C\sum_\alpha\int_{K_\alpha^*}\left|D_{c_\alpha}^m f(z)\right|^p d\lambda_n(z)\leq C\|f\|_{B_p}^p,
\end{aligned}$$

by the finite overlap condition 5 in Lemma 2.8. The general case with $a_\alpha\in K_\alpha$ is similar. This completes the proof of Proposition 6.5.

7. Besov spaces on trees

The theory of interpolating sequences is greatly simplified in the setting of abstract trees. A crucial advantage in the tree setting is that the appropriate derivatives of reproducing kernels for $B_p(\mathcal{T})$ are nonnegative, or in the case of the holomorphic Besov spaces considered in the next section, satisfy a suitable substitute, a fact which holds in the ball Besov space $B_p(\mathbb{B}_n)$ only for the vanishingly small range $1<p<1+\frac{1}{n-1}$. As we will see in Remark 7.13 however, the restriction map from $B_p(\mathbb{B}_n)$ to $B_p(\mathcal{T}_n)$ fails to be bounded unless $p>2n$. This limits the usefulness of the abstract Besov spaces on the Bergman tree for the study of $B_p(\mathbb{B}_n)$, and motivates the holomorphic version $HB_p(\mathcal{T}_n)$ introduced in the next section. On the other hand, the simple and elegant theory of the abstract spaces $B_p(\mathcal{T})$ will prove useful in guiding the development of, and motivating proofs for the holomorphic spaces $HB_p(\mathcal{T}_n)$.

We begin with the definition of Besov spaces on an abstract tree $\mathcal{T}$. Define the difference operator $\triangle$ on the tree by

$$\triangle f(\alpha)=f(\alpha)-f(A\alpha),\qquad \alpha\in\mathcal{T},$$

where $A\alpha$ denotes the predecessor or immediate *A*ncestor of α. Define the pointwise multiplier operator 2^d on the tree by

$$2^d f(\alpha)=2^{d(\alpha)}f(\alpha),\qquad \alpha\in\mathcal{T},$$

where $d(\alpha)=d(o,\alpha)$ denotes the distance in the tree from the root o to α.

DEFINITION 7.1. *For $1 < p < \infty$ and $m \geq 0$, define the Besov space $B_{p,m}(\mathcal{T})$ on a tree $\mathcal{T}$ to consist of all sequences $f = \{f(\alpha)\}_{\alpha \in \mathcal{T}}$ such that*

$$\|f\|_{B_{p,m}(\mathcal{T})} = \left(\sum_{\alpha \in \mathcal{T}: d(\alpha) \geq m} \left| \left(2^{-d}\right)^m \left(2^d \triangle\right)^m f(\alpha) \right|^p \right)^{\frac{1}{p}} + \sum_{d(\alpha) \leq m-1} |f(\alpha)| < \infty.$$

Note that in comparing this definition to the standard definition of Besov spaces in the unit ball of $\mathbb{C}^n$, the term 2^{-d} plays the role of $\left(1 - |z|^2\right)$, and $2^d \triangle$ plays the role of gradient. It turns out that the Carleson measures on $B_{p,m}(\mathcal{T}_n)$ are identical for all $m \geq 1$.

LEMMA 7.2. *Let $1 < p < \infty$ and $m \geq 1$. Then $\{\rho(\alpha)\}_{\alpha \in \mathcal{T}}$ is a $B_{p,m}(\mathcal{T})$-Carleson measure if and only if $\{\rho(\alpha)\}_{\alpha \in \mathcal{T}}$ is a $B_{p,1}(\mathcal{T})$-Carleson measure, which in turn holds if and only if*

$$\sum_{\beta \in \mathcal{T}: \beta \geq \alpha} I^* \rho(\beta)^{p'} \leq C^{p'} I^* \rho(\alpha), \qquad \alpha \in \mathcal{T}.$$

PROOF. If we write $g(\alpha) = \left(2^{-d}\right)^m \left(2^d \triangle\right)^m f(\alpha)$ for $d(\alpha) \geq m$ and then invert the operators, we have

(7.1)

$$\begin{aligned}
&f(\alpha) \\
&= \left(I 2^{-d}\right)^m \left(2^d\right)^m g(\alpha) \\
&= \sum_{\beta_1 \leq \alpha} 2^{-d(\beta_1)} \sum_{\beta_2 \leq \beta_1} 2^{-d(\beta_2)} \ldots \sum_{\beta_{m-1} \leq \beta_{m-2}} 2^{-d(\beta_{m-1})} \sum_{\gamma \leq \beta_{m-1}} 2^{-d(\gamma)} 2^{md(\gamma)} g(\gamma) \\
&= \sum_{\gamma \leq \beta_{m-1} \leq \ldots \leq \beta_2 \leq \beta_1 \leq \alpha} 2^{-d(\beta_1) - d(\beta_2) - d(\beta_{m-1})} 2^{(m-1)d(\gamma)} g(\gamma) \\
&= C_1 \sum_{\gamma \leq \beta_{m-1} \leq \ldots \leq \beta_2 \leq \alpha} 2^{-2d(\beta_2) - d(\beta_3) \ldots - d(\beta_{m-1})} 2^{(m-1)d(\gamma)} g(\gamma) \\
&= C_2 \sum_{\gamma \leq \beta_{m-1} \leq \ldots \leq \beta_3 \leq \alpha} 2^{-3d(\beta_3) \ldots - d(\beta_{m-1})} 2^{(m-1)d(\gamma)} g(\gamma) \\
&\quad \vdots \\
&= C_{m-1} \sum_{\gamma \leq \beta_{m-1} \leq \alpha} 2^{-(m-1)d(\beta_{m-1})} 2^{(m-1)d(\gamma)} g(\gamma) \approx Ig(\alpha).
\end{aligned}$$

Now $\{\rho(\alpha)\}_{\alpha \in \mathcal{T}}$ is a $B_{p,m}(\mathcal{T})$-Carleson measure if and only if

$$\left(\sum_{\alpha \in \mathcal{T}} |f(\alpha)|^p \rho(\alpha) \right)^{\frac{1}{p}} \leq C \left(\sum_{\alpha \in \mathcal{T}} \left| \left(2^{-d}\right)^m \left(2^d \triangle\right)^m f(\alpha) \right|^p \right)^{\frac{1}{p}} + \sum_{d(\alpha) \leq m-1} |f(\alpha)|,$$

which is equivalent to

$$\left(\sum_{\alpha \in \mathcal{T}} \left| \left(I 2^{-d}\right)^m \left(2^d\right)^m g(\alpha) \right|^p \rho(\alpha) \right)^{\frac{1}{p}} \leq C \left(\sum_{\alpha \in \mathcal{T}} |g(\alpha)|^p \right)^{\frac{1}{p}}.$$

From (7.1) we obtain that this latter inequality holds if and only if

$$\left(\sum_{\alpha\in\mathcal{T}}|Ig(\alpha)|^p\rho(\alpha)\right)^{\frac{1}{p}}\leq C\left(\sum_{\alpha\in\mathcal{T}}|g(\alpha)|^p\right)^{\frac{1}{p}},$$

which in turn holds if and only if $\{\rho(\alpha)\}_{\alpha\in\mathcal{T}}$ is a $B_{p,1}(\mathcal{T})$-Carleson measure. The final assertion follows from the equivalence of (1.7) and (1.10).

As is the case for Besov spaces on the unit ball, the norms $\|f\|_{B_{p,m}(\mathcal{T})}$ and $\|f\|_{B_{p,m'}(\mathcal{T})}$ are equivalent for m,m' large enough. Just how large depends on the notion of *upper dimension* $\overline{n}(\mathcal{T})$ of a tree $\mathcal{T}$, defined in Section 2 by

$$\overline{n}(\mathcal{T})=\limsup_{\ell\to\infty}\log_2\left(\sup_{\alpha\in\mathcal{T}}\mathit{card}\{\beta\in\mathcal{T}:\beta\geq\alpha\text{ and }d(\beta)\leq d(\alpha)+\ell\}\right)^{\frac{1}{\ell}}.$$

LEMMA 7.3. *For* $m>m'$, *we have* $\|f\|_{B_{p,m}(\mathcal{T})}\leq\|f\|_{B_{p,m'}(\mathcal{T})}$. *If* $\overline{n}=\overline{n}(\mathcal{T})$ *is the upper dimension of the tree* $\mathcal{T}$, *then* $\|f\|_{B_{p,m}(\mathcal{T})}\approx\|f\|_{B_{p,m'}(\mathcal{T})}$ *for* $m,m'>\frac{\overline{n}}{p}$.

PROOF. Let $g(\alpha)=\left(2^{-d}\right)^{m'}\left(2^d\bigtriangleup\right)^{m'}f(\alpha)$ so that for $d(\alpha)\geq m'$ we have $f(\alpha)=\left(I2^{-d}\right)^{m'}\left(2^d\right)^{m'}g(\alpha)$ Then

$$\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m f(\alpha)=\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m\left(I2^{-d}\right)^{m'}\left(2^d\right)^{m'}g(\alpha).$$

If $m>m'$, then

$$\left|\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m f(\alpha)\right|=\left|\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^{m-m'}\left(2^d\right)^{m'}g(\alpha)\right|\leq C\sum_{k=0}^{m-m'}\left|g\left(A^k\alpha\right)\right|,$$

and so

$$\begin{aligned}\|f\|_{B_{p,m}(\mathcal{T})}&=\left(\sum_{\alpha\in\mathcal{T}}\left|\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m f(\alpha)\right|^p\right)^{\frac{1}{p}}\\&\leq C\left(\sum_{\alpha\in\mathcal{T}}|g(\alpha)|^p\right)^{\frac{1}{p}}=\|f\|_{B_{p,m'}(\mathcal{T})}.\end{aligned}$$

On the other hand, if $m<m'$, then using (7.1) with m replaced by $m'-m$, we obtain

$$\begin{aligned}\left|\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m f(\alpha)\right|&=\left|\left(2^{-d}\right)^m\left(I2^{-d}\right)^{m'-m}\left(2^d\right)^{m'-m}\left(2^d\right)^m g(\alpha)\right|\\&\leq C\left(2^{-d}\right)^m I\left(2^d\right)^m|g(\alpha)|\\&=C\sum_{\beta\leq\alpha}2^{m[d(\beta)-d(\alpha)]}|g(\beta)|\\&=C\sum_{\beta\in\mathcal{T}}K(\alpha,\beta)|g(\beta)|,\end{aligned}$$

where the kernel $K(\alpha,\beta)$ is given by $\chi_{[o,\alpha]}(\beta)2^{m[d(\beta)-d(\alpha)]}$. We now apply Schur's test, Lemma 5.17, with auxiliary function with $h(\beta)=2^{td(\beta)}$. We have

$$\sum_{\beta\in\mathcal{T}}K(\alpha,\beta)h(\beta)^{p'}=\sum_{\beta\leq\alpha}2^{\left(m+p't\right)d(\beta)}2^{-md(\alpha)}\leq Ch(\alpha)^{p'}$$

provided $m+p't>0$. To estimate the complementary sum

$$\sum_{\alpha\in\mathcal{T}} K(\alpha,\beta)\, h(\alpha)^p = \sum_{\alpha\geq\beta} 2^{(pt-m)d(\alpha)} 2^{md(\beta)},$$

we use the device of splitting the sum $\sum_{\alpha\geq\beta}$ into "sparse" pieces $\sum_{\substack{\alpha\in\mathcal{T}\\ d(\alpha)\in d(\beta)+\ell\mathbb{N}+j}}$ for $0\leq j<\ell$, where ℓ is chosen so large in Definition 2.7 that given $\varepsilon>0$,

$$\begin{aligned}\log_2 (N_\ell)^{\frac{1}{\ell}} &= \log_2 \left(\sup_{\alpha\in\mathcal{T}} card\{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)=d(\alpha)+\ell\}\right)^{\frac{1}{\ell}} \\ &< \overline{n}(\mathcal{T})+\varepsilon.\end{aligned}$$

This is similar to the "sparse" argument used surrounding (3.19) to obtain the necessity of the tree condition for the Carleson embedding on the ball. We then have

$$\begin{aligned}\sum_{\alpha\in\mathcal{T}} K(\alpha,\beta)\, h(\alpha)^p &\leq C_\ell \sum_{k=0}^{\infty} \left(2^{\overline{n}(\mathcal{T})+\varepsilon}\right)^{k\ell} 2^{(pt-m)[d(\beta)+k\ell]} 2^{md(\beta)} \\ &\leq C_\ell h(\beta)^p,\end{aligned}$$

provided $\overline{n}(\mathcal{T})+\varepsilon+pt-m<0$. Thus if we can choose $-\frac{m}{p'}<t<\frac{m-\overline{n}(\mathcal{T})-\varepsilon}{p}$ for some $\varepsilon>0$, Schur's test shows that

$$\begin{aligned}\|f\|_{B_{p,m}(\mathcal{T})} &= \left(\sum_{\alpha\in\mathcal{T}} \left|\left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m f(\alpha)\right|^p\right)^{\frac{1}{p}} \\ &\leq C\left(\sum_{\alpha\in\mathcal{T}} \left|\sum_{\beta\in\mathcal{T}} K(\alpha,\beta)\,|g(\beta)|\right|^p\right)^{\frac{1}{p}} \\ &\leq C\left(\sum_{\alpha\in\mathcal{T}} |g(\alpha)|^p\right)^{\frac{1}{p}} \\ &= C\,\|f\|_{B_{p,m'}(\mathcal{T})}\end{aligned}$$

as required. But this choice of t is possible if and only if $-\frac{m}{p'}<\frac{m-\overline{n}(\mathcal{T})-\varepsilon}{p}$ for some $\varepsilon>0$, or $m>\frac{\overline{n}(\mathcal{T})}{p}$.

With a small abuse of notation the norm $\|f\|_{B_{p,m}(\mathcal{T})}$ will be denoted simply by $\|f\|_{B_p(\mathcal{T})}$ when it is understood that $m>\frac{\overline{n}(\mathcal{T})}{p}$. Otherwise, we will write $B_{p,m}(\mathcal{T})$ to emphasize the dependence on m. We note that in general $B_{p,m}(\mathcal{T})\neq B_{p,m'}(\mathcal{T})$ if $m<m'$ and $m\leq\frac{\overline{n}}{p}$. For example, in the case $m=1$ and $m'=2$, define f on the Bergman tree $\mathcal{T}_n$, with $\theta=\frac{\ln 2}{2}$ as in (3.3), by $f(\alpha)=2^{-d(\alpha)}$. Then $\bigtriangleup f(\alpha)=-2^{-d(\alpha)}$ and

$$2^{2d}\left(2^d \bigtriangleup\right)^2 f(\alpha) = 2^{-d} \bigtriangleup \left(-2^d 2^{-d}\right)(\alpha) = 0$$

for $d(\alpha) \geq 2$. Thus $f \in B_{p,2}(\mathcal{T}_n)$ for all $1 < p < \infty$. On the other hand, using the "sparse" argument again, $f \in B_{p,1}(\mathcal{T}_n)$ if and only if $p > n$:

$$\begin{aligned}\|f\|_{B_{p,1}(\mathcal{T}_n)} &= \left(\sum_{\alpha \in \mathcal{T}_n : d(\alpha) \geq 1} |\triangle f(\alpha)|^p\right)^{\frac{1}{p}} + |f(o)| \\ &= \left(\sum_{\alpha \in \mathcal{T}_n} 2^{-pd(\alpha)}\right)^{\frac{1}{p}} + |f(o)| \\ &\leq C_\ell \left(\sum_{k=0}^{\infty} 2^{(n+\varepsilon)k\ell} 2^{-pk\ell}\right)^{\frac{1}{p}} + |f(o)|,\end{aligned}$$

if $\log_2 (N_\ell)^{\frac{1}{\ell}} < n + \varepsilon$, since by Lemma 2.8, the dimension of the Bergman tree $\mathcal{T}_n$ is n when $\theta = \frac{\ln 2}{2}$.

We can now characterize the pointwise multipliers on $B_{p,m}(\mathcal{T})$.

Lemma 7.4. *Let $m > \frac{\overline{n}(\mathcal{T})}{p}$. Then $f \in M_{B_{p,m}(\mathcal{T})}$ if and only if f is bounded and $\left\{\left|\left(2^{-d}\right)^m \left(2^d \triangle\right)^m f(\alpha)\right|^p\right\}_{\alpha \in \mathcal{T}}$ is a $B_{p,m}(\mathcal{T})$-Carleson measure.*

Proof. (the case $m = 2 > \frac{\overline{n}(\mathcal{T})}{p}$) For the sufficiency, let $a_n = f(\alpha)$, $a_{n-1} = f(A\alpha)$, $a_{n-2} = f\left(A^2\alpha\right)$ and similarly $b_n = \varphi(\alpha)$, $b_{n-1} = \varphi(A\alpha)$, $b_{n-2} = \varphi\left(A^2\alpha\right)$ for $\varphi \in B_{p,2}(\mathcal{T})$. Then

$$\begin{aligned}&2^{-d} \triangle 2^d \triangle (f\varphi)(\alpha) \\ &\quad = 2^{-d(0,\alpha)} \left\{\frac{(f\varphi)(\alpha) - (f\varphi)(P\alpha)}{2^{-d(0,\alpha)}} - \frac{(f\varphi)(P\alpha) - (f\varphi)\left(P^2\alpha\right)}{2^{-d(0,P\alpha)}}\right\} \\ &\quad = (a_n b_n - a_{n-1}b_{n-1}) - \frac{1}{2}(a_{n-1}b_{n-1} - a_{n-2}b_{n-2}) \\ &\quad = a_n \left\{(b_n - b_{n-1}) - \frac{1}{2}(b_{n-1} - b_{n-2})\right\} \\ &\qquad + b_{n-2}\left\{(a_n - a_{n-1}) - \frac{1}{2}(a_{n-1} - a_{n-2})\right\} \\ &\qquad + \frac{3}{2}(a_n - a_{n-1})(b_{n-1} - b_{n-2}) \\ &\quad = f(\alpha)\left(2^{-d} \triangle 2^d \triangle \varphi\right)(\alpha) + \varphi\left(A^2\alpha\right)\left(2^{-d} \triangle 2^d \triangle f\right)(\alpha) \\ &\qquad + \frac{3}{2}(\triangle f)(\alpha)(\triangle \varphi)(A\alpha)\ .\end{aligned}$$

Since f is bounded and $\varphi \in B_{p,2}(\mathcal{T})$, we have $f(\alpha)\left(2^{-d} \triangle 2^d \triangle \varphi\right)(\alpha) \in \ell^p$. Since $B_{2p,2}(\mathcal{T}) \approx B_{2p,1}(\mathcal{T}) \supset B_{p,1}(\mathcal{T})$ upon appealing to the equivalence of norms in Lemma 7.3 (note that $1 > \frac{\overline{n}(\mathcal{T})}{2p}$), we obtain from the Cauchy-Schwartz inequality that

$$\left|\frac{3}{2}(\triangle f)(\alpha)(\triangle \varphi)(P\alpha)\right| \leq |(\triangle f)(\alpha)|^2 + |(\triangle \varphi)(P\alpha)|^2 \in \ell^p.$$

Thus the ℓ^p norm of $\varphi\left(A^2\alpha\right)\left(2^{-d}\bigtriangleup 2^d\bigtriangleup f\right)(\alpha)$ is controlled by the $B_{p,2}(\mathcal{T})$ norm of φ, and this shows that $\left\{\left|2^{-d}\bigtriangleup 2^d\bigtriangleup f(\alpha)\right|^p\right\}_{\alpha\in\mathcal{T}}$ is a $B_{p,2}(\mathcal{T})$-Carleson measure. The case $m>2$ is similar.

The necessity of the boundedness of f is standard. One can reverse the above argument to obtain the necessity of the Carleson embedding.

7.1. Interpolating sequences on trees. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence of points in a tree $\mathcal{T}$ having finite dimension $n=n(\mathcal{T})$. In the present subsubsection we will state without proof the equivalence of weighted ℓ^p interpolation for the Besov spaces $B_{p,m}(\mathcal{T})$, $m>\frac{n}{p}$, on the sequence $\{\alpha_j\}_{j=1}^\infty$, ℓ^∞ interpolation for their multiplier spaces $M_{B_{p,m}(\mathcal{T})}$ on $\{\alpha_j\}_{j=1}^\infty$, and the separation and Carleson embeddings on the tree,

$$d(\alpha_i,o)\le Cd(\alpha_i,\alpha_j)\ \text{ and }\ \sum_{j=1}^\infty d(o,\alpha_j)^{1-p}\,\delta_{\alpha_j}\ \text{ is a Carleson measure for } B_{p,m}(\mathcal{T}).$$

In order to give a precise statement involving additional equivalent conditions, we need to introduce duality pairings and reproducing formulas for $B_{p,m}(\mathcal{T})$ and $B_{p',m}(\mathcal{T})$.

For $m\ge 1$, define the pairing

$$\langle F,G\rangle_m=\sum_{\alpha\in\mathcal{T}}\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m F(\alpha)\,\overline{\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m G(\alpha)}.$$

In the case $m=1$, $\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m=\bigtriangleup$ and the reproducing kernel $k_\alpha^1(\beta)=k^1(\alpha,\beta)$ with respect to the pairing

$$\langle F,G\rangle_1=\sum_{\alpha\in\mathcal{T}_n}\bigtriangleup F(\alpha)\,\overline{\bigtriangleup G(\beta)},$$

for the Besov space $B_{p,1}(\mathcal{T})$ on the tree $\mathcal{T}$ is given by

$$k_\alpha^1(\beta)=\chi_{[0,\alpha]}(\beta)\,d(0,\beta)+\chi_{S(\alpha)}(\beta)\,d(0,\alpha),\qquad \alpha,\beta\in\mathcal{T}.$$

Indeed, we have

$$\bigtriangleup k_\alpha^1(\beta)=\chi_{[0,\alpha]}(\beta)\tag{7.2}$$

and so

$$\left\langle F,k_\alpha^1\right\rangle_{\mathcal{T}}=\sum_{\beta\in\mathcal{T}}\bigtriangleup F(\beta)\,\overline{\bigtriangleup k_\alpha^1(\beta)}=\sum_{0\le\beta\le\alpha}\bigtriangleup F(\beta)=F(\alpha).$$

The important property here is that the kernel $\bigtriangleup k_\alpha^1$ in (7.2) is nonnegative and pointwise comparable to $\chi_{[0,\alpha]}$, and this is easily generalized to the reproducing kernels k_α^m for all $m\ge 1$:

$$\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m k_\alpha^m(\beta)\approx\chi_{[0,\alpha]}(\beta)\ge 0,\qquad \alpha,\beta\in\mathcal{T}.\tag{7.3}$$

For example, when $m=2$, $\left(2^{-d}\right)^m\left(2^d\bigtriangleup\right)^m=2^{-d}\bigtriangleup 2^d\bigtriangleup$ and the reproducing kernel $k_\alpha^2(\beta)=k^2(\alpha,\beta)$ with respect to the pairing

$$\langle F,G\rangle_2=\sum_{\alpha\in\mathcal{T}}2^{-d}\bigtriangleup 2^d\bigtriangleup F(\alpha)\,\overline{2^{-d}\bigtriangleup 2^d\bigtriangleup G(\beta)},$$

for the Besov space $B_{p,2}(\mathcal{T})$ on the tree $\mathcal{T}$ satisfies

$$2^{-d} \bigtriangleup 2^{d} \bigtriangleup k_{\alpha}^{2}(\gamma) = \chi_{[0,\alpha]}(\gamma) \left\{ \sum_{\beta \in [\gamma,\alpha]} 2^{d(\gamma)-d(\beta)} \right\}, \qquad \alpha, \beta \in \mathcal{T}. \tag{7.4}$$

Indeed, with $f = 2^{-d} \bigtriangleup 2^{d} \bigtriangleup F$, we have $F = I2^{-d}I2^{d}f$ and

$$F(\alpha) = I2^{-d}I2^{d}f(\alpha) = \sum_{\beta \leq \alpha} 2^{-d(\beta)} \sum_{\gamma \leq \beta} 2^{d(\gamma)} f(\gamma) = \sum_{\gamma \leq \beta} \left\{ \sum_{\beta \in [\gamma,\alpha]} 2^{d(\gamma)-d(\beta)} \right\} f(\gamma).$$

Thus we have

$$\begin{aligned} \left\langle F, k_{\alpha}^{2} \right\rangle_{B_{p,2}(\mathcal{T})} &= \sum_{\gamma \in \mathcal{T}} 2^{-d} \bigtriangleup 2^{d} \bigtriangleup F(\gamma) \overline{2^{-d} \bigtriangleup 2^{d} \bigtriangleup k_{\alpha}^{2}(\gamma)} \\ &= \sum_{0 \leq \gamma \leq \alpha} \left\{ \sum_{\beta \in [\gamma,\alpha]} 2^{d(\gamma)-d(\beta)} \right\} f(\gamma) = F(\alpha). \end{aligned}$$

We have the following theorem.

THEOREM 7.5. *Let $1 < p < \infty$ and $m > \frac{n}{p}$ where $n = n(\mathcal{T})$ is the dimension of the tree $\mathcal{T}$. Then the dual space of $B_{p,m}(\mathcal{T})$ can be identified with $B_{p',m}(\mathcal{T})$ under the pairing $\langle \cdot, \cdot \rangle_m$, and the reproducing kernel k_w^m for this pairing satisfies (7.3).*

Now we suppose that $m > \frac{n}{p}, \frac{n}{p'}$ and suppress the dependence of $k_\alpha = k_\alpha^m$ and $B_p = B_p(\mathcal{T}) = B_{p,m}(\mathcal{T})$ on m and $\mathcal{T}$. Here is the abstract tree analogue of Böe's interpolation theorem. Note that by (7.3), we have

$$\|k_\alpha\|_{B_{p'}} = \left\| \left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m k_\alpha^m \right\|_{\ell^{p'}} \approx d(o,\alpha)^{\frac{1}{p'}}, \qquad \alpha \in \mathcal{T}.$$

THEOREM 7.6. *Let $1 < p < \infty$, $m > \frac{n}{p}, \frac{n}{p'}$ and k_α be the reproducing kernel for B_p relative to the pairing $\langle \cdot, \cdot \rangle_m$ given in Theorem 7.5 above. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence in the tree $\mathcal{T}$. Then the following conditions are equivalent.*

(1) *$\{\alpha_j\}_{j=1}^\infty$ interpolates M_{B_p}:*

$$\text{The map } f \to \{f(\alpha_j)\}_{j=1}^\infty \text{ takes } M_{B_p} \text{ boundedly into and onto } \ell^\infty. \tag{7.5}$$

(2) *$\{k_{\alpha_j}\}_{j=1}^\infty$ is an unconditional basic sequence in $B_{p'}$:*

$$\left\| \sum_{j=1}^\infty b_j k_{\alpha_j} \right\|_{B_{p'}} \leq C \left\| \sum_{j=1}^\infty a_j k_{\alpha_j} \right\|_{B_{p'}}, \qquad \text{whenever } |b_j| \leq |a_j|. \tag{7.6}$$

(3) *The following norm equivalence holds:*

$$\left\| \sum_{j=1}^\infty a_j \frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{B_{p'}}} \right\|_{B_{p'}} \approx \left(\sum_{j=1}^\infty |a_j|^{p'} \right)^{\frac{1}{p'}}. \tag{7.7}$$

(4) *$\{\alpha_j\}_{j=1}^\infty$ interpolates B_p:*

$$\text{The map } f \to \left\{ \frac{f(z_j)}{\left\|k_{\alpha_j}\right\|_{B_{p'}}} \right\}_{j=1}^\infty \text{ takes } B_p \text{ boundedly into and onto } \ell^p. \tag{7.8}$$

(5) *The following separation condition and Carleson embedding hold:*

$$d(\alpha_i, o) \leq C d(\alpha_i, \alpha_j) \ \text{ and} \tag{7.9}$$

$$\sum_{j=1}^{\infty} d(o, \alpha_j)^{1-p} \delta_{\alpha_j} \ \text{ is a Carleson measure for } B_{p,m}(\mathcal{T}). \tag{7.10}$$

REMARK 7.7. The interpolations in (7.5) and (7.8) can be taken to be linear, i.e. there are bounded linear maps $R : \ell^\infty \to M_{B_p}$ and $S : \ell^p \to B_p$ that yield right inverses to the restriction maps in (7.5) and (7.8) respectively.

We give a brief sketch of those parts of the proof that we will need in the more refined setting of holomorphic Besov spaces on trees. The proofs that (7.5) implies (7.6) implies (7.7) implies (7.8) implies (7.9) follow the corresponding arguments in Böe [**Boe**]. The details are standard, with the exception of the implication (7.6) implies (7.7). The proof of this uses Böe's Lemma 3.1 in [**Boe**] together with the crucial property (7.3) that the difference operator $(2^{-d})^m (2^d \triangle)^m$ applied to the reproducing kernel on the tree is nonnegative for all $1 < p < \infty$. Indeed, assuming (7.6) we have, with $\{r_j(t)\}$ being the Radamacher functions on $[0,1]$,

$$\begin{aligned}
\left\| \left\| a_j \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{\ell^1} \right\|_{\ell^{p'}}^{p'} &= \left\| \sum_{j=1}^{\infty} |a_j| \frac{k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{B_{p'}}^{p'} \\
&\leq C \int_0^1 \left\| \sum_{j=1}^{\infty} |a_j| \, r_j(t) \frac{k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}(\mathcal{T})}} \right\|_{B_{p'}}^{p'} dt \\
&= C \sum_{\alpha \in \mathcal{T}} \int_0^1 \left| \sum_{j=1}^{\infty} |a_j| \, r_j(t) \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}(\alpha)}{\|k_{\alpha_j}\|_{B_{p'}}} \right|^{p'} dt \\
&\approx \left\| \left\| a_j \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{\ell^2} \right\|_{\ell^{p'}}^{p'}.
\end{aligned}$$

Continuing to follow Böe, we now use the inequality $\|\cdot\|_{\ell^2} \leq \|\cdot\|_{\ell^1} \|\cdot\|_{\ell^\infty}$ along with the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}
\left\| \left\| a_j \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{\ell^1} \right\|_{\ell^{p'}}^{p'} &\leq C \left\| \left\| a_j \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{\ell^2} \right\|_{\ell^{p'}}^{p'} \\
&\leq C \left\| \left\| a_j \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{\ell^1} \right\|_{\ell^{p'}}^{\frac{p'}{2}} \\
&\quad \times \left\| \left\| a_j \frac{(2^{-d})^m (2^d \triangle)^m k_{\alpha_j}}{\|k_{\alpha_j}\|_{B_{p'}}} \right\|_{\ell^\infty} \right\|_{\ell^{p'}}^{\frac{p'}{2}},
\end{aligned}$$

which yields

$$\left\|\sum_{j=1}^{\infty} a_j \frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{B_{p'}}}\right\|_{B_{p'}}^{p'} \approx \left\|\left\|a_j \frac{\left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{B_{p'}}}\right\|_{\ell^1}\right\|_{\ell^{p'}}^{p'}$$

$$\approx \left\|\left\|a_j \frac{\left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{B_{p'}}}\right\|_{\ell^\infty}\right\|_{\ell^{p'}}^{\frac{p'}{2}}$$

$$\approx \left\|\left\|a_j \frac{\left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{B_{p'}}}\right\|_{\ell^{p'}}\right\|_{\ell^{p'}}^{\frac{p'}{2}} = \left(\sum_{j=1}^{\infty} \left|a_j\right|^{p'}\right)^{\frac{1}{p'}}$$

since

$$\left\|\frac{\left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{B_{p'}}}\right\|_{\ell^{p'}} = 1, \qquad 1 \leq j < \infty.$$

It remains to prove that the separation condition and Carleson embedding in (7.9) are sufficient for the multiplier interpolation in (7.5). This can be proved by following the argument given for the ball in Section 5. See also Theorem 26 in Section 6 of [**ArRoSa**], where the implication (7.9) implies (7.8) is proved in the case $m = 1$. Since this result is not needed for subsequent developments, we omit the proof.

7.2. The restriction map. Let $p > \widehat{n}$ where

$$\widehat{n} = \begin{cases} 1 & \text{if } n = 1 \\ 2n & \text{if } n > 1 \end{cases}.$$

Note that in this case we can take $m = 1$ in both the definition of $B_p(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$ and $B_p(\mathcal{T}_n)$ on the tree $\mathcal{T}_n$. We have from the case $m = 1$ of Proposition 6.5, or from Theorem 6.30 of [**Zhu**], the following estimate on the oscillation of a $B_p(\mathbb{B}_n)$ function on Bergman balls of bounded radius.

LEMMA 7.8. *The oscillation inequality*

$$\left\{\sum_{\alpha\in\mathcal{T}_n}\left(\max_{z_1,z_2\in K_\alpha}\left|f(z_1)-f(z_2)\right|\right)^p\right\}^{\frac{1}{p}} \leq C\left\|f\right\|_{B_p(\mathbb{B}_n)}, \tag{7.11}$$

holds for $p > \widehat{n}$.

The following local version of Lemma 7.8, also essentially contained in Proposition 6.5, will prove useful in estimating Carleson measure norms.

LEMMA 7.9. *The local oscillation inequality*

$$\left\{\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\max_{z_1,z_2\in K_\beta}\left|f(z_1)-f(z_2)\right|\right)^p\right\}^{\frac{1}{p}}$$

$$\leq C\left(\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\int_{B_d(c_\beta,C_2)}\left|\widetilde{\nabla}f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}, \qquad \alpha\in\mathcal{T}_n,$$

holds for $p > \widehat{n}$*.*

The local oscillation inequality is not in [**Zhu**], and we prefer to give here a proof of Lemma 7.8 using the invariant derivative, that immediately yields Lemma 7.9 as well, and avoids the use of the almost invariant holomorphic derivatives D_a in Proposition 6.5.

PROOF. (of Lemmas 7.8 and 7.9 without using Proposition 6.5) Denote by K_α^* the union of the Carleson box K_α and its neighbours at most M boxes away. From part 1. of Lemma 2.8, and provided we choose M sufficiently large, we obtain that there are constants A_1, A_2 and A_2^* in $[0,1)$ with $1 - A_2^* = \frac{1}{2}(1 - A_2)$ depending only on n such that

$$\begin{aligned} B(c_\alpha, A_1) &\subset K_\alpha \subset B(c_\alpha, A_2), \\ B(c_\alpha, A_2^*) &\subset K_\alpha^*, \end{aligned}$$

for $\alpha \in \mathcal{T}_n$. We first establish the estimate

$$\max_{z_1, z_2 \in K_0} |f(z_1) - f(z_2)| \le C \left(\int_{K_0^*} \left| \widetilde{\nabla} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}}, \tag{7.12}$$

for the root Carleson box $K_0 = \frac{1}{4}\mathbb{B}_n$, and where $\left|\widetilde{\nabla} f\right|$ denotes the invariant gradient length of f defined by

$$\left| \widetilde{\nabla} f(z) \right| = |\nabla (f \circ \varphi_z)(0)|.$$

Note that while the vector $\widetilde{\nabla} f(z) = \nabla (f \circ \varphi_z)(0)$ is not invariant under the action of the automorphism group, its length $\left|\widetilde{\nabla} f(z)\right|$ *is* invariant, i.e.

$$\left| \widetilde{\nabla} (f \circ \psi) \right| = \left| \left(\widetilde{\nabla} f \right) \circ \psi \right|, \qquad \psi \in Aut(\mathbb{B}_n). \tag{7.13}$$

Indeed, given $\psi \in Aut(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$, let $w = \psi(z)$. Then $\varphi_w \circ \psi \circ \varphi_z(0) = 0$ and so $\varphi_w \circ \psi \circ \varphi_z = U$ is unitary. Since g and $g \circ U$ have the same (ordinary) gradient length at the origin for any unitary transformation U, we obtain

$$\begin{aligned} \left| \widetilde{\nabla} (f \circ \psi)(z) \right| &= |\nabla (f \circ \psi \circ \varphi_z)(0)| = |\nabla (f \circ \varphi_w \circ U)(0)| \\ &= |\nabla (f \circ \varphi_w)(0)| = \left| \left(\widetilde{\nabla} f \right)(w) \right| \\ &= \left| \left(\left(\widetilde{\nabla} f \right) \circ \psi \right)(z) \right|. \end{aligned}$$

To obtain (7.12), we use the standard Euclidean inequality,

$$\int_B |f(z) - f(w)|\, dw \le C \int_B |z - w|^{-1} |\nabla f(w)|\, dw,$$

valid for continuously differentiable f on a Euclidean ball B, together with Bergman's formula (applied to the derivative) scaled to the ball $\frac{3}{4}\mathbb{B}_n$,

$$f'(w) = c_n \int_{\frac{3}{4}\mathbb{B}_n} \frac{f'(\xi)}{\left(\frac{3}{4} - \overline{\xi} \cdot w\right)^{n+1}} d\xi,$$

valid for f holomorphic on $\frac{3}{4}\mathbb{B}_n$, to obtain that for $|z| < \frac{1}{4}$ and $f \in B_p(\mathbb{B}_n)$,

$$\begin{aligned}\left| f(z) - \frac{1}{\left|\frac{1}{4}\mathbb{B}_n\right|}\int_{\frac{1}{4}\mathbb{B}_n} f(w)\, dw \right| &\leq \frac{1}{\left|\frac{1}{4}\mathbb{B}_n\right|}\int_{\frac{1}{4}\mathbb{B}_n} |f(z) - f(w)|\, dw \\ &\leq C \int_{\frac{1}{4}\mathbb{B}_n} |z-w|^{-1} |f'(w)|\, dw \\ &\leq C \int_{\frac{1}{4}\mathbb{B}_n} |z-w|^{-1} \left\{ \int_{\frac{3}{4}\mathbb{B}_n} |f'(\xi)|\, d\xi \right\} dw \\ &\leq C \int_{\frac{3}{4}\mathbb{B}_n} |f'(\xi)|\, d\xi \\ &\leq C \left(\int_{\frac{3}{4}\mathbb{B}_n} |f'(\xi)|^p\, d\xi \right)^{\frac{1}{p}}.\end{aligned}$$

This establishes (7.12) since $|f'(\xi)| \approx \left|\widetilde{\nabla} f(\xi)\right|$ and $d\xi \approx d\lambda_n(\xi)$ for $\xi \in \frac{3}{4}\mathbb{B}_n$.

Using the holomorphic homeomorphisms φ_w of the ball, we now show that (7.12) implies the estimate

$$\max_{z_1,z_2 \in K_\alpha} |f(z_1) - f(z_2)| \leq C \left(\int_{K_\alpha^*} \left|\widetilde{\nabla} f(\xi)\right|^p d\lambda_n(\xi) \right)^{\frac{1}{p}}, \quad \alpha \in \mathcal{T}_n. \tag{7.14}$$

Recall that c_α denotes the "center" of K_α. Since φ_{c_α} maps T_α to T_0 (at least approximately) with inverse φ_{c_α}, we can apply (7.12) to the function $g = f \circ \varphi_w$ with $w = c_\alpha$ and then use the invariance of $\widetilde{\nabla}$ (see (7.13)) and λ_n together with the change of variable $\xi = \varphi_w(z)$ to obtain

$$\begin{aligned}\max_{z_1,z_2 \in K_\alpha} |f(z_1) - f(z_2)|^p &= \max_{\xi_1,\xi_2 \in K_0} |g(\xi_1) - g(\xi_2)|^p \\ &\leq C \int_{K_0^*} \left|\widetilde{\nabla}(f \circ \varphi_w)(\xi)\right|^p d\lambda_n(\xi) \\ &= C \int_{K_0^*} \left|\widetilde{\nabla} f(\varphi_w(\xi))\right|^p d\lambda_n(\xi) \\ &= C \int_{K_\alpha^*} \left|\widetilde{\nabla} f(z)\right|^p d\lambda_n(z).\end{aligned}$$

This proves (7.14).

We now apply (7.14) to obtain (7.11) as follows:

$$\begin{aligned}\sum_{\alpha \in \mathcal{T}_n} \left(\max_{z_1,z_2 \in K_\alpha} |f(z_1) - f(z_2)| \right)^p &\leq C \sum_{\alpha \in \mathcal{T}_n} \int_{K_\alpha^*} \left|\widetilde{\nabla} f(z)\right|^p d\lambda_n(z) \\ &\leq C \left(\int_{\mathbb{B}_n} \left|\widetilde{\nabla} f(z)\right|^p d\lambda_n(z) \right)^{\frac{1}{p}},\end{aligned}$$

by the finite overlap condition 5. in Lemma 2.8. By Theorem 6.11 in [**Zhu**], the above expression is equivalent to the Besov space norm $\|f\|_{B_p(\mathbb{B}_n)}$ for $p > \widehat{n}$, and this completes the proof of Lemma 7.8. Lemma 7.9 follows by adding up estimates in (7.14):

$$\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\max_{z_1,z_2\in K_\beta}\left|f\left(z_1\right)-f\left(z_2\right)\right|\right)^p \leq C\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\int_{K_\beta^*}\left|\widetilde{\nabla}f\left(z\right)\right|^p d\lambda_n\left(z\right),\qquad \alpha\in\mathcal{T}_n.$$

Corollary 7.10. *Let $p>\widehat{n}$. Then the restriction map*

$$Tf=\left\{f\left(\alpha\right)\right\}_{\alpha\in\mathcal{T}_n},\qquad \text{where } Tf\left(\alpha\right)=f\left(c_\alpha\right),$$

is bounded from $B_p\left(\mathbb{B}_n\right)$ to $B_p\left(\mathcal{T}_n\right)$. If in addition $n=1$ and $1<p<\infty$, then T is also bounded from $M_{B_p(\mathbb{B}_1)}$ to $M_{B_p(\mathcal{T}_1)}$.

The reason the dimension is restricted to $n=1$ for the boundedness of the restriction map on the multiplier space, is that this restriction requires that Carleson embeddings be characterized by the tree condition. We have only established this latter result for $p<2+\frac{1}{n-1}$, which together with the restriction $p>\widehat{n}=2n>2+\frac{1}{n-1}$ for $n\geq 2$, leaves $n=1$ as the only possible dimension. Nevertheless, in the interest of motivating future proofs, we will present our arguments in as general a setting as possible.

Proof. With the tree difference operator $\left(2^{-d}\right)\left(2^d\vartriangle\right)=\vartriangle$ as in Definition 7.1 above with $m=1$, we have

$$\left|\vartriangle f\left(\alpha\right)\right|\leq\max_{z_1,z_2\in K_\alpha^*}\left|f\left(z_1\right)-f\left(z_2\right)\right|,\tag{7.15}$$

where K_α^* is the Bergman ball centered at c_α with radius C. Then from a modification of (7.11) where K_α is replaced by K_α^*, we obtain

$$\begin{aligned}\|f\|_{B_p(\mathcal{T}_n)}&=\left(\sum_{\alpha\in\mathcal{T}_n:d(\alpha)\geq1}\left|\vartriangle f\left(\alpha\right)\right|^p\right)^{\frac{1}{p}}+\left|f\left(o\right)\right|\\&\leq C\left(\sum_{\alpha\in\mathcal{T}_n:d(\alpha)\geq1}\left|\max_{z_1,z_2\in K_\alpha^*}\left|f\left(z_1\right)-f\left(z_2\right)\right|\right|^p\right)^{\frac{1}{p}}+\left|f\left(o\right)\right|\\&\leq C\left\|f\right\|_{B_p(\mathbb{B}_n)}.\end{aligned}$$

To handle the restriction map T on the multiplier space $M_{B_p(\mathbb{B}_1)}$, we need Stegenga's characterization of multipliers on the disk in terms of Carleson embeddings on the disk. More generally, we first record a variant of our multiplier theorem on the ball using the invariant gradient length.

Lemma 7.11. *Let $\varphi\in H^\infty\left(\mathbb{B}_n\right)\cap B_p\left(\mathbb{B}_n\right)$ and $p>\widehat{n}$. Then φ is a multiplier on $B_p\left(\mathbb{B}_n\right)$ if and only if*

$$\left|\widetilde{\nabla}\varphi\left(z\right)\right|^p d\lambda_n\left(z\right)$$

is a $B_p\left(\mathbb{B}_n\right)$-Carleson measure on $\mathbb{B}_n$.

PROOF. Using the product rule

$$\begin{aligned}\widetilde{\nabla}(fg)(z) &= \nabla[(fg)\circ\varphi_z](0) = \nabla[(f\circ\varphi_z)(g\circ\varphi_z)](0)\\ &= f\circ\varphi_z(0)\nabla(g\circ\varphi_z)(0)+\nabla(f\circ\varphi_z)(0)(g\circ\varphi_z)(0)\\ &= f(z)\widetilde{\nabla}g(z)+\widetilde{\nabla}f(z)g(z),\end{aligned}$$

the case $m=1$ of the proof of our multiplier theorem applies virtually verbatim.

Now let $f \in M_{B_p(\mathbb{B}_1)}$ and set

$$d\mu(z) = \sum_{\alpha\in\mathcal{T}_n} \chi_{K_\alpha}(z)\int_{K_\alpha^*}\left|\widetilde{\nabla}f(\zeta)\right|^p d\lambda_1(\zeta).$$

The above lemma shows that μ is a $B_p(\mathbb{B}_1)$-Carleson measure. Define the discretization of μ in the usual way by

$$\mu(\alpha) = \int_{K_\alpha^*}\left|\widetilde{\nabla}f(\zeta)\right|^p d\lambda_1(\zeta).$$

Since $p < 2+\frac{1}{n-1}$ when $n=1$, it then follows from Theorem 3.1 that $\{\mu(\alpha)\}_{\alpha\in\mathcal{T}_1}$ is a $B_{p,1}(\mathcal{T}_1)$-Carleson measure. Now set

$$\omega(\alpha) = |\triangle\, f(\alpha)|^p.$$

From (7.15) and a modification of Remark 7.9, we obtain

$$\begin{aligned}I^*\omega(\alpha) &= \sum_{\beta\in\mathcal{T}_1:\beta\geq\alpha}|\triangle\, f(\alpha)|^p\\ &\leq C\sum_{\beta\in\mathcal{T}_1:\beta\geq\alpha}\left(\max_{z_1,z_2\in K_\alpha^*}|f(z_1)-f(z_2)|\right)^p\\ &\leq C\sum_{\alpha\in\mathcal{T}_1:\beta\geq\alpha}\int_{B_d(c_\beta,C_2)}\left|\widetilde{\nabla}f(z)\right|^p d\lambda_1(z)\\ &= CI^*\nu(\alpha).\end{aligned}$$

It now follows that $\{\omega(\alpha)\}_{\alpha\in\mathcal{T}_1}$ is a $B_{p,1}(\mathcal{T}_1)$-Carleson measure with norm bounded by that of the $B_p(\mathbb{B}_1)$-Carleson measure μ. Finally, the tree multiplier characterization in Theorem 7.4, shows that $Tf \in M_{B_{p,1}(\mathcal{T}_1)} = M_{B_p(\mathcal{T}_1)}$ with $\|Tf\|_{M_{B_p(\mathcal{T}_1)}} \leq C\|f\|_{M_{B_p(\mathbb{B}_1)}}$.

REMARK 7.12. Let $p > \widehat{n}$. *Assume* that the restriction map in Corollary 7.10 is bounded from $M_{B_p(\mathbb{B}_n)}$ to $M_{B_p(\mathcal{T}_n)}$. If $\{z_j\}_{j=1}^\infty \subset \mathbb{B}_n$ interpolates $M_{B_p(\mathbb{B}_n)}$, i.e.

(7.16) The map $f \to \{f(z_j)\}_{j=1}^\infty$ takes M_{B_p} boundedly into and onto ℓ^∞,

and if we construct the Bergman tree $\mathcal{T}_n$ so that $\{c_\alpha\}_{\alpha\in\mathcal{T}_n}$ contains $\{z_j\}_{j=1}^\infty$, say with $z_j = c_{\alpha_j}$, then it follows that $\{\alpha_j\}_{j=1}^\infty$ interpolates $M_{B_p(\mathcal{T}_n)}$, i.e. that (7.5) holds with $\mathcal{T}_n$ in place of $\mathcal{T}$:

(7.17) The map $f \to \{f(\alpha_j)\}_{j=1}^\infty$ takes $M_{B_p(\mathcal{T}_n)}$ boundedly into and onto ℓ^∞.

To see that (7.17) holds if (7.16) does, suppose that $\{\xi_j\}_{j=1}^{\infty} \in \ell^{\infty}$. Using (5.6) we can find $\varphi \in M_{B_p(\mathbb{B}_n)}$ satisfying

$$\varphi(z_j) = \xi_j, \quad 1 \le j < \infty,$$
$$\|\varphi\|_{M_{B_p(\mathbb{B}_n)}} \le C \left\| \{\xi_j\}_{j=1}^{\infty} \right\|_{\infty}.$$

Now define f on the tree $\mathcal{T}_n$ by

$$f(\alpha) = \varphi(c_\alpha), \quad \alpha \in \mathcal{T}_n.$$

Then we have

$$f(\alpha_j) = \varphi\left(c_{\alpha_j}\right) = \varphi(z_j) = \xi_j$$

and our assumption on the restriction map shows that

$$\|f\|_{M_{B_p(\mathcal{T}_n)}} \le C \|\varphi\|_{M_{B_p(\mathbb{B}_n)}},$$

thus completing the proof of (7.17).

REMARK 7.13. It would be desirable to extend the conclusion of the previous remark to all $1 < p < \infty$ and $n > 1$, or at least to the case $1 < p < 2 + \frac{1}{n-1}$ where we have the equivalence of Carleson embeddings with the tree condition. The argument above breaks down for higher order differences since the analogue of (7.15) fails to hold. In fact, the restrictions of linear functions on the ball fail to belong to $B_p(\mathcal{T}_n)$ for $p \le 2n$, $n \ge 2$ (on the other hand, we showed above that the analogous linear functions $f(\alpha) = 2^{-d(\alpha)}$ on the tree belong to $B_{p,2}(\mathcal{T}_n)$ for all $1 < p < \infty$). Indeed, if $f(z) = z_1$, then for most $\alpha \in \mathcal{T}_n$, in particular for those α at a distance at least $c > 0$ from the complex line $\mathbb{C}(1, 0, ..., 0)$, we have

$$\sum_{\beta \in \mathcal{C}(\alpha)} |f(\beta) - f(\alpha)| = \sum_{\beta \in \mathcal{C}(\alpha)} |\beta_1 - \alpha_1| \approx e^{-d(\alpha)\theta},$$

where $\mathcal{C}(\alpha)$ denotes the set of children of α. By property 4 of Lemma 2.8, we have

$$\#\{\alpha \in \mathcal{T}_n : d(\alpha) = N\} \approx e^{2nN\theta},$$

and thus

$$\begin{aligned}
\|f\|_{B_p(\mathcal{T}_n)}^p &= \|f\|_{B_{p,1}(\mathcal{T}_n)}^p \\
&\ge c \sum_{\alpha \in \mathcal{T}_n} \left(\sum_{\beta \in \mathcal{C}(\alpha)} |f(\beta) - f(\alpha)| \right)^p \\
&\ge c \sum_{\alpha \in \mathcal{T}_n} \left(e^{-d(\alpha)\theta} \right)^p \approx \sum_{N=1}^{\infty} \sum_{\alpha \in \mathcal{T}_n : d(\alpha) = N} e^{-pN\theta} \\
&\approx \sum_{N=1}^{\infty} e^{2nN\theta} e^{-pN\theta} = \infty
\end{aligned}$$

if $2n - p \ge 0$.

What is needed now is a definition of Besov space on a tree that involves complex structure sufficient for higher order differences, or "derivatives", to be properly defined. This is introduced in the following subsection.

7.3. Structured trees. In this subsection we introduce an alternative definition of $B_{p,1}(\mathcal{T}_n)$ that is better adapted for generalization to those higher order differences that reflect the underlying complex structure of the Bergman tree. We must first interpret the fact that for a holomorphic function F in $B_p(\mathbb{B}_n)$, the differences $F(\beta) - F(\alpha)$ are *related* as β ranges over the children of a fixed α; namely they are close to $F'(\alpha)(\beta - \alpha)$. Thus we wish to define in a natural way the notion of a complex derivative f' of a complex-valued function f on the Bergman tree $\mathcal{T}_n$. It is convenient at this point to consider trees more general than $\mathcal{T}_n$, namely those with a complex structure.

An n-dimensional *complex structure* $\mathcal{V}$ on a tree $\mathcal{T}$ is a collection of n-vectors $\mathcal{V} = \{\mathbf{v}_\alpha\}_{\alpha\in\mathcal{T}}$, $\mathbf{v}_\alpha \in \mathbb{C}^n$. We can "immerse" the structured tree $(\mathcal{T},\mathcal{V})$ in $\mathbb{C}^n$ by identifying $\alpha \in \mathcal{T}$ with the point $c(\alpha) = \mathbf{v}_o + \sum_{o<\beta\leq\alpha} \mathbf{v}_\beta \in \mathbb{C}^n$. For example, the standard embedding of the Bergman tree $\mathcal{T}_n$ in the ball arises in this way from the complex structure $\mathcal{V} = \{c_\alpha - c_{A\alpha}\}_{\alpha\in\mathcal{T}_n}$ on $\mathcal{T}_n$. In general however, the map $\alpha \to c(\alpha)$ need not be one-to-one, hence the term "immerse". Additional properties of $\mathcal{V}$ will be required below.

Define the (backward) difference operator $\triangle$ on functions f mapping the tree $\mathcal{T}$ to $\mathbb{C}$ by

$$\triangle f(\alpha) = f(\alpha) - f(A\alpha), \qquad \alpha \in \mathcal{T},$$

where $A\alpha$ denotes the predecessor, or immediate *A*ncestor, of α (we reserve P for projection). We denote the set of children of $\alpha \in \mathcal{T}$ by $\mathcal{C}(\alpha)$. We now assume that $\dim(\mathcal{T})$ is finite, so that in particular, there is an upper bound N for the branching number of the tree, i.e. $\#\mathcal{C}(\alpha) \leq N$ for all $\alpha \in \mathcal{T}_n$. Let $\{\alpha^j\}_{j=1}^{\#\mathcal{C}(\alpha)} = \mathcal{C}(\alpha)$ be an enumeration of the children of α. Then for $\alpha \in \mathcal{T}$ we define the linear map $L_\alpha : \mathbb{C}^n \to \mathbb{C}^N$ by

$$L_\alpha(\mathbf{w}) = (\mathbf{w} \cdot \mathbf{v}_{\alpha^j})_{j=1}^N,$$

with the convention that $\alpha^j = \alpha$ and $\mathbf{v}_{\alpha^j} = 0$ if $\#\mathcal{C}(\alpha) < j \leq N$. We now make the assumption that L_α is *one-to-one* for all $\alpha \in \mathcal{T}$. Then given a complex-valued function f on $\mathcal{T}$ we can define its *complex derivative* $f'(\alpha) \in \mathbb{C}^n$ as follows. Let $\mathcal{P}_\alpha$ denote orthogonal projection of $\mathbb{C}^N$ onto the range of L_α, and let $\mathcal{Q}_\alpha = I - \mathcal{P}_\alpha$. Denote by $\mathcal{D}_\alpha f$ the N-vector of (forward) differences of f:

$$\mathcal{D}_\alpha f = \left(f\left(\alpha^j\right) - f(\alpha)\right)_{j=1}^N = \left(\triangle f\left(\alpha^j\right)\right)_{j=1}^N \in \mathbb{C}^N,$$

where $\triangle f\left(\alpha^j\right) = f(\alpha) - f(\alpha) = 0$ if $\#\mathcal{C}(\alpha) < j \leq N$ by our convention. Then we define

$$f'(\alpha) = L_\alpha^{-1}\mathcal{P}_\alpha(\mathcal{D}_\alpha f),$$

so that

$$\left(f\left(\alpha^j\right) - f(\alpha)\right)_{j=1}^N = (f'(\alpha) \cdot \mathbf{v}_{\alpha^j})_{j=1}^N + \mathcal{Q}_\alpha(\mathcal{D}_\alpha f). \tag{7.18}$$

In the case of the natural complex structure introduced above on the Bergman tree $\mathcal{T}_n$, we have $\mathbf{v}_{\alpha^j} = \alpha^j - \alpha$ and (7.18) is thus an analogue of Taylor's formula of degree one on $\mathcal{T}_n$. We now make this more precise in the special case of the Bergman tree $\mathcal{T}_n$.

For the Bergman tree $\mathcal{T}_n$ we can choose $N = Ce^{2n\theta}$, with $\lambda \leq 1$ and $\theta > 1$ to be chosen below, by property 4 of Lemma 2.8. We also define the difference

$$\triangle \alpha = \alpha - A\alpha, \qquad \alpha \in \mathcal{T}_n,$$

where we identify α with the center c_α of the Bergman kube K_α. With the convention as above, we view the set of differences

$$\mathcal{D}_\alpha f = \left(\triangle\ f\left(\alpha^j\right)\right)_{j=1}^N$$

as a vector of complex numbers of length N, i.e. $\mathcal{D}_\alpha f \in \mathbb{C}^N$. The device of choosing N larger than the branching number at any element of the tree is simply a matter of convenience. We could just as well have worked with $\mathbb{C}^N$ replaced replaced by $\mathbb{C}^{\#\mathcal{C}(\alpha)}$ at each element α, but at the expense of more complicated notation. We also consider the corresponding family of differences

$$\left(\triangle\ \alpha^j\right)_{j=1}^N,$$

as a vector of points in $\mathbb{C}^n$ of length N, i.e. in $\left(\mathbb{C}^n\right)^N$.

The linear map L_α defined above sends $\mathbf{v} \in \mathbb{C}^n$ to the point

$$L_\alpha \mathbf{v} = \left(\mathbf{v} \cdot \left(\alpha^j - \alpha\right)\right)_{j=1}^N \in \mathbb{C}^N.$$

Note that for the Bergman tree, the map L_α is one-to-one since the collection of n-vectors $\left(\alpha^j - \alpha\right)_{j=1}^N$ has rank n if $\theta > 1$ is chosen large enough. Recall that $\mathcal{P}_\alpha$ is the orthogonal projection of $\mathbb{C}^N$ onto the range of L_α (which has dimension n since L_α is one-to-one) and $\mathcal{Q}_\alpha = I - \mathcal{P}_\alpha$. The complex derivative $f'(\alpha)$ of f at the point α is then the unique vector $\mathbf{v}$ such that

$$L_\alpha \mathbf{v} = \mathcal{P}_\alpha\left(\mathcal{D}_\alpha f\right).$$

Thus we have

$$L_\alpha f'(\alpha) = \mathcal{P}_\alpha\left(\mathcal{D}_\alpha f\right) = \left(f'(\alpha) \cdot \left(\alpha^j - \alpha\right)\right)_{j=1}^N.$$

Now denote the radial and tangential components of $f'(\alpha)$ by $f'(\alpha) P_\alpha$ and $f'(\alpha) Q_\alpha$ respectively, where $P_\alpha z = \frac{z \cdot \overline{\alpha}}{|\alpha|^2}\alpha$ as in (2.1), and $Q_\alpha = I - P_\alpha$. Here we are viewing $f'(\alpha)$ as belonging to the space $\mathcal{L}\left(\mathbb{C}^n, \mathbb{C}\right)$ of linear maps from $\mathbb{C}^n$ to $\mathbb{C}$, and $\mathcal{P}_\alpha, \mathcal{Q}_\alpha$ as belonging to the corresponding space of linear maps $\mathcal{L}\left(\mathbb{C}^n, \mathcal{L}\left(\mathbb{C}^n, \mathbb{C}\right)\right)$. Thus we have decomposed the difference set $\mathcal{D}_\alpha f$ as

$$\begin{aligned} (7.19)\quad \mathcal{D}_\alpha f &= \left(\triangle\ f\left(\alpha^j\right)\right)_{j=1}^N \\ &= \left(f'(\alpha) P_\alpha \cdot \left(\alpha^j - \alpha\right)\right)_{j=1}^N + \left(f'(\alpha) Q_\alpha \cdot \left(\alpha^j - \alpha\right)\right)_{j=1}^N + \mathcal{Q}_\alpha\left(\mathcal{D}_\alpha f\right). \end{aligned}$$

In our alternative definition of the Besov space $B_{p,1}\left(\mathcal{T}_n\right)$, we weight the various components of this decomposition in accordance with the complex structure the Bergman tree $\mathcal{T}_n$ inherits from its embedding in the unit ball $\mathbb{B}_n$.

DEFINITION 7.14. *For $1 < p < \infty$, define the* holomorphic *Besov space $HB_{p,1}\left(\mathcal{T}_n\right)$ on $\mathcal{T}_n$ to consist of all complex-valued sequences $f = \{f(\alpha)\}_{\alpha \in \mathcal{T}_n}$ such that*

$$\begin{aligned} \|f\|_{HB_{p,1}(\mathcal{T}_n)}^p = |f(o)|^p &+ \sum_{\alpha \in \mathcal{T}_n} \left|e^{-2d(\alpha)\theta} f'(\alpha) P_\alpha + e^{-d(\alpha)\theta} f'(\alpha) Q_\alpha\right|^p \\ &+ \sum_{\alpha \in \mathcal{T}_n} \left|\mathcal{Q}_\alpha \mathcal{D}_\alpha f\right|^p \\ &< \infty. \end{aligned}$$

REMARK 7.15. The expression

$$\left|e^{-2d(\alpha)\theta}\mathbf{v}P_\alpha + e^{-d(\alpha)\theta}\mathbf{v}Q_\alpha\right|$$

is the tree analogue of

$$\sqrt{\left(B\left(\alpha\right)\mathbf{v}\right)\cdot\overline{\mathbf{v}}} = \sqrt{\left(\frac{P_\alpha\mathbf{v}}{\left(1-\left|\alpha\right|^2\right)^2} + \frac{Q_\alpha\mathbf{v}}{1-\left|\alpha\right|^2}\right)\cdot\overline{\mathbf{v}}},$$

where $B\left(z\right) = \frac{1}{n+1}\frac{\partial^2}{\partial\overline{z_i}\partial z_j}\log\frac{1}{\left(1-\left|z\right|^2\right)^{n+1}}$ is the Bergman Riemannian metric on tangent vectors $\mathbf{v}$ at the point z in $\mathbb{B}_n$ that leads to the Bergman distance β - see (2.14) and Chapter 1.5 of [**Zhu**].

The equivalence of the norms $\left\|\cdot\right\|_{B_{p,1}(\mathcal{T}_n)}$ and $\left\|\cdot\right\|_{HB_{p,1}(\mathcal{T}_n)}$ for $1<p<\infty$ and θ sufficiently large follows from the next result.

DEFINITION 7.16. *For a vector* $\mathbf{v}\in\mathbb{C}^n$ *and* $\alpha\in\mathcal{T}_n$, *let*

$$\left|\mathbf{v}\right|_\alpha = \left|e^{-2d(\alpha)\theta}\mathbf{v}P_\alpha + e^{-d(\alpha)\theta}\mathbf{v}Q_\alpha\right| = \sqrt{\left(B\left(\alpha\right)\mathbf{v}\right)\cdot\overline{\mathbf{v}}}.$$

LEMMA 7.17. *For* θ *sufficiently large in the construction of the Bergman tree* $\mathcal{T}_n$, *and for all* $1<p<\infty$, *we have*

$$\left(\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{D}_\alpha f\right|^p\right)^{\frac{1}{p}} \approx \left(\sum_{\alpha\in\mathcal{T}_n}\left|f'\left(\alpha\right)\right|_\alpha^p\right)^{\frac{1}{p}} + \left(\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_\alpha\mathcal{D}_\alpha f\right|^p\right)^{\frac{1}{p}}.$$

PROOF. Since $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ (respectively P_α and Q_α) are orthogonal projections on $\mathbb{C}^N$ (respectively $\mathbb{C}^n$), we have

$$\begin{aligned}
\left|\mathcal{D}_\alpha f\right|^2 &= \left|\mathcal{P}_\alpha\mathcal{D}_\alpha f\right|^2 + \left|\mathcal{Q}_\alpha\mathcal{D}_\alpha f\right|^2\\
&= \left|L_\alpha f'\left(\alpha\right)\right|^2 + \left|\mathcal{Q}_\alpha\mathcal{D}_\alpha f\right|^2\\
&\approx \left|e^{-2d(\alpha)\theta}f'\left(\alpha\right)P_\alpha\right|^2 + \left|e^{-d(\alpha)\theta}f'\left(\alpha\right)Q_\alpha\right|^2 + \left|\mathcal{Q}_\alpha\mathcal{D}_\alpha f\right|^2\\
&= \left|f'\left(\alpha\right)\right|_\alpha^2 + \left|\mathcal{Q}_\alpha\mathcal{D}_\alpha f\right|^2,
\end{aligned}$$

where the third line follows from

$$\begin{aligned}
\left|\mathcal{P}_\alpha\mathcal{D}_\alpha f\right|^2 &= \left|L_\alpha f'\left(\alpha\right)\right|^2\\
&= \left|\left\{f'\left(\alpha\right)P_\alpha\cdot P_\alpha\left(\alpha^j-\alpha\right) + f'\left(\alpha\right)Q_\alpha\cdot Q_\alpha\left(\alpha^j-\alpha\right)\right\}_{j=1}^N\right|^2\\
&\approx \left|e^{-2d(\alpha)\theta}f'\left(\alpha\right)P_\alpha\right|^2 + \left|e^{-d(\alpha)\theta}f'\left(\alpha\right)Q_\alpha\right|^2.
\end{aligned}$$

To see this last equivalence, we note using (2.14) that both

$$\sum_{j=1}^N\left|P_\alpha\left(\alpha^j-\alpha\right)\right| \geq ce^{-2d(\alpha)\theta}, \tag{7.20}$$

$$\sum_{j=1}^N\left|Q_\alpha\left(\alpha^j-\alpha\right)\right| \geq ce^{-d(\alpha)\theta}.$$

The first inequality is obvious. The second inequality follows if $\theta > 1$ is chosen sufficiently large, since the set of projections onto the sphere $\mathcal{S}_{d(\alpha)\theta}$ of the children $\mathcal{C}(\alpha)$ is $e^{-2\theta}C_2$-dense in the qube $Q_j^{d(\alpha)} = K_\alpha \cap \mathcal{S}_{d(\alpha)\theta}$ corresponding to K_α. Indeed, by property 4 of Proposition 2.8, there are roughly $e^{2n\theta}$ children of α whose projections onto the Bergman sphere $S_{d(\alpha)\theta}$ all lie in $Q_j^{d(\alpha)}$. Since the Bergman distance β is preserved by automorphisms, we see upon mapping matters to the origin that these projections are roughly $e^{-2\theta}$-dense in the Bergman distance. With this established for $e^{-2\theta}$ sufficiently small, we now see that the vectors $\left\{Q_\alpha\left(\alpha^j - \alpha\right)\right\}_{j=1}^N$ are sufficiently well distributed that (7.20) holds uniformly in α.

7.3.1. *An abstract approach.* The purpose of this short subsubsection is to illustrate the flexibility of defining spaces via "derivatives" on trees, by including a variety of classical spaces within a generalization of this framework. We will not use the material here in the sequel.

Given a tree $\mathcal{T}$ with branching number bounded by N, we can more generally than above, suppose that we are given an m-dimensional complex vector space W, and for each $\alpha \in \mathcal{T}$, Hilbert space norms $[\cdot]_\alpha$ and $\{\cdot\}_\alpha$ on W and $\mathbb{C}^N$ respectively, and a one-to-one linear map L_α from W to $\mathbb{C}^N$. Then we can define a derivative $f'(\alpha) \in W$ by

$$f'(\alpha) = L_\alpha^{-1}\mathcal{P}_\alpha(\mathcal{D}_\alpha f),$$

where $\mathcal{D}_\alpha f$ is the (forward) difference set defined as above, and $\mathcal{P}_\alpha$ is orthogonal projection onto the range of L_α. With $\mathcal{Q}_\alpha = I - \mathcal{P}_\alpha$, define a norm on $\mathbb{C}^N$ by

$$|w|_\alpha^2 = \left[L_\alpha^{-1}\mathcal{P}_\alpha(w)\right]_\alpha^2 + \{\mathcal{Q}_\alpha(w)\}_\alpha^2.$$

Then we define a Besov space norm $\|f\|_{B_p(\mathcal{T})}$ by

$$\|f\|_{B_p(\mathcal{T})}^p = |f(o)|^p + \sum_{\alpha \in \mathcal{T}} |f'(\alpha)|_\alpha^p.$$

In the case $\mathcal{T} = \mathcal{T}_n$ and $W = \mathbb{C}^n$, with L_α defined as above and

$$\begin{aligned} [\mathbf{v}]_\alpha &= |\mathbf{v}|_\alpha = \left|e^{-2d(\alpha)\theta}\mathbf{v}P_\alpha + e^{-d(\alpha)\theta}\mathbf{v}Q_\alpha\right|, \qquad \mathbf{v} \in \mathbb{C}^n, \\ \{w\}_\alpha &= |w|, \qquad w \in \mathbb{C}^N, \end{aligned}$$

we obtain that $B_p(\mathcal{T})$ is the holomorphic Besov space $HB_{p,1}(\mathcal{T}_n)$ defined above. If however we take $W = \mathbb{C}^N$, $L_\alpha = I$ and the norms $[\cdot]_\alpha$ and $\{\cdot\}_\alpha$ to be the Euclidean norm on $\mathbb{C}^N$, then $B_p(\mathcal{T})$ is the abstract space $B_{p,1}(\mathcal{T})$ defined earlier.

For another example, suppose that $\mathcal{T}$ is a homogeneous tree with branching number N. Take W to be the orthogonal complement in $\mathbb{C}^N$ of the one-dimensional subspace $\mathbb{C}(1,1,...,1)$ generated by $(1,1,...,1)$, and let $[\cdot]_\alpha$ be the restriction of the Euclidean norm to W. Let L_α be the natural inclusion of W into $\mathbb{C}^N$. Finally, let $\{\cdot\}_\alpha$ be the trivial norm on $\mathbb{C}^N$ that is infinite on all nonzero vectors and vanishes on the zero vector. Then $B_p(\mathcal{T})$ is the martingale of all ℓ^p functions on the tree $\mathcal{T}$ satisfying

$$f(\alpha) = \frac{1}{N}\sum_{\beta \in \mathcal{C}(\alpha)} f(\beta), \qquad \alpha \in \mathcal{T}.$$

For yet another example, take $\mathcal{T} = \mathcal{T}_1$, $W = \mathbb{C}$, $L_\alpha v = \left(v\left(\alpha^j - \alpha\right)\right)_{j=1}^2$ and

$$[v]_\alpha = \left|e^{-3d(\alpha)\theta} v\right|, \qquad v \in \mathbb{C},$$
$$\{w\}_\alpha = |w|, \qquad w \in \mathbb{C}^2.$$

The resulting space $B_2(\mathcal{T})$ normed by

$$\|f\|^2_{B_2(\mathcal{T})} = |f(o)|^2 + \sum_{\alpha \in \mathcal{T}_1} \left|e^{-3d(\alpha)\theta} f'(\alpha)\right|^2 + \sum_{\alpha \in \mathcal{T}_1} |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^2$$

is a tree model for the Hardy space

$$H^2(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) : |F(0)|^2 + \int_{\mathbb{D}} \left| \left(1 - |z|^2\right)^{\frac{3}{2}} F'(z) \right|^2 \frac{dz}{\left(1 - |z|^2\right)^2} < \infty \right\}$$

on the unit disk. Note that $|\mathcal{D}_\alpha f| \approx \left|e^{-2d(\alpha)\theta} f'(\alpha)\right| + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|$, so that the tree model $B_2(\mathcal{T})$ for the Hardy space permits the differences of f to grow rapidly in the holomorphic direction, and is thus much larger than the abstract space $B_{2,1}(\mathcal{T}_1)$. Finally, by including higher order derivatives, this example can be extended to higher dimensions to provide a tree model for the space H^2_n of Arveson [**Arv**], that consists of all $F \in H(\mathbb{B}_n)$ whose radial derivative $\mathcal{R}^m F$ satisfies

$$\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^m \mathcal{R}^m F(z) \right|^2 \frac{dz}{\left(1 - |z|^2\right)^n} = \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m + \frac{1}{2}} \mathcal{R}^m F(z) \right|^2 d\lambda_n(z) < \infty,$$

where $m = 1 + \left[\frac{n-1}{2}\right]$ (the Hardy space $H^2(\mathbb{D})$ on the disk is the case $m = n = 1$).

In the next section we will construct holomorphic Besov spaces on Bergman trees $\mathcal{T}_n$ that model the Besov spaces $B_p(\mathbb{B}_n)$ on the ball.

8. Holomorphic Besov spaces on Bergman trees

Our goal in this lengthy section is to obtain a definition of a holomorphic Besov space $HB_{p,m}(\mathcal{T}_n)$ on the Bergman tree $\mathcal{T}_n$ so that the restriction map (as in Corollary 7.10) is bounded from $B_p(\mathbb{B}_n)$ to $HB_{p,m}(\mathcal{T}_n)$ in the range $p > \frac{2n}{m}$, while retaining as many of the properties of the abstract Besov space $B_{p,m}(\mathcal{T}_n)$ as possible. One essential property we wish to retain is that Carleson measures for $HB_{p,m}(\mathcal{T}_n)$ be characterized by the tree condition (3.2), as that is the condition needed to prove the sufficiency implication for multiplier interpolation on the ball. Another essential property is an appropriate positivity of "derivatives" of reproducing kernels for $HB_{p,m}(\mathcal{T}_n)$, as that is needed to prove the necessity of the tree condition for multiplier interpolation on the ball in the difficult range $1 + \frac{1}{n-1} \leq p < 2$. See the discussion at the end of Section 5.

Recall that in the previous section we showed in Corollary 7.10 that the restriction map is bounded from $B_p(\mathbb{B}_n)$ to $B_{p,1}(\mathcal{T}_n)$ in the range $p > \widehat{n} = \frac{2n}{1}$. Of course the Carleson measures for $B_{p,1}(\mathcal{T}_n)$ are characterized by the tree condition, and the first order difference of the reproducing kernel for $B_{p,1}(\mathcal{T}_n)$ is nonnegative (the analogues of these latter two properties actually hold for *all* $B_{p,m}(\mathcal{T}_n)$ by Lemma 7.2 and (7.3)). This demonstrates that the abstract Besov space $B_{p,1}(\mathcal{T}_n)$

has the properties desired of our holomorphic Besov space for $p > 2n$, and in view of Lemma 7.17, we have

$$HB_{p,1}(\mathcal{T}_n) = B_{p,1}(\mathcal{T}_n)\,. \tag{8.1}$$

However, the proof of the restriction theorem given in Corollary 7.10 is not amenable to generalization to higher order differences, and in fact the abstract Besov spaces $B_{p,m}(\mathcal{T}_n)$ on trees do not capture the higher order derivatives of holomorphic functions in the ball. Indeed, higher order tree differences vanish on appropriate polynomial functions of $r^{-d(\alpha)}$ on the tree, but *not* on the restrictions to the tree of polynomials on the ball, even though the corresponding derivatives are identically zero on the ball. In particular, recall from Remark 7.13 that linear functions on the ball do not restrict to $B_{p,m}(\mathcal{T}_n)$ for $p \leq 2n$ for any $m \geq 1$. Thus we now proceed, as we did for $m = 1$ in our definition of $HB_{p,1}(\mathcal{T}_n)$ above, to model our definition of $HB_{p,m}(\mathcal{T}_n)$ after the almost invariant holomorphic derivatives D_a^m used in the seminorms (6.7) for $B_p(\mathbb{B}_n)$.

We begin with the holomorphic Besov space $HB_{p,1}(\mathcal{T}_n)$ already defined in Subsection 7.3, and derive its reproducing kernels relative to the duality pairing induced by the norm $\|\cdot\|_{HB_{p,1}(\mathcal{T}_n)}$, along with their positivity properties. In preparation for an inductive definition of the higher order Besov spaces $HB_{p,m}(\mathcal{T}_n)$, we must also derive the analogous theory of the Besov spaces $HB_{p,1}^{(k)}(\mathcal{T}_n)$ of k-tensors defined on $\mathcal{T}_n$, as we will view higher order complex tree derivatives $f^{(k)}(\alpha)$ as tensor-valued functions on the tree. To expedite this process, it is advantageous to consider first the order zero case, and develop the required tensor apparatus for ℓ^p spaces, the order zero analogue of a holomorphic Besov space. Then we proceed to define $HB_{p,m}(\mathcal{T}_n)$ inductively for $m \geq 2$ and establish the appropriate positivity properties of their reproducing kernels, which will require a careful choice of the structural constants λ and θ in the construction of the Bergman tree $\mathcal{T}_n$, along with an additional modification of the centers of the Bergman balls. Finally, we establish for these spaces the Carleson measure theorem and the restriction theorem, and then complete the proof of the multiplier interpolation loop for $1 < p < 2 + \frac{1}{n-1}$, that was left open at the end of Section 5.

8.1. The order zero and order one holomorphic Besov spaces. Recall that for $1 < p < \infty$ we defined the order 1 holomorphic Besov space $HB_{p,1}(\mathcal{T}_n)$ on $\mathcal{T}_n$ in Definition 7.14 to consist of all complex-valued sequences $f = \{f(\alpha)\}_{\alpha\in\mathcal{T}_n}$ such that

$$\begin{aligned}\|f\|_{HB_{p,1}(\mathcal{T}_n)}^p = |f(o)|^p &+ \sum_{\alpha\in\mathcal{T}_n}\left|r^{-d(\alpha)}f'(\alpha)P_\alpha + r^{-\frac{d(\alpha)}{2}}f'(\alpha)Q_\alpha\right|^p \\ &+ \sum_{\alpha\in\mathcal{T}_n}|\mathcal{Q}_\alpha\mathcal{D}_\alpha f|^p \\ < \infty,&\end{aligned}$$

where we have written

$$r = e^{2\theta} \tag{8.2}$$

for convenience, so that by (2.14),

$$1 - |\alpha|^2 \approx e^{-2\beta(0,\alpha)} \approx e^{-2\theta d(\alpha)} = r^{-d(\alpha)}. \tag{8.3}$$

In comparing this definition with Definition 7.1 for the real Besov space $B_{p,1}(\mathcal{T}_n)$ on the Bergman tree $\mathcal{T}_n$,

$$\|f\|^p_{B_{p,1}(\mathcal{T}_n)} = |f(o)|^p + \sum_{\alpha \in \mathcal{T}_n} |\mathcal{D}_\alpha f|^p < \infty,$$

recall that the set of differences $\mathcal{D}_\alpha f$ can be written as the linear sum of the two pieces $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ and $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$; the first piece $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ lying in the range of L_α (the holomorphic part), and the second piece $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$ orthogonal to the range of L_α. The first piece $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ can be further decomposed using the identities

$$\begin{aligned} L_\alpha f'(\alpha) &= \mathcal{P}_\alpha \mathcal{D}_\alpha f, \\ f'(\alpha) &= L_\alpha^{-1} \mathcal{P}_\alpha \mathcal{D}_\alpha f, \end{aligned}$$

where the second one follows since L_α is one-to-one and $\mathcal{P}_\alpha$ is orthogonal projection onto the range of L_α. We then decompose $f'(\alpha)$ as $f'(\alpha) P_\alpha + f'(\alpha) Q_\alpha$. Now L_α has the $N \times n$ matrix representation $\left[\alpha^j_k - \alpha_k\right]_{1 \leq j \leq N, 1 \leq k \leq n}$ where $\alpha^j = \left(\alpha^j_k\right)^n_{k=1}$ and $\alpha = (\alpha_k)^n_{k=1}$, and so resembles the nonisotropic linear operator

$$\mathbf{R}^{-d(\alpha)} \equiv r^{-d(\alpha)} P_\alpha + r^{-\frac{d(\alpha)}{2}} Q_\alpha,$$

whose action on a vector $v \in \pounds(\mathbb{C}^n, \mathbb{C})$ is given by $\mathbf{R}^{-d(\alpha)} v \equiv r^{-d(\alpha)} v P_\alpha + r^{-\frac{d(\alpha)}{2}} v Q_\alpha$. Thus we formally have that

$$\begin{aligned} \mathcal{D}_\alpha f &= L_\alpha f'(\alpha) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f \\ &\sim \mathbf{R}^{-d(\alpha)} f'(\alpha) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f. \end{aligned}$$

We actually proved that $|\mathcal{D}_\alpha f| \approx \left|\mathbf{R}^{-d(\alpha)} f'(\alpha)\right| + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|$ in the course of proving Lemma 7.17.

Thus locally we measure the holomorphic parts $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ of the differences $\mathcal{D}_\alpha f$ by the Bergman Riemannian metric, where the radial directions $f'(\alpha) P_\alpha$ are weighted by $r^{-d(\alpha)}$, and the tangential directions $f'(\alpha) Q_\alpha$ weighted by $\sqrt{r^{-d(\alpha)}} = r^{-\frac{d(\alpha)}{2}}$. This is analogous to the definition (6.1) of the almost invariant holomorphic derivative D_α given in the ball by

$$D_a f(z) = -f'(z) \left\{ \left(1 - |a|^2\right) P_a + \left(1 - |a|^2\right)^{\frac{1}{2}} Q_a \right\}, \qquad a \in \mathbb{B}_n.$$

We then take the $\ell^p(\mathcal{T}_n)$ norm of these local measures. We measure the antiholomorphic parts $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$ of the differences $\mathcal{D} f(\alpha)$ by the $\ell^p(\mathcal{T})$-norm. We now consider reproducing kernels associated to these norms.

8.1.1. *Reproducing kernels.* The reproducing kernel k_α for the space $\ell^p(\mathcal{T}_n)$ is trivial: $k_\alpha(\gamma) = \chi_{\{\alpha\}}(\gamma)$, the delta function at α. We must work harder to obtain the reproducing kernel for $HB_{p,1}(\mathcal{T}_n)$. We first observe that we can recover a function $f \in HB_{p,1}(\mathcal{T}_n)$ from its differences

$$\begin{aligned} \mathcal{D}_\alpha f &= \mathcal{P}_\alpha \mathcal{D}_\alpha f + \mathcal{Q}_\alpha \mathcal{D}_\alpha f \\ &= L_\alpha f'(\alpha) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f \end{aligned}$$

and hence also from its derivatives and antiholomorphic differences $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$. In order to give an explicit formula, we write

$$\mathcal{P}_\alpha \mathcal{D}_\alpha f = \left\{\mathcal{P}_\alpha \mathcal{D}_\alpha f\left(\alpha^j\right)\right\}^N_{j=1} \text{ and } \mathcal{Q}_\alpha \mathcal{D}_\alpha f = \left\{\mathcal{Q}_\alpha \mathcal{D}_\alpha f\left(\alpha^j\right)\right\}^N_{j=1},$$

so that $\triangle f\left(\alpha^{j}\right)=\mathcal{P}_{\alpha}\mathcal{D}_{\alpha}f\left(\alpha^{j}\right)+\mathcal{Q}_{\alpha}\mathcal{D}_{\alpha}f\left(\alpha^{j}\right)$, where $\alpha=A\alpha^{j}$. Thus $\mathcal{P}_{\alpha}\mathcal{D}_{\alpha}f\left(\alpha^{j}\right)$ is the j^{th} component of the N-vector $\mathcal{P}_{\alpha}\mathcal{D}_{\alpha}f=L_{\alpha}f'\left(\alpha\right)$, i.e.

$$\mathcal{P}_{A\beta}\mathcal{D}_{A\beta}f\left(\beta\right)=f'\left(A\beta\right)\cdot\left(\beta-A\beta\right),$$

and we have the tree version of the Taylor expansion of order 1 at the point $A\beta$;

$$f\left(\beta\right)=f\left(A\beta\right)+f'\left(A\beta\right)\cdot\left(\beta-A\beta\right)+\mathcal{Q}_{A\beta}\mathcal{D}_{A\beta}f\left(\beta\right).$$

For $\alpha\in\mathcal{T}_n$ we write the geodesic to α as $[o,\alpha]=\{o,\alpha_1,\alpha_2,...,\alpha_m\}$ (note the different use of the terminology α_j here). An explicit formula for $f\left(\alpha\right)$ is now given by

$$\begin{aligned}(8.4)\qquad f\left(\alpha\right)=f\left(\alpha_m\right)&=\sum_{k=1}^{m}\triangle f\left(\alpha_k\right)+f\left(o\right)\\&=f\left(o\right)+\sum_{k=0}^{m}\mathcal{P}_{\alpha_{k-1}}\mathcal{D}_{\alpha_{k-1}}f\left(\alpha_k\right)+\sum_{k=0}^{m}\mathcal{Q}_{\alpha_{k-1}}\mathcal{D}_{\alpha_{k-1}}f\left(\alpha_k\right)\\&=f\left(o\right)+\sum_{k=0}^{m}f'\left(\alpha_{k-1}\right)\cdot\left(\alpha_k-\alpha_{k-1}\right)+\sum_{k=0}^{m}\mathcal{Q}_{\alpha_{k-1}}\mathcal{D}_{\alpha_{k-1}}f\left(\alpha_k\right).\end{aligned}$$

We can rewrite this as

$$(8.5)\qquad f\left(\alpha\right)=f\left(o\right)+\sum_{\gamma<\alpha}f'\left(\gamma\right)\cdot\left(\gamma_\alpha-\gamma\right)+\sum_{\gamma<\alpha}\mathcal{Q}_{\gamma}\mathcal{D}_{\gamma}f\left(\gamma_\alpha\right),$$

where γ_α denotes the child of γ lying on the geodesic $[o,\alpha]$, so that $\gamma_\alpha=\alpha_k$ if $\gamma=\alpha_{k-1}$. Note that an immediate consequence of (8.5) is the inequality

$$(8.6)\qquad \left|f\left(\alpha\right)\right|\leq C\left\|f\right\|_{HB_{p,1}(\mathcal{T}_n)}d\left(\alpha\right)^{\frac{1}{p'}}.$$

Indeed, using $I=P_\gamma^2+Q_\gamma^2$ we have the formula

$$(8.7)\qquad f'\left(\gamma\right)\cdot\left(\gamma_\alpha-\gamma\right)=f'\left(\gamma\right)P_\gamma\cdot P_\gamma\left(\gamma_\alpha-\gamma\right)+f'\left(\gamma\right)Q_\gamma\cdot Q_\gamma\left(\gamma_\alpha-\gamma\right),$$

and hence the estimate

$$\begin{aligned}\left|f'\left(\gamma\right)\cdot\left(\gamma_\alpha-\gamma\right)\right|&\leq\left|f'\left(\gamma\right)P_\gamma\cdot P_\gamma\left(\gamma_\alpha-\gamma\right)\right|+\left|f'\left(\gamma\right)Q_\gamma\cdot Q_\gamma\left(\gamma_\alpha-\gamma\right)\right|\\&\leq\left|f'\left(\gamma\right)P_\gamma\right|r^{-d(\gamma)}+\left|f'\left(\gamma\right)Q_\gamma\right|r^{-\frac{d(\gamma)}{2}}\\&\approx\left|r^{-d(\gamma)}f'\left(\gamma\right)P_\gamma+r^{-\frac{d(\gamma)}{2}}f'\left(\gamma\right)Q_\gamma\right|=\left|f'\left(\gamma\right)\right|_\gamma.\end{aligned}$$

Plugging this estimate into (8.5) and using Holder's inequality yields (8.6).

Now consider the case $p=2$. Since P_α and Q_α are orthogonal projections with vanishing product $P_\gamma Q_\gamma=P_\gamma\left(I-P_\gamma\right)=P_\gamma-P_\gamma^2=0$, polarization shows that the inner product $\langle\langle f,g\rangle\rangle_1$ for the Hilbert space $HB_{2,1}\left(\mathcal{T}_n\right)$, with norm $\left\|\cdot\right\|_{HB_{2,1}(\mathcal{T}_n)}$ as in Definition 7.14, is given by

(8.8)

$$\begin{aligned}\langle\langle f,g\rangle\rangle_1&=f\left(o\right)\overline{g\left(o\right)}\\&\quad+\sum_{\alpha\in\mathcal{T}_n}\left\{r^{-2d(\alpha)}f'\left(\alpha\right)P_\alpha\cdot\overline{g'\left(\alpha\right)P_\alpha}+r^{-d(\alpha)}f'\left(\alpha\right)Q_\alpha\cdot\overline{g'\left(\alpha\right)Q_\alpha}\right\}\\&\quad+\sum_{\alpha\in\mathcal{T}_n}\mathcal{Q}_\alpha\mathcal{D}_\alpha f\cdot\overline{\mathcal{Q}_\alpha\mathcal{D}_\alpha g}.\end{aligned}$$

Note that the dot product in the second line of (8.8) is n-dimensional, while that in the third line is N-dimensional.

By inequality (8.6), point evaluation at $\alpha \in \mathcal{T}_n$ is a continuous linear functional on $HB_{2,1}(\mathcal{T}_n)$. Thus there is a unique $k_\alpha^{(1)} \in HB_{2,1}(\mathcal{T}_n)$ such that $f(\alpha) = \left\langle\left\langle f, k_\alpha^{(1)} \right\rangle\right\rangle_1$, which written out explicitly is

(8.9)
$$\begin{aligned} f(\alpha) &= f(o)\,\overline{k_\alpha^{(1)}(o)} \\ &\quad + \sum_{\gamma \in \mathcal{T}_n} \left\{ r^{-2d(\gamma)} f'(\gamma) P_\gamma \cdot \overline{k_\alpha^{(1)\prime}(\gamma) P_\gamma} + r^{-d(\gamma)} f'(\gamma) Q_\gamma \cdot \overline{k_\alpha^{(1)\prime}(\gamma) Q_\gamma} \right\} \\ &\quad + \sum_{\gamma \in \mathcal{T}_n} \mathcal{Q}_\gamma \mathcal{D}_\gamma f \cdot \overline{\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}}. \end{aligned}$$

Just as for the real Besov spaces $B_{p,1}(\mathcal{T}_n)$, we can construct by hand a function $k_\alpha^{(1)} \in HB_{p,1}(\mathcal{T}_n)$ so that the right hand side of (8.5) matches the expression in (8.9).

Indeed, for $\alpha \in \mathcal{T}_n$ with geodesic $[o, \alpha] = \{o, \alpha_1, \alpha_2, ..., \alpha_m = \alpha\}$, we choose

$$k_\alpha^{(1)\prime}(\gamma) = \begin{cases} r^{2d(\gamma)}\overline{P_\gamma(\gamma_\alpha - \gamma)} + r^{d(\gamma)}\overline{Q_\gamma(\gamma_\alpha - \gamma)} & \text{if} \quad \gamma = \alpha_{k-1} \\ 0 & \text{if} \quad \gamma \notin [o, \alpha] \end{cases}. \tag{8.10}$$

Using the identity (8.7), we then have

(8.11)
$$\begin{aligned} & f'(\gamma) \cdot (\gamma_\alpha - \gamma) \\ &= f'(\gamma) P_\gamma \cdot P_\gamma(\gamma_\alpha - \gamma) + f'(\gamma) Q_\gamma \cdot Q_\gamma(\gamma_\alpha - \gamma) \\ &= r^{-2d(\gamma)} f'(\gamma) P_\gamma \cdot r^{2d(\gamma)} P_\gamma(\gamma_\alpha - \gamma) + r^{-d(\gamma)} f'(\gamma) Q_\gamma \cdot r^{d(\gamma)} Q_\gamma(\gamma_\alpha - \gamma) \\ &= r^{-2d(\gamma)} f'(\gamma) P_\gamma \cdot \overline{k_\alpha^{(1)\prime}(\gamma) P_\gamma} + r^{-d(\gamma)} f'(\gamma) Q_\gamma \cdot \overline{k_\alpha^{(1)\prime}(\gamma) Q_\gamma}, \end{aligned}$$

upon applying the formulas

$$P_\gamma \overline{z} = \overline{z P_\gamma}, \qquad Q_\gamma \overline{z} = \overline{z Q_\gamma}, \tag{8.12}$$

to (8.10). Indeed, (8.12) yields

$$\begin{aligned} \overline{k_\alpha^{(1)\prime}(\gamma) P_\gamma} &= P_\gamma \overline{k_\alpha^{(1)\prime}(\gamma)} \\ &= P_\gamma \left\{ r^{2d(\gamma)} P_\gamma(\gamma_\alpha - \gamma) + r^{d(\gamma)} Q_\gamma(\gamma_\alpha - \gamma) \right\} \\ &= r^{2d(\gamma)} P_\gamma(\gamma_\alpha - \gamma) \end{aligned}$$

and

$$\begin{aligned} \overline{k_\alpha^{(1)\prime}(\gamma) Q_\gamma} &= Q_\gamma \overline{k_\alpha^{(1)\prime}(\gamma)} \\ &= Q_\gamma \left\{ r^{2d(\gamma)} P_\gamma(\gamma_\alpha - \gamma) + r^{d(\gamma)} Q_\gamma(\gamma_\alpha - \gamma) \right\} \\ &= r^{d(\gamma)} Q_\gamma(\gamma_\alpha - \gamma), \end{aligned}$$

and the formula $P_\gamma \overline{z} = \overline{z P_\gamma}$ in (8.12) follows from

$$\left\langle \overline{z P_\gamma}, w \right\rangle = \langle \overline{z}, P_\gamma w \rangle = \left\langle \overline{P_\gamma w}, z \right\rangle = \overline{\langle P_\gamma w, \overline{z} \rangle} = \overline{\langle w, P_\gamma \overline{z} \rangle} = \langle P_\gamma \overline{z}, w \rangle,$$

where $\langle a, b \rangle = a \cdot \overline{b}$ is the usual inner product in $\mathbb{C}^n$.

The vectors $\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}$ must be chosen to lie in the orthogonal complement of the range of L_γ. Let $M_\gamma = rangeL_\gamma$ and let $M_\gamma^\perp$ be its orthogonal complement. For $\gamma = \alpha_{k-1}$ we must also have the identity

$$\mathcal{Q}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} f \cdot \overline{\mathcal{Q}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} k_\alpha^{(1)}} = \mathcal{Q}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} f\left(\alpha_k\right)$$

or

$$\mathcal{Q}_\gamma \mathcal{D}_\gamma f \cdot \overline{\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}} = \mathcal{Q}_\gamma \mathcal{D}_\gamma f\left(\gamma_\alpha\right), \quad \gamma < \alpha, \tag{8.13}$$

for all f, which implies that for $\gamma < \alpha$, $\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}$ is the unique vector V_γ in $M_\gamma^\perp$ whose inner product gives rise to the "γ_α-coordinate" linear functional. Recall that the γ_α-coordinate points to the path along which α lies. In fact we have

$$\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} = V_\gamma = \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}$$

where $\mathbf{e}_{\gamma_\alpha}$ is the coordinate vector in $\mathbb{C}^N$ in the direction of the γ_α coordinate. Indeed,

$$\langle w, \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha} \rangle = \langle \mathcal{Q}_\gamma w, \mathbf{e}_{\gamma_\alpha} \rangle = \langle w, \mathbf{e}_{\gamma_\alpha} \rangle$$

if $w \in M_\gamma^\perp$ since $\mathcal{Q}_\gamma$ is a projection onto $M_\gamma^\perp$. For future reference we note that

$$\left| \mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} \right| = \left| \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha} \right| \leq 1, \quad \gamma \leq \alpha, \tag{8.14}$$

since $\mathcal{Q}_\gamma$ is an orthogonal projection in $\mathbb{C}^N$. Thus we define

$$\mathcal{Q}_\gamma \mathcal{D} k_\alpha^{(1)}\left(\gamma\right) = \begin{cases} \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}, & \text{if} \quad \gamma \in [o, \alpha) \\ 0 & \text{if} \quad \gamma \notin [o, \alpha) \end{cases}, \tag{8.15}$$

so that

$$\sum_{\gamma \in \mathcal{T}_n} \mathcal{Q}_\gamma \mathcal{D} f\left(\gamma\right) \cdot \overline{\mathcal{Q}_\gamma \mathcal{D} k_\alpha^{(1)}\left(\gamma\right)} = \sum_{k=1}^m \mathcal{Q}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} f\left(\alpha_k\right) = \sum_{\gamma < \alpha} \mathcal{Q}_\gamma \mathcal{D}_\gamma f\left(\gamma_\alpha\right).$$

Finally, we set

$$k_\alpha^{(1)}\left(o\right) = 1 \tag{8.16}$$

so that

$$f\left(o\right) \overline{k_\alpha^{(1)}\left(o\right)} = f\left(o\right).$$

Combining these definitions and observations with (8.4) and (8.9) we obtain

$$f\left(\alpha\right) = \left\langle \left\langle f, k_\alpha^{(1)} \right\rangle \right\rangle_1.$$

Using the fact that the representation given in (8.5),

$$g\left(\alpha\right) = g\left(o\right) + \sum_{\gamma < \alpha} g'\left(\gamma\right) \cdot \left(\gamma_\alpha - \gamma\right) + \sum_{\gamma < \alpha} \mathcal{Q}_\gamma \mathcal{D}_\gamma g\left(\gamma_\alpha\right), \tag{8.17}$$

is uniquely determined by the condition that

$$\mathcal{Q}_\gamma \mathcal{D}_\gamma g\left(\gamma_\alpha\right) \in M_\gamma^\perp, \quad \gamma < \alpha,$$

(since $g'\left(\gamma\right) \cdot \left(\gamma_\alpha - \gamma\right)$ is obviously in M_γ) we thus see that there does indeed exist a unique function $k_\alpha^{(1)}$ satisfying properties (8.10), (8.15) and (8.16). Indeed, if

$\beta \in [0, \alpha]$, then with $g = k_\alpha^{(1)}$ in (8.17) we have

(8.18)

$$\begin{aligned} &k_\alpha^{(1)}(\beta) \\ &= k_\alpha^{(1)}(o) + \sum_{\gamma<\beta} k_\alpha^{(1)\prime}(\gamma) \cdot (\gamma_\beta - \gamma) + \sum_{\gamma<\beta} \mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}(\gamma_\beta) \\ &= 1 + \sum_{\gamma<\beta} \left\{ r^{2d(\gamma)} \overline{P_\gamma(\gamma_\alpha - \gamma)} + r^{d(\gamma)} \overline{Q_\gamma(\gamma_\alpha - \gamma)} \right\} \cdot (\gamma_\beta - \gamma) + \sum_{\gamma<\beta} \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}(\gamma_\beta) \end{aligned}$$

where $\mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}(\gamma_\beta)$ denotes the γ_β-coordinate of the N-vector $\mathcal{Q}_\gamma \mathbf{e}_{\gamma_\beta}$. If $\beta \neq \alpha$ is at distance exactly one from the geodesic $[o, \alpha)$, then $A\beta \in [o, \alpha)$ and the formula for $k_\alpha^{(1)}(\beta)$ is identical to that above except that the final term in the sum $\sum_{\gamma<\beta} \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}(\gamma_\beta)$ is now $\mathcal{Q}_{A\beta} \mathbf{e}_{A\beta_\alpha}(\beta)$ instead of $\mathcal{Q}_{A\beta} \mathbf{e}_{A\beta_\alpha}(A\beta_\alpha)$. The function $k_\alpha^{(1)}$ is then determined for all remaining β by the requirement that $k_\alpha^{(1)}$ be constant on all successor sets $S(\gamma)$ with vertex γ at distance exactly one from the geodesic $[o, \alpha)$.

These calculations generalize to yield duality and reproducing kernels for the holomorphic Besov spaces $HB_{p,1}(\mathcal{T}_n)$, $1 < p < \infty$. Indeed, Hölder's inequality yields

$$|\langle\langle f, g \rangle\rangle_1| \leq \|f\|_{HB_{p,1}(\mathcal{T}_n)} \|g\|_{HB_{p',1}(\mathcal{T}_n)},$$

and to see that

$$\|f\|_{HB_{p,1}(\mathcal{T}_n)} = \sup_{\|g\|_{HB_{p',1}(\mathcal{T}_n)}=1} |\langle\langle f, g \rangle\rangle_1|, \tag{8.19}$$

we choose G to be the unique function satisfying

$$\begin{aligned} G(o) &= f(o) |f(o)|^{p-2}, \\ G'(\gamma) &= f'(\gamma) |f'(\gamma)|^{p-2}, \\ \mathcal{Q}_\gamma \mathcal{D}_\gamma G &= \mathcal{Q}_\gamma \mathcal{D}_\gamma f |\mathcal{Q}_\gamma \mathcal{D}_\gamma f|^{p-2}. \end{aligned}$$

Thus

$$\mathcal{D}G(\alpha) = \{G'(\alpha)(\beta_j - \alpha)\}_{j=1}^N + \mathcal{Q}_\alpha \mathcal{D}f(\alpha) |\mathcal{Q}_\alpha \mathcal{D}f(\alpha)|^{p-2}$$

where $\mathcal{Q}_\alpha \mathcal{D}f(\alpha) |\mathcal{Q}_\alpha \mathcal{D}f(\alpha)|^{p-2} \in M_\alpha^\perp$ and $\{G'(\alpha)(\beta_j - \alpha)\}_{j-1}^N \in M_\alpha$. With this choice we have $\|G\|_{HB_{p',1}(\mathcal{T}_n)} = \|f\|_{HB_{p,1}(\mathcal{T}_n)}^{p-1}$ since $(p-1)p' = p$, as well as

$$\begin{aligned} \langle\langle f, G \rangle\rangle_1 &= |f(o)|^2 |f(o)|^{p-2} + \sum_{\substack{\alpha \in \mathcal{T}_n: \\ d(\alpha) \geq 1}} \left\{ |f'(\alpha) P_\alpha|^2 + |f'(\alpha) Q_\alpha|^2 \right\} |\triangle\, f'(\alpha)|^{p-2} \\ &\qquad + \sum_{\alpha \in \mathcal{T}_n} |\mathcal{Q}_\gamma \mathcal{D}_\gamma f|^2 |\mathcal{Q}_\gamma \mathcal{D}_\gamma f|^{p-2} \\ &= \|f\|_{HB_{p,1}(\mathcal{T}_n)}^p, \end{aligned}$$

since $|f'(\alpha) P_\alpha|^2 + |f'(\alpha) Q_\alpha|^2 = |f'(\alpha)|^2$. Then taking $g = \frac{G}{\|G\|_{HB_{p',1}(\mathcal{T}_n)}}$ we obtain (8.19). We summarize these results in the following Proposition.

Proposition 8.1. *Let $1 < p < \infty$. Then the dual space of $HB_{p,1}(\mathcal{T}_n)$ can be identified with $HB_{p',1}(\mathcal{T}_n)$ under the pairing $\langle\langle \cdot, \cdot \rangle\rangle_1$ given in (8.8), and the* *reproducing kernel* $k_\alpha^{(1)}(\gamma)$ *for this pairing is the unique function* $k_\alpha^{(1)}$ *satisfying (8.10), (8.15) and (8.16), and given explicitly in (8.18).*

The corresponding formula (7.2) for $\triangle\ k_\alpha^1$, the difference operator applied to the reproducing kernel k_α^1 for the abstract Besov space $B_{p,1}(\mathcal{T}_n)$, consists entirely of nonnegative entries, a feature that plays prominently in deriving the Carleson embedding property from multiplier interpolation using Böe's "curious lemma". The terms $k_\alpha^{(1)}(o)$, $k_\alpha^{(1)\prime}(\gamma)$ and $\mathcal{Q}_\gamma\mathcal{D}_\gamma k_\alpha^{(1)}$ arising in the above formula do not consist entirely of nonnegative entries, but the following two properties will serve as a suitable substitute:

(8.20)
$$\begin{aligned}&\left|r^{-d(\gamma)}k_\alpha^{(1)\prime}(\gamma)\,P_\gamma+r^{-\frac{d(\gamma)}{2}}k_\alpha^{(1)\prime}(\gamma)\,Q_\gamma\right|\\&\qquad\approx r^{-d(\gamma)}\operatorname{Re}\left\{\overline{\gamma}\cdot r^{2d(\gamma)}P_\gamma(\gamma_\alpha-\gamma)+\overline{\gamma}\cdot r^{d(\gamma)}Q_\gamma(\gamma_\alpha-\gamma)\right\}\\&\qquad\approx r^{-d(\gamma)}\operatorname{Re}\left(\gamma\cdot k_\alpha^{(1)\prime}(\gamma)\right)\\&\qquad\approx 1;\\&\left|\mathcal{Q}_\gamma\mathcal{D}_\gamma k_\alpha^{(1)}\right|\leq 1.\end{aligned}$$

Analogues of these properties will be used in Subsection 9.1 to complete the proof of equivalence of multiplier interpolation with the separation and tree conditions when $1<p<2+\frac{1}{n-1}$.

To see the equivalence in the first line of (8.20), we compute that

$$\begin{aligned}\overline{\gamma}\cdot P_\gamma(\gamma_\alpha-\gamma)&=\overline{\gamma}\cdot\frac{\overline{\gamma}\cdot(\gamma_\alpha-\gamma)}{|\gamma|^2}\gamma\\&=\overline{\gamma}\cdot(\gamma_\alpha-\gamma)\end{aligned}$$

has real part approximately $r^{-d(\gamma)}$, that

$$|P_\gamma(\gamma_\alpha-\gamma)|\geq cr^{-d(\gamma)},$$

and that

$$\begin{aligned}|\overline{\gamma}\cdot Q_\gamma(\gamma_\alpha-\gamma)|&\leq|Q_\gamma(\gamma_\alpha-\gamma)|\\&\leq|\gamma_\alpha-\gamma|\\&\leq Cr^{-\frac{d(\gamma)}{2}}.\end{aligned}$$

Using (8.10), we thus obtain

$$\begin{aligned}r^{-d(\gamma)}\operatorname{Re}\left(\gamma\cdot k_\alpha^{(1)\prime}(\gamma)\right)&=r^{-d(\gamma)}\operatorname{Re}\left(\overline{\gamma}\cdot\overline{k_\alpha^{(1)\prime}(\gamma)}\right)\\&=r^{-d(\gamma)}\operatorname{Re}\left\{\overline{\gamma}\cdot r^{2d(\gamma)}P_\gamma(\gamma_\alpha-\gamma)+\overline{\gamma}\cdot r^{d(\gamma)}Q_\gamma(\gamma_\alpha-\gamma)\right\}\\&\approx r^{-d(\gamma)}\left\{r^{d(\gamma)}+O\left(r^{\frac{d(\gamma)}{2}}\right)\right\}\\&\approx 1.\end{aligned}$$

Using (8.10) again, we obtain from (8.12) and the above that

$$\begin{aligned}
&\left| r^{-d(\gamma)} k_\alpha^{(1)\prime}(\gamma) P_\gamma + r^{-\frac{d(\gamma)}{2}} k_\alpha^{(1)\prime}(\gamma) Q_\gamma \right|^2 \\
&\qquad = \left| r^{-d(\gamma)} k_\alpha^{(1)\prime}(\gamma) P_\gamma \right|^2 + \left| r^{-\frac{d(\gamma)}{2}} k_\alpha^{(1)\prime}(\gamma) Q_\gamma \right|^2 \\
&\qquad = \left| r^{-d(\gamma)} P_\gamma \overline{k_\alpha^{(1)\prime}(\gamma)} \right|^2 + \left| r^{-\frac{d(\gamma)}{2}} Q_\gamma \overline{k_\alpha^{(1)\prime}(\gamma)} \right|^2 \\
&\qquad = \left| r^{d(\gamma)} P_\gamma (\gamma_\alpha - \gamma) \right|^2 + \left| r^{\frac{d(\gamma)}{2}} Q_\gamma (\gamma_\alpha - \gamma) \right|^2 \\
&\qquad \geq c > 0,
\end{aligned}$$

which completes the proof of the first line in (8.20). The inequality in the second line of (8.20) is (8.14).

An example. We close this subsubsection by computing $k_\alpha^{(1)}$ for the simple case of the Bergman tree $\mathcal{T}_1$ in terms of the geometric embedding of $\mathcal{T}_1$ in the unit disk, and then verifying (8.20) in this case. The branching number for the tree $\mathcal{T}_1$ is 2. Fix $\alpha \in \mathcal{T}_1$ with geodesic $[o, \alpha] = \{o, \alpha_1, \alpha_2, ..., \alpha_m = \alpha\}$ as above, and let $\gamma = \alpha_{k-1}$ with children $\gamma_1 = \alpha_k$ and γ_2. Then

$$\begin{aligned}
\triangle\, \gamma_1 &= \gamma_1 - \gamma, \\
\triangle\, \gamma_2 &= \gamma_2 - \gamma, \\
\mathcal{E}(\gamma) &= \{\triangle\, \gamma_1, \triangle\, \gamma_2\},
\end{aligned}$$

and for $v \in \mathbb{C}$,

$$L_\gamma v = \{v \triangle \gamma_1, v \triangle \gamma_2\}.$$

Thus

$$\begin{aligned}
M_\gamma &= \{v \triangle \gamma_1, v \triangle \gamma_2 : v \in \mathbb{C}\} = \mathbb{C}\{\triangle\, \gamma_1, \triangle\, \gamma_2\}, \\
M_\gamma^\perp &= \mathbb{C}\{-\overline{\triangle\, \gamma_2}, \overline{\triangle\, \gamma_1}\},
\end{aligned}$$

since if $V = v\{\triangle\, \gamma_1, \triangle\, \gamma_2\} \in M_\gamma$ and $W = w\{-\overline{\triangle\, \gamma_2}, \overline{\triangle\, \gamma_1}\} \in M_\gamma^\perp$, then

$$\langle V, W \rangle = v\overline{w}\{(\triangle\, \gamma_1)(-\triangle\, \gamma_2) + (\triangle\, \gamma_2)(\triangle\, \gamma_1)\} = 0.$$

We now claim that

$$\begin{aligned}
\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} &= \frac{\triangle\, \gamma_2}{|\triangle\, \gamma_1|^2 + |\triangle\, \gamma_2|^2} \{-\overline{\triangle\, \gamma_1}, \overline{\triangle\, \gamma_2}\} \\
&= \left\{ \frac{|\triangle\, \gamma_2|^2}{|\triangle\, \gamma_1|^2 + |\triangle\, \gamma_2|^2}, \frac{-\overline{\triangle\, \gamma_1}\, \triangle\, \gamma_2}{|\triangle\, \gamma_1|^2 + |\triangle\, \gamma_2|^2} \right\}.
\end{aligned}$$

Indeed, we have $\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} \in M_\gamma^\perp$ from the definition, and if

$$W = \{W_1, W_2\} = w\{-\overline{\triangle\, \gamma_2}, \overline{\triangle\, \gamma_1}\} \in M_\gamma^\perp,$$

then the left side of (8.13) is

$$\begin{aligned}
\left\langle W, \mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} \right\rangle &= w(-\overline{\triangle\, \gamma_2}) \overline{\frac{|\triangle\, \gamma_2|^2}{|\triangle\, \gamma_1|^2 + |\triangle\, \gamma_2|^2}} + w(\overline{\triangle\, \gamma_1}) \overline{\frac{-\overline{\triangle\, \gamma_1}\, \triangle\, \gamma_2}{|\triangle\, \gamma_1|^2 + |\triangle\, \gamma_2|^2}} \\
&= w(-\overline{\triangle\, \gamma_2}) = W_1,
\end{aligned}$$

which is the right side of (8.13) since $\gamma_1 = \alpha_k \in [o, \alpha]$.

Since the projection P_γ is the identity in dimension $n = 1$, we also have

$$k_\alpha^{(1)\prime}(\gamma) = r^{2d(\gamma)}\overline{(\gamma_\alpha - \gamma)},$$

and since it is geometrically evident that $\operatorname{Re}\gamma \cdot \overline{\gamma_\alpha - \gamma} \approx |\gamma| |\gamma_\alpha - \gamma|$, we now have

$$\operatorname{Re}\left(\gamma \cdot k_\alpha^{(1)\prime}(\gamma)\right) = r^{2d(\gamma)} \operatorname{Re}\gamma \cdot \overline{\gamma_\alpha - \gamma} \approx r^{d(\gamma)} \approx \left|k_\alpha^{(1)\prime}(\gamma)\right|,$$

$$|\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha|^2 \leq \frac{|\triangle\, \gamma_2|^4 + |\triangle\, \gamma_1|^2 |\triangle\, \gamma_2|^2}{\left(|\triangle\, \gamma_1|^2 |\triangle\, \gamma_2|^2\right)^2} \leq 1,$$

which is (8.20) for the Bergman tree $\mathcal{T}_1$.

Using (8.18), we give an explicit formula for $k_\alpha(\beta)$, $\alpha, \beta \in \mathcal{T}_1$. When $\beta \in [o, \alpha]$ we have

$$k_\alpha^{(1)}(\beta) = 1 + \sum_{\gamma < \beta} r^{2d(\gamma)}\overline{(\gamma_\alpha - \gamma)}(\gamma_\beta - \gamma) + \sum_{\gamma < \beta} \frac{\left|\gamma_\beta^\perp - \gamma\right|^2}{|\gamma_\beta - \gamma|^2 + \left|\gamma_\beta^\perp - \gamma\right|^2},$$

where $\gamma_\beta^\perp$ is the child of γ not lying in $[o, \beta]$. The formula for $k_\alpha^{(1)}\left(\beta_\alpha^\perp\right)$, where $\beta \in [o, \alpha)$ and $\beta_\alpha^\perp$ is the child of β not lying in $[o, \alpha]$, is given by

$$k_\alpha^{(1)}\left(\beta_\alpha^\perp\right) = 1 + \sum_{o < \gamma \leq \beta} r^{2d(\gamma)}\overline{(\gamma_\alpha - \gamma)}\left(\gamma_{\beta_\alpha^\perp} - \gamma\right)$$
$$. + \sum_{o < \gamma < \beta} \frac{\left|\gamma_\beta^\perp - \gamma\right|^2}{|\gamma_\beta - \gamma|^2 + \left|\gamma_\beta^\perp - \gamma\right|^2} - \frac{\overline{(\beta_\alpha - \beta)}\left(\beta_\alpha^\perp - \beta\right)}{|\beta_\alpha - \beta|^2 + |\beta_\alpha^\perp - \beta|^2}.$$

The values of $k_\alpha^{(1)}$ remain constant on the successor sets $S(\alpha)$ and $S\left(\beta_\alpha^\perp\right)$ for $\beta \in [o, \alpha)$, and this completes the evaluation of $k_\alpha^{(1)}(\beta)$ for all $\beta \in \mathcal{T}_1$.

8.1.2. *Tensor-valued functions.* In order to extend our definitions to tensor-valued functions on the Bergman tree, it is convenient to consider first the simplest case of order zero.

Definition 8.2. *Let $1 < p < \infty$. For a $\mathbb{C}$-valued function $f(\alpha)$ defined for $\alpha \in \mathcal{T}_n$, define*

$$\|f\|_{HB_{p,0}(\mathcal{T}_n)} = \|f\|_{\ell^p(\mathcal{T}_n)} = \left(\sum_{\alpha \in \mathcal{T}_n} |f(\alpha)|^p\right)^{\frac{1}{p}}.$$

Define an operator $\mathbf{R}^{-d}$ on $\mathbb{C}^n$-valued functions $\mathbf{v}$ on the tree $\mathcal{T}_n$ by

$$\left(\mathbf{R}^{-d}\mathbf{v}\right)(\alpha) = r^{-d(\alpha)}\mathbf{v}(\alpha) P_\alpha + r^{-\frac{d(\alpha)}{2}}\mathbf{v}(\alpha) Q_\alpha.$$

For vectors $\mathbf{v} \in \mathbb{C}^n$, let

$$|\mathbf{v}|_\alpha = \left|\left(\mathbf{R}^{-d}\mathbf{v}\right)(\alpha)\right| = \left|e^{-2d(\alpha)\theta}\mathbf{v}P_\alpha + e^{-d(\alpha)\theta}\mathbf{v}Q_\alpha\right| \approx \sqrt{(B(\alpha)\mathbf{v}) \cdot \overline{\mathbf{v}}},$$

and for a $\mathbb{C}^n$-valued function $\mathbf{v}(\alpha)$ defined for $\alpha \in \mathcal{T}_n$, define

$$\|\mathbf{v}\|_{HB_{p,0}^{(1)}(\mathcal{T}_n)} = \||\mathbf{v}|_\alpha\|_{\ell^p(\mathcal{T}_n)} = \left(\sum_{\alpha \in \mathcal{T}_n} |\mathbf{v}|_\alpha^p\right)^{\frac{1}{p}}.$$

REMARK 8.3. Recall from (8.1) that we have $HB_{p,1}(\mathcal{T}_n) = B_{p,1}(\mathcal{T}_n)$. Let $\mathcal{QD}f(\alpha) = \mathcal{Q}_\alpha \mathcal{D}_\alpha f$. Then using Definitions 8.2 and 7.14, we have the following observation that will provide the basis for extending the definition of $HB_{p,0}(\mathcal{T}_n)$ to $HB_{p,m}(\mathcal{T}_n)$ for larger m:

$$\begin{aligned}\|f\|^p_{HB_{p,1}(\mathcal{T}_n)} &= |f(o)|^p + |f'(o)|^p + \sum_{\alpha\in\mathcal{T}_n} |f'(\alpha)|^p_\alpha + \sum_{\alpha\in\mathcal{T}_n} |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^p \\ &= |f(o)|^p + \|f'\|^p_{HB^{(1)}_{p,0}(\mathcal{T}_n)} + \|\mathcal{QD}f\|^p_{\ell^p(\mathcal{T}_n)}.\end{aligned}$$

In Definition 8.2 above, we defined the order zero holomorphic Besov space $HB^{(1)}_{p,0}(\mathcal{T}_n)$ on $\mathcal{T}_n$ for $\mathbb{C}^n$-valued functions $\mathbf{v}(\alpha)$ using the nonisotropic norm

$$|\mathbf{v}|_\alpha = \left| r^{-d(\alpha)} \mathbf{v} P_\alpha + r^{-\frac{d(\alpha)}{2}} \mathbf{v} Q_\alpha \right|,$$

and where $\mathbf{v}(\alpha)$ was interpreted as a covariant tensor of order 1 acting on the "tangent space" $\mathbb{C}^n$ at α. We now wish to extend this definition to order zero holomorphic Besov spaces $HB^{(t)}_{p,0}(\mathcal{T}_n)$ of symmetric covariant tensors of order t, or t-tensors, on $\mathcal{T}_n$. First we review the tensor setup.

Let E_α be the n-dimensional Hilbert space $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_\alpha)$ whose inner product is given by

$$\langle v^1, v^2\rangle_\alpha = r^{-2d(\alpha)} \langle v^1, p\rangle \overline{\langle v^2, p\rangle} + r^{-d(\alpha)} \sum_{i=2}^n \langle v^1, q_i\rangle \overline{\langle v^2, q_i\rangle}$$

where $\{p, q_2, ..., q_n\}$ is an orthonormal basis of $\mathbb{C}^n$ with $M_\alpha = \mathit{span}\{p\}$ and $M_\alpha^\perp = \mathit{span}\{q_2, ..., q_n\}$. Thus $p = e^{is}\frac{\alpha}{|\alpha|}$ for some $s \in \mathbb{R}$, and we will take $s = 0$ so that

$$p = \frac{\alpha}{|\alpha|}. \tag{8.21}$$

In terms of the operator $\mathbf{R}^{-d}$ we have

$$\langle v^1, v^2\rangle_\alpha = \langle \mathbf{R}^{-d} v^1, \mathbf{R}^{-d} v^2\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{C}^n$. For $t \ge 1$, denote by $\mathcal{E}^{(t)}_\alpha$ the vector space of *symmetric* multilinear maps, or symmetric t-tensors, from the product space $E^t_\alpha = E_\alpha \times ... \times E_\alpha$ (t times) to the complex numbers $\mathbb{C}$.

Using the identification of E^*_α with E_α under the *Euclidean* inner product $\langle\cdot,\cdot\rangle$ (not $\langle\cdot,\cdot\rangle_\alpha$), every symmetric t-tensor $\mathbf{A} \in \mathcal{E}^{(t)}_\alpha$ can be written

$$\mathbf{A} = \sum_{1\le i_1\le i_2\le ...\le i_t\le n} a^{i_1,i_2,...,i_t} e^{i_1} \otimes ... \otimes e^{i_t},$$

where e^1 (respectively e^j, $j \ge 2$) is the Euclidean dual vector to $e_1 = p$ (respectively $e_j = q_j$, $j \ge 2$) so that $e^j(e_i) = \langle e_i, e_j\rangle = \delta^i_j$. The vectors e_i depend on α, but we will usually suppress this dependence. We have

$$\mathbf{A}\left[v^1, ..., v^t\right] = \sum_{1\le i_1\le i_2\le ...\le i_t\le n} a^{i_1,i_2,...,i_t} \langle v^1, e_{i_1}\rangle \times ... \times \langle v^t, e_{i_t}\rangle. \tag{8.22}$$

We now define an inner product $\langle\cdot,\cdot\rangle^{(t)}_\alpha$ on $\mathcal{E}^{(t)}_\alpha$ by

$$\langle e^{i_1} \otimes ... \otimes e^{i_t}, e^{j_1} \otimes ... \otimes e^{j_t}\rangle^{(t)}_\alpha = \prod_{k=1}^t \langle e^{i_k}, e^{j_k}\rangle_\alpha = \prod_{k=1}^t \left\langle \mathbf{R}^{-d(\alpha)} e_{i_k}, \mathbf{R}^{-d(\alpha)} e_{j_k}\right\rangle,$$

and then extend the definition by linearity so that
(8.23)

$$\langle \mathbf{A}, \mathbf{B} \rangle_\alpha^{(t)} = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_t \le n} a^{i_1, i_2, \ldots, i_t} \overline{b^{i_1, i_2, \ldots, i_t}} \eta_{i_1, i_2, \ldots, i_t}(\alpha) = \sum_i a^i \overline{b^i} \eta_i(\alpha).$$

where we write $i = (i_1, \ldots, i_t)$, $j = (j_1, \ldots, j_t)$ and

$$\begin{aligned} \eta_i(\alpha) &= \eta_{i_1, i_2, \ldots, i_t}(\alpha) \\ &= \left\langle e^{i_1}, e^{i_1} \right\rangle_\alpha \times \ldots \times \left\langle e^{i_t}, e^{i_t} \right\rangle_\alpha \\ &= \left\langle \mathbf{R}^{-d(\alpha)} e_{i_1}, \mathbf{R}^{-d(\alpha)} e_{i_1} \right\rangle \times \ldots \times \left\langle \mathbf{R}^{-d(\alpha)} e_{i_t}, \mathbf{R}^{-d(\alpha)} e_{i_t} \right\rangle, \end{aligned} \tag{8.24}$$

and where $\mathbf{A} = \sum_i a^i e^{i_1} \otimes \ldots \otimes e^{i_t}$ and $\mathbf{B} = \sum_i b^i e^{i_1} \otimes \ldots \otimes e^{i_t}$. We extend this inner product to t-tensor-valued functions $\mathbf{A}$ and $\mathbf{B}$ on the Bergman tree $\mathcal{T}_n$ in the obvious way:

$$\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_0^{(t)} = \sum_{\gamma \in \mathcal{T}_n} \langle \mathbf{A}(\gamma), \mathbf{B}(\gamma) \rangle_\gamma^{(t)}. \tag{8.25}$$

We denote by $|\mathbf{A}|_\alpha$ the norm of $\mathbf{A}$ in the Hilbert space $\mathcal{E}_\alpha^{(t)}$ when there is no confusion regarding the order t of the tensor in question. For example, if $F \in H(\mathbb{B}_n)$ and $\alpha \in \mathbb{B}_n$, then $F''(\alpha) \in \mathcal{E}_\alpha^{(2)}$ is a symmetric 2-tensor, or equivalently an $n \times n$ matrix $[F''(\alpha)]$ relative to the basis $\{e_1, \ldots, e_n\}$ satisfying $F''(\alpha)[v \otimes w] = v'[F''(\alpha)] w$, and whose norm squared in $\mathcal{E}_\alpha^{(2)}$ is given by

$$\begin{aligned} |F''(\alpha)|_\alpha^2 &= \langle F''(\alpha), F''(\alpha) \rangle_\alpha^{(t)} \\ &= \left\langle \sum_{i,j=1}^n \frac{\partial^2 F}{\partial z_i \partial z_j}(\alpha) e^i \otimes e^j, \sum_{k,\ell=1}^n \frac{\partial^2 F}{\partial z_k \partial z_\ell}(\alpha) e^k \otimes e^\ell \right\rangle_\alpha^{(t)} \\ &= \sum_{i,j,k,\ell=1}^n \frac{\partial^2 F}{\partial z_i \partial z_j}(\alpha) \overline{\frac{\partial^2 F}{\partial z_k \partial z_\ell}(\alpha)} \left\langle e^i, e^k \right\rangle_\alpha \overline{\left\langle e^j, e^\ell \right\rangle_\alpha} \\ &= \sum_{i,j=1}^n \left| \frac{\partial^2 F}{\partial z_i \partial z_j}(\alpha) \right|^2 \eta_{(i,j)}, \end{aligned}$$

where

$$\eta_{(i,j)} = \begin{cases} r^{-2d(\alpha)} & \text{if} \quad i = j = 1 \\ r^{-\frac{3}{2}d(\alpha)} & \text{if} \quad i = 1, 2 \le j \le n \\ r^{-\frac{3}{2}d(\alpha)} & \text{if} \quad j = 1, 2 \le i \le n \\ r^{-d(\alpha)} & \text{if} \quad 2 \le i, j \le n \end{cases}.$$

This is of course comparable to the operator norm

$$\sup_{|v|, |w| \le 1} \left| F''(\alpha) \left[\mathbf{R}^{-d} v, \mathbf{R}^{-d} w \right] \right|^2$$

where $\left| F''(\alpha) \left[\mathbf{R}^{-d} v, \mathbf{R}^{-d} w \right] \right|^2$ is equal to

$$\begin{aligned} &\left| r^{-2d(\alpha)} F''(\alpha) [P_\alpha v, P_\alpha w] \right|^2 + \left| r^{-\frac{3}{2}d(\alpha)} F''(\alpha) [P_\alpha v, Q_\alpha w] \right|^2 \\ &\qquad + \left| r^{-\frac{3}{2}d(\alpha)} F''(\alpha) [Q_\alpha v, P_\alpha w] \right|^2 + \left| r^{-d(\alpha)} F''(\alpha) [Q_\alpha v, Q_\alpha w] \right|^2. \end{aligned}$$

In similar fashion we define the Hilbert space $\mathcal{E}_\alpha^{(s,t)}$ of symmetric (s,t)-tensors that are covariant of order s and contravariant of order t (see for example chapter 4 of Spivak [**Spi**]). Then $\mathcal{E}_\alpha^{(t)} = \mathcal{E}_\alpha^{(t,0)}$ and we will stop referring to tensors as covariant or contravariant. We define the tensor product of a (s_1,t_1)-tensor $\mathbf{B}$ and a (s_2,t_2)-tensor $\mathbf{A}$ to be the (s_1+s_2,t_1+t_2)-tensor $\mathbf{B}\otimes\mathbf{A}$ in the usual way,

$$\begin{aligned}\mathbf{B}\otimes\mathbf{A}&\left[v^1,...,v^{s_1},w^1,...,w^{t_1},x^1,...,x^{s_2},y^1,...,y^{t_2}\right]\\ &= \mathbf{B}\left[v^1,...,v^{s_1},w^1,...,w^{t_1}\right]\times\mathbf{A}\left[x^1,...,x^{s_2},y^1,...,y^{t_2}\right],\end{aligned}$$

as well as the Euclidean contraction $\mathbf{B}\wedge\mathbf{A}$ of an (s,t)-tensor $\mathbf{B}$ and a t-tensor $\mathbf{A}$ (see immediately below for this definition). We will see later that reproducing kernels for Besov spaces of t-tensor-valued functions can be interpreted as (t,t)-tensor-valued functions on $\mathcal{T}_n$.

We define the *α-contraction* $\mathbf{B}\wedge_\alpha\mathbf{A}$ of an (s,t)-tensor

$$\mathbf{B} = \sum_{1\le i_1\le...\le i_t\le n} b_{j_1,...,j_t}^{i_1,...,i_s} e^{i_1}\otimes...\otimes e^{i_s}\otimes e_{j_1}\otimes...\otimes e_{j_t}$$

and a t-tensor

$$\mathbf{A} = \sum_{1\le i_1\le...\le i_t\le n} a^{i_1,i_2,...,i_t} e^{i_1}\otimes...\otimes e^{i_t}$$

to be the s-tensor given by

$$\begin{aligned}\mathbf{B}\wedge_\alpha\mathbf{A} &= \sum_{1\le i_1\le...\le i_t\le n} b_{j_1,...,j_t}^{i_1,...,i_s} a^{j_1,...,j_t}\left\langle e_{j_1},e_{j_1}\right\rangle_\alpha\times...\times\left\langle e_{j_t},e_{j_t}\right\rangle_\alpha e^{i_1}\otimes...\otimes e^{i_s}\\ &= \sum_i b_j^i a^j \eta_j(\alpha)\, e^{i_1}\otimes...\otimes e^{i_s},\end{aligned}$$

where by the summation convention, we also sum over the repeated upper and lower indices $j_1,...,j_t$. The Euclidean contraction $\mathbf{B}\wedge\mathbf{A}$ is given by $\sum_i b_j^i a^j e^{i_1}\otimes...\otimes e^{i_s}$ without the η_j. Thus $\mathbf{B}\wedge\mathbf{A}[v_1,...,v_s]$ is the contraction (trace if $t=1$) of the linear map λ given by

$$\lambda(w_1,...,w_t,u_1,...,u_t) = \mathbf{B}\otimes\mathbf{A}(v_1,...,v_s,w_1,...,w_t,u_1,...,u_t)$$

(see page 4-27 in [**Spi**]). Note that if we interpret v^j as a $(0,1)$-tensor (contravariant of order 1), then from (8.22) we have

$$\mathbf{A}\left[v^1,...,v^t\right] = \mathbf{A}\wedge\left(v^1\otimes...\otimes v^t\right). \tag{8.26}$$

Definition 8.4. *We define the "inner product" $\langle\mathbf{A},\mathbf{B}\rangle_\alpha^{(t)}$ of a t-tensor $\mathbf{A}$ and a (t,t)-tensor $\mathbf{B}$ to be the t-tensor given by*

$$\langle\mathbf{A},\mathbf{B}\rangle_\alpha^{(t)} = \overline{\mathbf{B}}\wedge_\alpha\mathbf{A}, \tag{8.27}$$

so that

$$\langle\mathbf{A},\mathbf{B}\rangle_\alpha^{(t)} = \sum_i \overline{b_j^i} a^j \eta_j(\alpha)\, e^{i_1}\otimes...\otimes e^{i_t}. \tag{8.28}$$

We also define an "inner product" $\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle_0^{(t)}$, for a t-tensor-valued function $\mathbf{A}$ and a (t,t)-tensor-valued function $\mathbf{B}$ on the tree $\mathcal{T}_n$, by

$$\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle_0^{(t)} = \sum_{\gamma\in\mathcal{T}_n}\langle\mathbf{A}(\gamma),\mathbf{B}(\gamma)\rangle_\gamma^{(t)}. \tag{8.29}$$

The use of the same notation $\langle \mathbf{A}, \mathbf{B} \rangle_\alpha^{(t)}$ in both (8.23) and (8.27) should not cause confusion as the former is a scalar and the latter a t-tensor, and similarly for the same notation $\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_0^{(t)}$ in both (8.25) and (8.29).

Order zero spaces of t-tensors. We now define the order zero holomorphic Besov spaces $HB_{p,0}^{(t)}(\mathcal{T}_n)$ of t-tensors on $\mathcal{T}_n$.

Definition 8.5. *For $1 < p < \infty$ and $t \in \mathbb{N}$, let $HB_{p,0}^{(t)}(\mathcal{T}_n)$ consist of all t-tensor-valued functions $\mathbf{A}(\alpha)$ defined for $\alpha \in \mathcal{T}_n$ such that the norm*

$$\begin{aligned}\|\mathbf{A}\|_{HB_{p,0}^{(t)}(\mathcal{T}_n)} &= \left\| |\mathbf{A}|_\alpha \right\|_{\ell^p(\mathcal{T}_n)} \\ &= \left(\sum_{\alpha \in \mathcal{T}_n} |\mathbf{A}(\alpha)|_\alpha^p \right)^{\frac{1}{p}}\end{aligned}$$

is finite.

The inner product for the Hilbert space $HB_{2,0}^{(t)}(\mathcal{T}_n)$ is given by (8.25), and the dual space of $HB_{p,0}^{(t)}(\mathcal{T}_n)$ can be identified with $HB_{p',0}^{(t)}(\mathcal{T}_n)$ under the pairing $\langle\langle \cdot, \cdot \rangle\rangle_0^{(t)}$.

Lemma 8.6. *For $1 < p < \infty$ and $t \in \mathbb{N}$, we have*

$$\begin{aligned}\left| \langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_0^{(t)} \right| &\leq \|\mathbf{A}\|_{HB_{p,0}^{(t)}(\mathcal{T}_n)} \|\mathbf{B}\|_{HB_{p',0}^{(t)}(\mathcal{T}_n)}, \\ \|\mathbf{B}\|_{HB_{p',0}^{(t)}(\mathcal{T}_n)} &= \sup_{\|\mathbf{A}\|_{HB_{p,0}^{(t)}(\mathcal{T}_n)} \leq 1} \left| \langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_0^{(t)} \right|.\end{aligned}$$

Moreover, in the case $p = 2$, the function $\Lambda_\alpha^{v^1,\dots,v^t}$ defined by

$$\Lambda_\alpha^{v^1,\dots,v^t} \mathbf{A} = \mathbf{A}(\alpha)\left[v^1, ..., v^t\right]$$

is a continuous linear functional on $HB_{2,0}^{(t)}(\mathcal{T}_n)$ for $\alpha \in \mathcal{T}_n$ and every choice of $v^1, ..., v^t$. Thus there is a unique $\mathbf{k}_\alpha^{v^1,\dots,v^t}$ in the Hilbert space $HB_{2,0}^{(t)}(\mathcal{T}_n)$ such that

$$\left\langle \left\langle \mathbf{A}, \mathbf{k}_\alpha^{v^1,\dots,v^t} \right\rangle \right\rangle_0^{(t)} = \Lambda_{v^1,\dots,v^t} \mathbf{A} = \mathbf{A}(\alpha)\left[v^1, ..., v^t\right]. \tag{8.30}$$

By this uniqueness, the function that sends $v^1, ..., v^t$ to the t-tensor $\mathbf{k}_\alpha^{v^1,\dots,v^t}(\gamma)$ is multi-conjugate linear in $v^1, ..., v^t$, and so there is a unique (t,t)-tensor $\mathbf{k}_\alpha^{(0,t)}$ such that

$$\mathbf{k}_\alpha^{v^1,\dots,v^t}(\gamma)\left[w^1, ..., w^t\right] = \mathbf{k}_\alpha^{(0,t)}(\gamma)\left[w^1, ..., w^t, \overline{v^1}, ..., \overline{v^t}\right],$$

which by (8.26) is

$$\mathbf{k}_\alpha^{v^1,\dots,v^t}(\gamma) = \mathbf{k}_\alpha^{(0,t)}(\gamma) \wedge \overline{v^1 \otimes ... \otimes v^t}.$$

We refer to this (t,t)-tensor-valued function $\mathbf{k}_\alpha^{(0,t)}$ as the reproducing kernel for the holomorphic Besov space of t-tensors $HB_{p,0}^{(t)}(\mathcal{T}_n)$ relative to the pairing $\langle\langle \cdot, \cdot \rangle\rangle_0^{(t)}$

since by (8.26), (8.27) and (8.29), we have

$$\begin{aligned}\left\langle\left\langle \mathbf{A},\mathbf{k}_\alpha^{(0,t)}\right\rangle\right\rangle_0^{(t)}\left[v^1,...,v^t\right] &= \sum_{\gamma\in\mathcal{T}_n}\left\langle \mathbf{A}(\gamma),\mathbf{k}_\alpha^{(0,t)}(\gamma)\right\rangle_\gamma^{(t)}\left[v^1,...,v^t\right]\\ &= \sum_{\gamma\in\mathcal{T}_n}\overline{\mathbf{k}_\alpha^{(0,t)}(\gamma)}\wedge_\gamma \mathbf{A}(\gamma)\left[v^1,...,v^t\right]\\ &= \sum_{\gamma\in\mathcal{T}_n}\overline{\mathbf{k}_\alpha^{(0,t)}(\gamma)}\wedge_\gamma \mathbf{A}(\gamma)\wedge v^1\otimes...\otimes v^t\\ &= \sum_{\gamma\in\mathcal{T}_n}\overline{\mathbf{k}_\alpha^{(0,t)}(\gamma)}\wedge v^1\otimes...\otimes v^t\wedge_\gamma \mathbf{A}(\gamma)\end{aligned}$$

since $\wedge$ and $\wedge_\gamma$ commute, as they act on different sets of variables. We then continue with

$$\begin{aligned}\left\langle\left\langle \mathbf{A},\mathbf{k}_\alpha^{(0,t)}\right\rangle\right\rangle_0^{(t)}\left[v^1,...,v^t\right] &= \sum_{\gamma\in\mathcal{T}_n}\left\langle \mathbf{A}(\gamma),\mathbf{k}_\alpha^{(0,t)}(\gamma)\wedge_\gamma\overline{v^1\otimes...\otimes v^t}\right\rangle_\gamma^{(t)}\\ &= \sum_{\gamma\in\mathcal{T}_n}\left\langle \mathbf{A}(\gamma),\mathbf{k}_\alpha^{v^1,...,v^t}(\gamma)\right\rangle_\gamma^{(t)}\\ &= \left\langle\left\langle \mathbf{A},\mathbf{k}_\alpha^{v^1,...,v^t}\right\rangle\right\rangle_0^{(t)}\\ &= \mathbf{A}(\alpha)\left[v^1,...,v^t\right],\end{aligned}$$

by (8.30). Thus we have shown that

$$\mathbf{A}(\alpha)=\left\langle\left\langle \mathbf{A},\mathbf{k}_\alpha^{(0,t)}\right\rangle\right\rangle_0^{(t)},\qquad \alpha\in\mathcal{T}_n. \tag{8.31}$$

The reproducing kernel $\mathbf{k}_\alpha^{(0,t)}$ is in fact given by the (t,t)-tensor

$$\sum_i \eta_i(\alpha)^{-1}e^{i_1}\otimes...\otimes e^{i_t}\otimes e_{i_1}\otimes...\otimes e_{i_t}$$

times the delta function at α, i.e.

$$\begin{aligned}\mathbf{k}_\alpha^{(0,t)}(\gamma) &= \chi_{\{\alpha\}}(\gamma)\sum_i \eta_i(\alpha)^{-1}\left\{e^{i_1}\otimes...\otimes e^{i_t}\right\}\otimes\left\{e_{i_1}\otimes...\otimes e_{i_t}\right\}\\ &= \chi_{\{\alpha\}}(\gamma)\sum_i\left\{\mathbf{R}^{d(\alpha)}e^{i_1}\otimes...\otimes\mathbf{R}^{d(\alpha)}e^{i_t}\right\}\\ &\qquad\otimes\left\{\mathbf{R}^{d(\alpha)}e_{i_1}\otimes...\otimes\mathbf{R}^{d(\alpha)}e_{i_t}\right\}.\end{aligned} \tag{8.32}$$

Indeed, by (8.27) and (8.28),

$$\begin{aligned}&\sum_{\gamma\in\mathcal{T}_n}\left\langle \mathbf{A}(\gamma),\mathbf{k}_\alpha^{(0,t)}(\gamma)\right\rangle_\gamma^{(t)} = \mathbf{k}_\alpha^{(0,t)}(\alpha)\wedge_\alpha \mathbf{A}(\alpha)\\ &\quad= \left\{\sum_i \eta_i(\alpha)^{-1}\left\{e^{i_1}\otimes...\otimes e^{i_t}\right\}\otimes\left\{e_{i_1}\otimes...\otimes e_{i_t}\right\}\right\}\wedge_\alpha\left\{\sum_i a^i e^{i_1}\otimes...\otimes e^{i_t}\right\},\\ &\quad= \sum_i \eta_i(\alpha)^{-1}a^i\eta_i(\alpha)e^{i_1}\otimes...\otimes e^{i_t}=\mathbf{A}(\alpha).\end{aligned}$$

Order one spaces of t-tensors. We next turn to defining the order one holomorphic Besov spaces $HB_{p,1}^{(t)}(\mathcal{T}_n)$ of t-tensors on $\mathcal{T}_n$. In Remark 8.3, we have already defined the scalar case $HB_{p,1}(\mathcal{T}_n)$ using the norm (to the p^{th} power)

$$\|f\|_{HB_{p,1}(\mathcal{T}_n)}^p = |f(o)|^p + \|f'\|_{HB_{p,0}^{(1)}(\mathcal{T}_n)}^p + \|\mathcal{Q}\mathcal{D}f\|_{\ell^p(\mathcal{T}_n)}^p .$$

In order to replace f with a tensor, we first need to define the complex derivative $\mathbf{A}'(\alpha)$ of a t-tensor-valued function $\mathbf{A}(\alpha)$ on the tree $\mathcal{T}_n$. The derivative $\mathbf{A}'$ will be a $(t+1)$-tensor-valued function on the tree defined in the same spirit as f'.

Define the forward difference $\mathcal{D}_\alpha \mathbf{A}$ of a t-tensor-valued function $\mathbf{A}(\alpha)$ in the obvious way. Define the linear map $L_\alpha^{(t)}$ from the space of $(t+1)$-tensors $\mathcal{E}_\alpha^{(t+1)}$ to the space $\left(\mathcal{E}_\alpha^{(t)}\right)^N$ by sending $\mathbf{v} \in \mathcal{E}_\alpha^{(t+1)}$ to

$$L_\alpha^{(t)}\mathbf{v} = \left(\mathbf{v}\wedge\left(\alpha^j - \alpha\right)\right)_{j=1}^N \in \left(\mathcal{E}_\alpha^{(t)}\right)^N ,$$

where $\mathbf{v}\wedge\left(\alpha^j-\alpha\right)$ denotes the t-tensor obtained by contracting the $(t+1)$-tensor $\mathbf{v}$ with the vector $\left(\alpha^j-\alpha\right)$ viewed as a $(0,1)$-tensor or contravariant 1-tensor, i.e.

$$\mathbf{v}\wedge\left(\alpha^j-\alpha\right)\left[v^1,...,v^t\right] = \mathbf{v}\left[\alpha^j-\alpha, v^1,...,v^t\right] ,$$

since if $\mathbf{v} = \sum v_k^i e_k \otimes \left\{e^{i_1}\otimes ... \otimes e^{i_t}\right\}$, then both sides of the above equation equal

$$\sum v_k^i \left\langle \alpha^j-\alpha, e_k\right\rangle \left\langle v^1, e^{i_1}\right\rangle \times ... \times \left\langle v^t, e^{i_t}\right\rangle .$$

In Subsection 8.5 below, we will show that by choosing θ sufficiently large in the construction of the Bergman tree in Subsubsection 2.2.1, and then modifying the centers, we can make the map $L_\alpha^{(t)}$ one-to-one for all $0 \le t \le M-1$ for any finite M - we only need to take $M = 2n$ for our purposes in this paper. Let $\mathcal{P}_\alpha^{(t)}$ be the orthogonal projection of $\left(\mathcal{E}_\alpha^{(t)}\right)^N$ onto the range $M_\alpha^{(t)}$ of $L_\alpha^{(t)}$ with respect to the inner product on the product Hilbert space $\mathcal{E}_\alpha^{(t)} \times ... \times \mathcal{E}_\alpha^{(t)}$ (N times). Let $\mathcal{Q}_\alpha^{(t)} = I - \mathcal{P}_\alpha^{(t)}$ be orthogonal projection onto $\left(M_\alpha^{(t)}\right)^\perp$. The complex derivative $\mathbf{A}'(\alpha)$ of $\mathbf{A}$ at the point α is then the unique $(t+1)$-tensor $\mathbf{v}$ such that

$$L_\alpha \mathbf{v} = \mathcal{P}_\alpha^{(t)}\left(\mathcal{D}_\alpha \mathbf{A}\right).$$

Thus we have

$$L_\alpha^{(t)}\mathbf{A}'(\alpha) = \mathcal{P}_\alpha^{(t)}\left(\mathcal{D}_\alpha\mathbf{A}\right) = \left(\mathbf{A}'(\alpha)\wedge\left(\alpha^j-\alpha\right)\right)_{j=1}^N ,$$

and the first order Taylor formula for t-tensor-valued functions

$$\mathcal{D}_\alpha\mathbf{A} = \left(\mathbf{A}'(\alpha)\wedge\left(\alpha^j-\alpha\right)\right)_{j=1}^N + \mathcal{Q}_\alpha^{(t)}\left(\mathcal{D}_\alpha\mathbf{A}\right).$$

We can now define the order one holomorphic Besov space $HB_{p,1}^{(t)}(\mathcal{T}_n)$ of t-tensors on $\mathcal{T}_n$. First we define

$$\left\|\mathcal{Q}^{(t)}\mathcal{D}\mathbf{A}\right\|_{\ell^p(\mathcal{T}_n)}^p = \sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_\alpha^{(t)}\left(\mathcal{D}_\alpha\mathbf{A}\right)\right|_\alpha^p$$

where $\left|\left(\mathbf{v}^1,...,\mathbf{v}^N\right)\right|_\alpha^2 = \sum_{j=1}^N\left|\mathbf{v}^j\right|_\alpha^2$ for $\left(\mathbf{v}^1,...,\mathbf{v}^N\right) \in \left(\mathcal{E}_\alpha^{(t)}\right)^N$. Note that at the root o, $|\cdot|_o$ is always a Euclidean norm.

Definition 8.7. *For $1 < p < \infty$ and $0 \leq t \leq M-1$, let $HB^{(t)}_{p,1}(\mathcal{T}_n)$ consist of all t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$ such that the norm (to the p^{th} power)*

$$\|\mathbf{A}\|^p_{HB^{(t)}_{p,1}(\mathcal{T}_n)} = |\mathbf{A}(o)|^p + \|\mathbf{A}'\|^p_{HB^{(t+1)}_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)}$$

is finite.

As in Proposition 8.1, we can obtain the duality of $HB^{(t)}_{p,1}(\mathcal{T}_n)$ and $HB^{(t)}_{p',1}(\mathcal{T}_n)$ relative to the inner product $\langle\langle\cdot,\cdot\rangle\rangle^{(t)}_1$ for the Hilbert space $HB^{(t)}_{2,1}(\mathcal{T}_n)$:

$$\begin{aligned}\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_1 &= \mathbf{A}(o)\overline{\mathbf{B}(o)} + \sum_{\alpha\in\mathcal{T}_n}\langle\mathbf{A}',\mathbf{B}'\rangle^{(t+1)}_\alpha + \sum_{\alpha\in\mathcal{T}_n}\mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{A}\cdot\overline{\mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{B}} \\ &= \mathbf{A}(o)\overline{\mathbf{B}(o)} + \langle\langle\mathbf{A}',\mathbf{B}'\rangle\rangle^{(t+1)}_0 + \sum_{\alpha\in\mathcal{T}_n}\mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{A}\cdot\overline{\mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{B}}.\end{aligned} \tag{8.33}$$

Lemma 8.8. *For $1 < p < \infty$ and $t \in \mathbb{N}$, we have*

$$\left|\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_1\right| \leq \|\mathbf{A}\|_{HB^{(t)}_{p,1}(\mathcal{T}_n)}\|\mathbf{B}\|_{HB^{(t)}_{p',1}(\mathcal{T}_n)},$$

$$\|\mathbf{B}\|_{HB^{(t)}_{p',1}(\mathcal{T}_n)} = \sup_{\|\mathbf{A}\|_{HB^{(t)}_{p,1}(\mathcal{T}_n)}\leq 1}\left|\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_1\right|.$$

Combining the arguments for the order one space $HB_{p,1}(\mathcal{T}_n)$ in Subsubsection 8.1.1 with the arguments above for the order zero space $HB^{(t)}_{p,0}(\mathcal{T}_n)$ of t-tensors, we can show there is a unique reproducing kernel $\mathbf{k}^{(1,t)}_\alpha$ for the holomorphic Besov space of t-tensors $HB^{(t)}_{p,1}(\mathcal{T}_n)$ relative to this pairing. Again, $\mathbf{k}^{(1,t)}_\alpha$ is a (t,t)-tensor-valued function on the tree satisfying

$$\mathbf{A}(\alpha) = \left\langle\left\langle\mathbf{A},\mathbf{k}^{(1,t)}_\alpha\right\rangle\right\rangle^{(t)}_1, \qquad \alpha\in\mathcal{T}_n,$$

for $\mathbf{A}\in HB^{(t)}_{p,1}(\mathcal{T}_n)$, where just as in Definition 8.4, the notation $\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_1$ represents a scalar if $\mathbf{B}$ is a t-tensor, and a t-tensor if $\mathbf{B}$ is a (t,t)-tensor.

8.2. The order m holomorphic Besov space. We cannot simply define the m^{th} order holomorphic Besov space $HB^{(t)}_{p,m}(\mathcal{T}_n)$ of t-tensor-valued functions inductively to consist of all f such that $f' \in HB^{(t+1)}_{p,m-1}(\mathcal{T}_n)$. Besides the question of how to handle the error term $\left\{\mathcal{Q}^{(t)}_\alpha(\mathcal{D}_\alpha\mathbf{A})\right\}_{\alpha\in\mathcal{T}_n}$, the restriction theorem from $B_p(\mathbb{B}_n)$ to $HB_{p,2}(\mathcal{T}_n)$ will fail because $\|f'\|_{HB^{(1)}_{p,1}(\mathcal{T}_n)}$ will not in general be controlled by $\|F\|_{B_p(\mathbb{B}_n)}$ when f is the restriction of $F\in B_p(\mathbb{B}_n)$ to the Bergman tree $\mathcal{T}_n$. The problem arises when we minimize the distance from $\mathcal{D}_\alpha f$ to M_α by letting $f'(\alpha) = L^{-1}_\alpha P_\alpha\mathcal{D}_\alpha f$. The resulting vector $f'(\alpha)$ is within order 2, but not within order 3, of the restriction $F'(\alpha)$ of F' to the tree. In order to circumvent this difficulty, we will simultaneously define the first, second and through to the m^{th} order derivatives $f'(\alpha)$, $f''(\alpha)$, ..., $f^{(m)}(\alpha)$ at α using a single orthogonal projection onto the range of an appropriate generalization of the operator L_α. We will also need to define holomorphic Besov spaces of t-tensor-valued functions as well, in order to implement an inductive definition. Recall that we defined a variant $L^{(t)}_\alpha$ of $L_\alpha \equiv L^{(0)}_\alpha$ in order to define the complex derivative of a t-tensor-valued function on the Bergman tree. We will now use the notation $L_\alpha^{(1,t)}$ to denote these operators, and use the notation $L_\alpha^{(m,t)}$ to define a corresponding linear operator that will allow us to simultaneously define first through m^{th} order complex derivatives of a t-tensor-valued function on the Bergman tree. Here is the setup for the scalar case, $t = 0$.

We define the operator $L_\alpha^{(m,0)}$ as the linear map taking the point $\mathbf{x} = \left(\mathbf{v}^1, ..., \mathbf{v}^m\right)$ in the product space $\mathcal{E}_\alpha^{(1)} \times \mathcal{E}_\alpha^{(2)} \times ... \times \mathcal{E}_\alpha^{(m)}$ to the point $L_\alpha^{(m,0)}\mathbf{x}$ given by

$$\begin{aligned} L_\alpha^{(m,0)}\mathbf{x} &= \left\{\sum_{\ell=1}^m \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell \left(\alpha^j - \alpha\right)\right\}\right\}_{j=1}^N \\ &= \left\{\sum_{\ell=1}^m \frac{1}{\ell!}\mathbf{v}^\ell \left[\alpha^j - \alpha, ..., \alpha^j - \alpha\right]\right\}_{j=1}^N \in \mathbb{C}^N, \end{aligned}$$

where the second equality follows from (8.26). Here $\otimes^\ell \left(\alpha^j - \alpha\right)$ denotes the $(0,\ell)$-tensor

$$\left(\alpha^j - \alpha\right) \otimes ... \otimes \left(\alpha^j - \alpha\right),$$

where $\left(\alpha^j - \alpha\right)$ is repeated ℓ times. Let $M_\alpha^{(m,0)} = range L_\alpha^{(m,0)}$ and denote by $\left(M_\alpha^{(m,0)}\right)^\perp$ the orthogonal complement of $M_\alpha^{(m,0)}$ in $\mathbb{C}^N$. Let $\mathcal{P}_\alpha^{(m,0)}$ denote orthogonal projection of $\mathbb{C}^N$ onto $M_\alpha^{(m,0)}$, let $\mathcal{Q}_\alpha^{(m,0)} = I - \mathcal{P}_\alpha^{(m,0)}$ be orthogonal projection of $\mathbb{C}^N$ onto $\left(M_\alpha^{(m,0)}\right)^\perp$, and define as usual

$$\mathcal{D}_\alpha f = \left(f\left(\alpha^j\right) - f(\alpha)\right)_{j=1}^N \in \mathbb{C}^N.$$

At this point we need to know that $L_\alpha^{(m,0)}$ is one-to-one. In fact, in order to prove the boundedness of the restriction map later, we will need the following inequality

$$c_m \sum_{\ell=1}^m \left|\mathbf{v}^\ell\right|_\alpha \leq \left|L_\alpha^{(m,0)}\left(\mathbf{v}^1, ..., \mathbf{v}^m\right)\right| \leq C_m \sum_{\ell=1}^m \left|\mathbf{v}^\ell\right|_\alpha, \tag{8.34}$$

for all $1 \leq m \leq M$, uniformly for $\alpha \in \mathcal{T}_n$ (we actually only need $M = 2n$ for our purposes). This is established in Subsection 8.5 below using a careful reconstruction of the Bergman tree $\mathcal{T}_n$ with parameter θ sufficiently large depending on M.

Assuming (8.34) for the moment, we see that $L_\alpha^{(m,0)}$ is one-to-one. If we now define the derivatives $\left(D_m f(\alpha), D_m^2 f(\alpha), ..., D_m^m f(\alpha)\right)$ by

$$L_\alpha^{(m,0)}\left(D_m f(\alpha), D_m^2 f(\alpha), ..., D_m^m f(\alpha)\right) = \mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha f,$$

then we have that

$$\left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f\right|^2 = \left|\mathcal{D}_\alpha f - \mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha f\right|^2$$

is given by

$$\sum_{j=1}^N \left| f\left(\alpha^j\right) - \left\{f(\alpha) + \sum_{\ell=1}^m \frac{1}{\ell!} D_m^\ell f(\alpha) \wedge \left\{\otimes^\ell\left(\alpha^j - \alpha\right)\right\}\right\}\right|^2,$$

and is the minimum value of $\left|\mathcal{D}_\alpha f - L_\alpha^{(m,0)}\mathbf{x}\right|^2$ over $\mathbf{x} = \left(\mathbf{v}^1, ..., \mathbf{v}^m\right)$ in the product space $\mathcal{E}_\alpha^{(1)} \times \mathcal{E}_\alpha^{(2)} \times ... \times \mathcal{E}_\alpha^{(m)}$, where $\left|\mathcal{D}_\alpha f - L_\alpha^{(m,0)}\mathbf{x}\right|^2$ is given by

$$\sum_{j=1}^N \left| f\left(\alpha^j\right) - \left\{ f(\alpha) + \sum_{\ell=1}^m \frac{1}{\ell!}\mathbf{v}^\ell(\alpha) \wedge \left\{\otimes^\ell \left(\alpha^j - \alpha\right)\right\}\right\}\right|^2 .$$

Thus if we write

$$\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f = \left\{\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f\left(\alpha^j\right)\right\}_{j=1}^N ,$$

we have the following m^{th} order Taylor expansion of f at $A\alpha$:

$$f(\alpha) = f(A\alpha) + \sum_{\ell=1}^m \frac{1}{\ell!} D_m^\ell f(A\alpha) \wedge \left\{\otimes^\ell (\alpha - A\alpha)\right\} + \mathcal{Q}_{A\alpha}^{(m,0)}\mathcal{D}_{A\alpha} f(\alpha) . \tag{8.35}$$

The remainder term $\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f$ satisfies the minimizing property

$$\left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f\right| = \min_{\mathbf{x}\in\mathcal{E}^{(1)}\times...\times\mathcal{E}^{(m)}} \left|\mathcal{D}_\alpha f - L_\alpha^{(m,0)}\mathbf{x}\right| . \tag{8.36}$$

Note that the complex derivatives $D_m^\ell f(\alpha)$ defined here depend on both the order m and the degree of differentiation ℓ, and are generally different for different m.

We now extend the definition of m^{th} order derivatives to t-tensor-valued functions A on the tree. Define the operator $L_\alpha^{(m,t)}$ as the linear map taking the point $\mathbf{x} = \left(\mathbf{v}^1, ..., \mathbf{v}^m\right)$ in the product space $\mathcal{E}_\alpha^{(t+1)} \times \mathcal{E}_\alpha^{(t+2)} \times ... \times \mathcal{E}_\alpha^{(t+m)}$ to the point $L_\alpha^{(m,t)}\mathbf{x}$ given by

$$L_\alpha^{(m,t)}\mathbf{x} = \left\{\sum_{\ell=1}^m \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell \left(\alpha^j - \alpha\right)\right\}\right\}_{j=1}^N \in \left(\mathcal{E}_\alpha^{(t)}\right)^N ,$$

where $\mathbf{v}\wedge\left\{\otimes^\ell\left(\alpha^j - \alpha\right)\right\}$ denotes the t-tensor obtained by contracting the $(t+\ell)$-tensor $\mathbf{v}$ with the $(0,\ell)$-tensor $\otimes^\ell\left(\alpha^j - \alpha\right)$, i.e.

$$\mathbf{v}\wedge\left(\otimes^\ell\left(\alpha^j - \alpha\right)\right)\left[v^1, ..., v^t\right] = \mathbf{v}\left[\alpha^j - \alpha, ..., \alpha^j - \alpha, v^1, ..., v^t\right] .$$

Let $M_\alpha^{(m,t)} = range L_\alpha^{(m,t)}$ and denote by $\left(M_\alpha^{(m,t)}\right)^\perp$ the orthogonal complement of $M_\alpha^{(m,t)}$ in $\left(\mathcal{E}_\alpha^{(t)}\right)^N$. Let $\mathcal{P}_\alpha^{(m,t)}$ denote orthogonal projection of $\left(\mathcal{E}_\alpha^{(t)}\right)^N$ onto $M_\alpha^{(m,t)}$ with respect to the inner product on the product Hilbert space $\mathcal{E}_\alpha^{(t)} \times ... \times \mathcal{E}_\alpha^{(t)}$ (N times), and let $\mathcal{Q}_\alpha^{(m,t)} = I - \mathcal{P}_\alpha^{(m,t)}$ be orthogonal projection of $\left(\mathcal{E}_\alpha^{(t)}\right)^N$ onto $\left(M_\alpha^{(m,t)}\right)^\perp$.

In Subsection 8.5 below, we also demonstrate the extension to t-tensors of inequality (8.34) above, namely that the map $L_\alpha^{(m,t)}$ satisfies the structural inequality

$$c_m \sum_{\ell=1}^m \left|\mathbf{v}^\ell\right|_\alpha \leq \left|L_\alpha^{(m,t)}\left(\mathbf{v}^1, ..., \mathbf{v}^m\right)\right|_\alpha \leq C_m \sum_{\ell=1}^m \left|\mathbf{v}^\ell\right|_\alpha \tag{8.37}$$

for all $0 \leq m + t \leq M$, uniformly for $\alpha \in \mathcal{T}_n$. This shows in particular that $L_\alpha^{(m,t)}$ is one-to-one. Suppose now that $\mathbf{A}$ is a t-tensor-valued function on the tree $\mathcal{T}_n$. We define the m-tuple $\left(D_m \mathbf{A}(\alpha), D_m^2 \mathbf{A}(\alpha), ..., D_m^m \mathbf{A}(\alpha)\right)$ of derivatives up to order m of $\mathbf{A}$ so that

$$L_\alpha^{(m,t)} \left(D_m \mathbf{A}(\alpha), D_m^2 \mathbf{A}(\alpha), ..., D_m^m \mathbf{A}(\alpha)\right) = \mathcal{P}_\alpha^{(m,t)} \mathcal{D}_\alpha \mathbf{A}.$$

As in the case $t = 0$, the remainder term $\mathcal{Q}_\alpha^{(m,t)} \mathcal{D}_\alpha \mathbf{A}$ satisfies the minimizing property

$$\left| \mathcal{Q}_\alpha^{(m,t)} \mathcal{D}_\alpha \mathbf{A} \right|_\alpha = \min_{\mathbf{x} \in \mathcal{E}_\alpha^{(1)} \times ... \times \mathcal{E}_\alpha^{(m)}} \left| \mathcal{D}_\alpha \mathbf{A} - L_\alpha^{(m,t)} \mathbf{x} \right|_\alpha . \tag{8.38}$$

We can now define by induction on m the order m holomorphic Besov space $HB_{p,m}^{(t)}(\mathcal{T}_n)$ of t-tensors on $\mathcal{T}_n$. As usual, we use

$$\left\| \mathcal{Q}^{(m,t)} \mathcal{D} \mathbf{A} \right\|_{\ell^p(\mathcal{T}_n)}^p = \sum_{\alpha \in \mathcal{T}_n} \left| \mathcal{Q}_\alpha^{(m,t)} \mathcal{D}_\alpha \mathbf{A} \right|_\alpha^p ,$$

where as above, we define

$$|w|_\alpha^2 = \sum_{j=1}^N |w_j|_\alpha^2 , \qquad w = (w_1, ..., w_N) \in \left(\mathcal{E}_\alpha^{(t)}\right)^N .$$

Definition 8.9. *For $1 < p < \infty$ and $0 \leq m + t \leq M$, let $HB_{p,m}^{(t)}(\mathcal{T}_n)$ consist of all t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$ such that the norm (to the p^{th} power)*

$$\|\mathbf{A}\|_{HB_{p,m}^{(t)}(\mathcal{T}_n)}^p = |\mathbf{A}(o)|^p + \sum_{\ell=1}^m \left\| D_m^\ell \mathbf{A} \right\|_{HB_{p,m-\ell}^{(t+\ell)}(\mathcal{T}_n)}^p + \left\| \mathcal{Q}^{(m,t)} \mathcal{D} \mathbf{A} \right\|_{\ell^p(\mathcal{T}_n)}^p$$

is finite. We write simply $HB_{p,m}(\mathcal{T}_n)$ for the scalar case $t = 0$.

8.2.1. *Higher order reproducing kernels for tensors and the positivity property.* In this subsubsection we establish the key positivity property of reproducing kernels that will permit us to use the technique of Böe's "curious lemma". It is this property that yields the fruit of our labour in developing the theory of holomorphic Besov spaces on trees. Let $p = 2$. Then the inner product corresponding to the Hilbert space norm $\|\mathbf{A}\|_{HB_{2,m}^{(t)}(\mathcal{T}_n)}$ defined on t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$ is defined by induction on m by

$$\begin{aligned} \langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_m^{(t)} = \mathbf{A}(o) \cdot \overline{\mathbf{B}(o)} &+ \sum_{\ell=1}^m \left\langle\left\langle D_m^\ell \mathbf{A}, D_m^\ell \mathbf{B} \right\rangle\right\rangle_{m-\ell}^{(t+\ell)} \\ &+ \sum_{\alpha \in \mathcal{T}_n} \left\langle \mathcal{Q}_\alpha^{(m,t)}(\mathcal{D}_\alpha \mathbf{A}), \mathcal{Q}_\alpha^{(m,t)}(\mathcal{D}_\alpha \mathbf{B}) \right\rangle_\alpha , \end{aligned}$$

where the cases $m = 0$ and $m = 1$ are defined in (8.25) and (8.33) respectively. We have the following duality relation.

PROPOSITION 8.10. *For $1 < p < \infty$ and $t \in \mathbb{N}$, we have*

$$\left|\langle\langle \mathbf{A}, \mathbf{B}\rangle\rangle_m^{(t)}\right| \leq \|\mathbf{A}\|_{HB_{p,m}^{(t)}(\mathcal{T}_n)} \|\mathbf{B}\|_{HB_{p',m}^{(t)}(\mathcal{T}_n)},$$
$$\|\mathbf{A}\|_{HB_{p,m}^{(t)}(\mathcal{T}_n)} = \sup_{\|\mathbf{B}\|_{HB_{p',m}^{(t)}(\mathcal{T}_n)} \leq 1} \left|\langle\langle \mathbf{A}, \mathbf{B}\rangle\rangle_m^{(t)}\right|.$$

PROOF. The case $m = 0$ is just the duality of ℓ^p and $\ell^{p'}$, and the case $m = 1$ is proved in Proposition 8.1 and Lemma 8.8, which treat respectively the cases $t = 0$ and $t \geq 1$. Consider now the case $m = 2$. The inequality above is Hölder's inequality. To see the equality above, let $[o, \alpha] = \{o, \alpha_1, ..., \alpha_M = \alpha\}$ be the geodesic from the root to α, and note that

$$\begin{aligned}
\mathbf{A}(\alpha) &= \mathbf{A}(o) + \sum_{j=1}^{M} [\mathbf{A}(\alpha_j) - \mathbf{A}(\alpha_{j-1})] \\
&= \mathbf{A}(o) + \sum_{j=1}^{M} D_2^1\mathbf{A}(\alpha_{j-1}) \wedge (\alpha_j - \alpha_{j-1}) \\
&\quad + \sum_{j=1}^{M} \frac{1}{2} D_2^2\mathbf{A}(\alpha_{j-1}) \wedge \{(\alpha_j - \alpha_{j-1}) \otimes (\alpha_j - \alpha_{j-1})\} \\
&\quad + \sum_{j=1}^{M} \mathcal{Q}_{\alpha_{j-1}}^{(2,t)} \mathcal{D}_{\alpha_{j-1}} \mathbf{A}(\alpha_j).
\end{aligned}$$

To handle the terms $D_2^1\mathbf{A}(\alpha_{j-1})$ we must first use the analogue of the above formula with $m = 1$, $D_2^1\mathbf{A}$ in place of $\mathbf{A}$, and α_N in place of α, $N < M$, to obtain

$$\begin{aligned}
D_2^1\mathbf{A}(\alpha_N) &= D_2^1\mathbf{A}(o) + \sum_{j=1}^{N} \left[D_2^1\mathbf{A}(\alpha_j) - D_2^1\mathbf{A}(\alpha_{j-1})\right] \\
&= D_2^1\mathbf{A}(o) + \sum_{j=1}^{N} D_1^1 D_2^1\mathbf{A}(\alpha_{j-1}) \wedge (\alpha_j - \alpha_{j-1}) \\
&\quad + \sum_{j=1}^{N} \mathcal{Q}_{\alpha_{j-1}}^{(1,t+1)} \mathcal{D}_{\alpha_{j-1}} D_2^1\mathbf{A}(\alpha_j).
\end{aligned}$$

Using this formula for $D_2^1\mathbf{A}$, we see that there is a unique $(t+1)$-tensor $\mathbf{G}$ satisfying

$$\begin{aligned}
\mathbf{G}(o) &= D_2^1\mathbf{A}(o)\left|D_2^1\mathbf{A}(o)\right|^{p-2}, \\
D_1^1\mathbf{G}(\gamma) &= D_1^1 D_2^1\mathbf{A}(\gamma)\left|D_1^1 D_2^1\mathbf{A}(\gamma)\right|_\gamma^{p-2}, \\
\mathcal{Q}_\gamma^{(1,t+1)}\mathcal{D}_\gamma\mathbf{G} &= \mathcal{Q}_\gamma^{(1,t+1)}\mathcal{D}_\gamma D_2^1\mathbf{A}\left|\mathcal{Q}_\gamma^{(1,t+1)}\mathcal{D}_\gamma D_2^1\mathbf{A}\right|_\gamma^{p-2},
\end{aligned}$$

(note the crucial property that $\mathcal{Q}_\gamma^{(1,t+1)}\mathcal{D}_\gamma\mathbf{G} \in \left(M_\gamma^{(1,t+1)}\right)^\perp$) so that by Definition 8.9,

$$\begin{aligned}
\|\mathbf{G}\|^{p'}_{HB^{(t+1)}_{p',1}(\mathcal{T}_n)} &= |\mathbf{G}(o)|^{p'} + \left\|D_1^1\mathbf{G}\right\|^{p'}_{HB^{(t+2)}_{p',0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,t+1)}\mathcal{D}\mathbf{G}\right\|^{p'}_{\ell^{p'}(\mathcal{T}_n)} \\
&= \left|D_2^1\mathbf{A}(o)\right|^p + \left\|D_1^1D_2^1\mathbf{A}\right\|^p_{HB^{(t+2)}_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,t+1)}\mathcal{D}D_2^1\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&= \left\|D_2^1\mathbf{A}\right\|^p_{HB^{(t+1)}_{p,1}(\mathcal{T}_n)}
\end{aligned}$$

since $(p-1)p' = p$. Using the above formula for $\mathbf{A}$, we see that there is a unique t-tensor $\mathbf{H}$ satisfying

$$\begin{aligned}
\mathbf{H}(o) &= \mathbf{A}(o)\,|\mathbf{A}(o)|^{p-2} \\
D_2^1\mathbf{H}(\gamma) &= \mathbf{G}(\gamma) \\
D_2^2\mathbf{H}(\gamma) &= D_2^2\mathbf{A}(\gamma)\left|D_2^2\mathbf{A}(\gamma)\right|^{p-2}_\gamma, \\
\mathcal{Q}_\gamma^{(2,t)}\mathcal{D}_\gamma\mathbf{H} &= \mathcal{Q}_\gamma^{(2,t)}\mathcal{D}_\gamma\mathbf{A}\left|\mathcal{Q}_\gamma^{(2,t)}\mathcal{D}_\gamma\mathbf{A}\right|^{p-2}_\gamma,
\end{aligned}$$

(note the crucial property that $\mathcal{Q}_\gamma^{(2,t)}\mathcal{D}_\gamma\mathbf{H} \in \left(M_\gamma^{(2,t)}\right)^\perp$) so that by Definition 8.9,

$$\begin{aligned}
\|\mathbf{H}\|^{p'}_{HB^{(t)}_{p',2}(\mathcal{T}_n)} &= |\mathbf{A}(o)|^p + \|\mathbf{G}\|^{p'}_{HB^{(t+1)}_{p',1}(\mathcal{T}_n)} \\
&\qquad + \left\|D_2^2\mathbf{A}\right\|^p_{HB^{(t+2)}_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&= |\mathbf{A}(o)|^p + \left\|D_2^1\mathbf{A}\right\|^p_{HB^{(t+1)}_{p,1}(\mathcal{T}_n)} \\
&\qquad + \left\|D_2^2\mathbf{A}\right\|^p_{HB^{(t+2)}_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&= \|\mathbf{A}\|^p_{HB^{(t)}_{p,2}(\mathcal{T}_n)}
\end{aligned}$$

again since $(p-1)p' = p$. A similar calculation also yields that

$$\langle\langle\mathbf{A},\mathbf{H}\rangle\rangle_2^{(t)} = \|\mathbf{A}\|^p_{HB^{(t)}_{p,2}(\mathcal{T}_n)},$$

and if we now take $\mathbf{B} = \frac{\mathbf{H}}{\|\mathbf{H}\|_{HB^{(t)}_{p',2}(\mathcal{T}_n)}}$, we obtain

$$\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle_2^{(t)} = \frac{\|\mathbf{A}\|^p_{HB^{(t)}_{p,2}(\mathcal{T}_n)}}{\|\mathbf{H}\|_{HB^{(t)}_{p',2}(\mathcal{T}_n)}} = \frac{\|\mathbf{A}\|^p_{HB^{(t)}_{p,2}(\mathcal{T}_n)}}{\|\mathbf{A}\|^{p-1}_{HB^{(t)}_{p,2}(\mathcal{T}_n)}} = \|\mathbf{A}\|_{HB^{(t)}_{p,2}(\mathcal{T}_n)},$$

which yields the equality in the statement of Proposition 8.10. The case $m \geq 3$ is treated in the same fashion, and this completes the proof of Proposition 8.10.

Denote by $\mathbf{k}_\alpha^{(m,t)}$ the reproducing kernel for $\alpha \in \mathcal{T}_n$ relative to this inner product, which exists by a modification of the argument in the case $m = 0$ immediately following Lemma 8.6 in Subsubsection 8.1.2. Then with the usual notation convention regarding inner products as in Definition 8.4, we have

$$\begin{aligned} \mathbf{A}(\alpha) &= \left\langle\left\langle \mathbf{A}, \mathbf{k}_\alpha^{(m,t)} \right\rangle\right\rangle_m^{(t)} \\ &= \mathbf{A}(o) \cdot \overline{\mathbf{k}_\alpha^{(m,t)}(o)} + \sum_{\ell=1}^{m} \left\langle\left\langle D_m^\ell \mathbf{A}, D_m^\ell \mathbf{k}_\alpha^{(m,t)} \right\rangle\right\rangle_{m-\ell}^{(t+\ell)} \\ &\quad + \sum_{\alpha \in \mathcal{T}_n} \left\langle \mathcal{Q}_\alpha^{(m,t)}(\mathcal{D}_\alpha \mathbf{A}), \mathcal{Q}_\alpha^{(m,t)}\left(\mathcal{D}_\alpha \mathbf{k}_\alpha^{(m,t)}\right)\right\rangle_\alpha . \end{aligned} \tag{8.39}$$

We can also recover $\mathbf{A}(\alpha)$ from $\mathbf{A}(o)$ together with the data

$$D_m^1 \mathbf{A}(\gamma), D_m^2 \mathbf{A}(\gamma), ..., D_m^m \mathbf{A}(\gamma)$$

and the remainder term $\mathcal{Q}_\gamma^{(m,t)} \mathcal{D}_\gamma \mathbf{A}$ for $\gamma \in [o, \alpha]$. Set $[o, \alpha] = \{o, \alpha_1, ..., \alpha_M = \alpha\}$ then we have

$$\begin{aligned} \mathbf{A}(\alpha) &= \mathbf{A}(o) + \sum_{j=1}^{M} [\mathbf{A}(\alpha_j) - \mathbf{A}(\alpha_{j-1})] \\ &= \mathbf{A}(o) + \sum_{j=1}^{M} D_m^1 \mathbf{A}(\alpha_{j-1}) \wedge (\alpha_j - \alpha_{j-1}) \\ &\quad + \sum_{j=1}^{M} \frac{1}{2} D_m^2 \mathbf{A}(\alpha_{j-1}) \wedge \{(\alpha_j - \alpha_{j-1}) \otimes (\alpha_j - \alpha_{j-1})\} \\ &\vdots \\ &\quad + \sum_{j=1}^{M} \frac{1}{m!} D_m^m \mathbf{A}(\alpha_{j-1}) \wedge \{\otimes^m (\alpha_j - \alpha_{j-1})\} \\ &\quad + \sum_{j=1}^{M} \mathcal{Q}_{\alpha_{j-1}}^{(m,t)} \mathcal{D}_{\alpha_{j-1}} \mathbf{A}(\alpha_j) . \end{aligned} \tag{8.40}$$

Using the reproducing kernels $\mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}$ to recover $D_m^\ell \mathbf{A}(A\beta)$, we can rewrite this as

$$\begin{aligned} \mathbf{A}(\alpha) &= \mathbf{A}(o) + \sum_{\ell=1}^{m} \sum_{o<\beta\leq\alpha} \frac{1}{\ell!} D_m^\ell \mathbf{A}(A\beta) \wedge \left\{\otimes^\ell (\beta - A\beta)\right\} \\ &\quad + \sum_{o<\beta\leq\alpha} \mathcal{Q}_{A\beta}^{(m,t)} \mathcal{D}_{A\beta} \mathbf{A}(\beta) \\ &= \mathbf{A}(o) + \sum_{\ell=1}^{m} \sum_{o<\beta\leq\alpha} \frac{1}{\ell!} \left\langle\left\langle D_m^\ell \mathbf{A}, \mathbf{k}_{A\beta}^{(m-\ell,t+\ell)} \right\rangle\right\rangle_{m-\ell}^{(t+\ell)} \wedge \left\{\otimes^\ell (\beta - A\beta)\right\} \\ &\quad + \sum_{o<\beta\leq\alpha} \mathcal{Q}_{A\beta}^{(m,t)} \mathcal{D}_{A\beta} \mathbf{A}(\beta) , \end{aligned}$$

and since $\wedge$ and $\wedge_{A\beta}$ commute, we have

$$\begin{aligned}\left\langle D_m^\ell \mathbf{A}(\gamma), \mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma)\right\rangle_\gamma^{(t+\ell)} \wedge \left\{\otimes^\ell (\beta - A\beta)\right\} &= \overline{\mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma)} \wedge_\gamma D_m^\ell \mathbf{A}(\gamma) \wedge \left\{\otimes^\ell (\beta - A\beta)\right\} \\ &= \overline{\mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma)} \wedge \left\{\otimes^\ell (\beta - A\beta)\right\} \wedge_\gamma D_m^\ell \mathbf{A}(\gamma) \\ &= \left\langle D_m^\ell \mathbf{A}(\gamma), \mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma) \wedge \left\{\otimes^\ell \overline{\beta - A\beta}\right\}\right\rangle_\gamma^{(t+\ell)},\end{aligned}$$

and so

$$\mathbf{A}(\alpha) = \mathbf{A}(o) + \sum_{\ell=1}^m \left\langle\left\langle D_m^\ell \mathbf{A}, \sum_{o<\beta\le\alpha} \frac{1}{\ell!} \mathbf{k}_{A\beta}^{(m-\ell,t+\ell)} \wedge \left(\otimes^\ell \overline{\beta - A\beta}\right)\right\rangle\right\rangle_{m-\ell}^{(t+\ell)} + \sum_{o<\beta\le\alpha} \mathcal{Q}_{A\beta}^{(m,t)} \mathcal{D}_{A\beta} \mathbf{A}(\beta). \tag{8.41}$$

By uniqueness of the representation formula (8.40) subject to the condition that

$$\mathcal{Q}_{\alpha_{j-1}}^{(m,t)} \mathcal{D}_{\alpha_{j-1}} \mathbf{A} \in M_{\alpha_{j-1}}^\perp, \qquad 1 \le j \le m,$$

(this uses that $L_{\alpha_{j-1}}^{(m,t)}$ is one-to-one as well as the uniqueness of the orthogonal decompositions into $M_{\alpha_{j-1}}$ and $M_{\alpha_{j-1}}^\perp$; compare with the discussion surrounding (8.17)), we see upon comparing (8.39) and (8.41) that we have the recursion formula

$$D_m^\ell \mathbf{k}_\alpha^{(m,t)} = \sum_{o<\beta\le\alpha} \frac{1}{\ell!} \mathbf{k}_{A\beta}^{(m-\ell,t+\ell)} \wedge \left(\otimes^\ell \overline{\beta - A\beta}\right), \qquad 1 \le \ell \le m, \tag{8.42}$$

as well as

$$\mathbf{A}(o) = \mathbf{A}(o) \cdot \overline{\mathbf{k}_\alpha^{(m,t)}(o)},$$
$$\sum_{o<\beta\le\alpha} \mathcal{Q}_{A\beta}^{(m,0)} \mathcal{D}_{A\beta} \mathbf{A}(\beta) = \sum_{\alpha\in\mathcal{T}_n} \left\langle \mathcal{Q}_\alpha^{(m,t)}(\mathcal{D}_\alpha \mathbf{A}), \mathcal{Q}_\alpha^{(m,t)}\left(\mathcal{D}_\alpha \mathbf{k}_\alpha^{(m,t)}\right)\right\rangle_\alpha.$$

Note that the left side (8.42) is a tensor of order $2t + \ell$, while that of the right side has order $2(t+\ell) - \ell$, the same order. We now use the recursion formula in (8.42) to establish by induction the following positivity property for derivatives of the reproducing kernels $\mathbf{k}_\alpha^{(m,t)}$.

Lemma 8.11. *Let $0 \le m+t \le M$. Then provided we choose λ small enough and θ large enough in the construction of the Bergman tree, we have for all $\alpha, \gamma \in \mathcal{T}_n$,*

$$r^{-md(\gamma)} \operatorname{Re}\left(D_m^m k_\alpha^{(m,0)}(\gamma) \wedge \{\otimes^m \gamma\}\right) \approx 1 \tag{8.43}$$

$$\left|D_m^\ell k_\alpha^{(m,t)}(\gamma)\right|_\gamma + \left|\mathcal{Q}_\gamma^{(m,t)}\left(\mathcal{D}_\gamma k_\alpha^{(m,t)}\right)\right|_\gamma \le \begin{cases} C & \text{for} \quad \gamma \le \alpha \\ 0 & \qquad \text{otherwise} \end{cases},$$

where $D_m^m k_\alpha^{(m,0)}(\gamma) \wedge \{\otimes^m \gamma\} = D_m^m k_\alpha^{(m,0)}(\gamma)[\gamma, ..., \gamma]$.

Proof. Using induction with (8.32) and the recursion formula (8.42), we see that $D_m^\ell \mathbf{k}_\alpha^{(m,t)}$ is supported in the geodesic $[o, \alpha]$ for $\ell \ge 1$. The case $m = 0$ of

(8.43) is trivial from (8.32), and for $m=1$ and $t=0$ (8.43) has been established in (8.20) above. Now consider the case $m=1$ and $t\geq 1$. Then from (8.42) we have

$$D_m\mathbf{k}_\alpha^{(1,t)}(\gamma)=\sum_{o<\beta\leq\alpha}\mathbf{k}_{A\beta}^{(0,t+1)}(\gamma)\wedge\overline{\beta-A\beta}.$$

Note that each side of the above equation is a $(2t+1)$-tensor. By (8.32), we have that $\mathbf{k}_{A\beta}^{(0,t+1)}(\gamma)$ is a $2(t+1)$-tensor that vanishes if $\gamma\neq A\beta$, and so

$$D_m\mathbf{k}_\alpha^{(1,t)}(\gamma)=\mathbf{k}_\gamma^{(0,t+1)}(\gamma)\wedge\overline{\gamma_\alpha-\gamma}.$$

Recall also from (8.32) that

$$\mathbf{k}_\gamma^{(0,t)}(\gamma)=\sum_i\left\{\mathbf{R}^{d(\gamma)}e^{i_1}\otimes...\otimes\mathbf{R}^{d(\gamma)}e^{i_t}\right\}\otimes\left\{\mathbf{R}^{d(\gamma)}e_{i_1}\otimes...\otimes\mathbf{R}^{d(\gamma)}e_{i_t}\right\}.$$

We now easily obtain the second line in (8.43) for $m=1$ and $t\geq 0$.

For $m=\ell\geq 2$ and $t=0$, the recursion formula (8.42) yields

$$\begin{aligned}D_m^m\mathbf{k}_\alpha^{(m,0)}(\gamma)&=\sum_{o<\beta\leq\alpha}\frac{1}{m!}\mathbf{k}_{A\beta}^{(0,m)}(\gamma)\wedge\left\{\otimes^m\overline{\beta-A\beta}\right\}\\&=\frac{1}{m!}\mathbf{k}_\gamma^{(0,m)}(\gamma)\wedge\left\{\otimes^m\overline{\gamma_\alpha-\gamma}\right\},\end{aligned}$$

and so

$$\begin{aligned}D_m^m\mathbf{k}_\alpha^{(m,0)}(\gamma)\wedge\{\otimes^m\gamma\}&=\frac{1}{m!}\mathbf{k}_\gamma^{(0,m)}(\gamma)\wedge\left\{\otimes^m\overline{\gamma_\alpha-\gamma}\right\}\wedge\{\otimes^m\gamma\}\\&=\frac{1}{m!}\sum_i\left\langle\mathbf{R}^{d(\gamma)}e^{i_1},\gamma_\alpha-\gamma\right\rangle\times...\times\left\langle\mathbf{R}^{d(\gamma)}e^{i_m},\gamma_\alpha-\gamma\right\rangle\\&\qquad\times\left\langle\mathbf{R}^{d(\gamma)}e_{i_1},\gamma\right\rangle\times...\times\left\langle\mathbf{R}^{d(\gamma)}e_{i_m},\gamma\right\rangle.\end{aligned}$$

Now we recall from (8.21) that $e_1=\frac{\gamma}{|\gamma|}$ and that e_j is orthogonal to γ for $j\geq 2$. Thus $\left\langle\mathbf{R}^{d(\gamma)}e_{i_k},\gamma\right\rangle=r^{d(\gamma)}|\gamma|\,\delta_{i_k}^1$ and so

$$\begin{aligned}D_m^m\mathbf{k}_\alpha^{(m,0)}(\gamma)\wedge\{\otimes^m\gamma\}&=\frac{1}{m!}\left\langle\mathbf{R}^{d(\gamma)}e^1,\gamma_\alpha-\gamma\right\rangle^m\left(r^{d(\gamma)}|\gamma|\right)^m\\&=\frac{r^{2md(\gamma)}}{m!}\langle\gamma,\gamma_\alpha-\gamma\rangle^m\\&=\frac{r^{2md(\gamma)}}{m!}\langle\gamma,P_\gamma\gamma_\alpha-\gamma\rangle^m.\end{aligned}$$

Let $\varepsilon>0$ be given. The vector $P_\gamma\gamma_\alpha$ lies in $P_\gamma K_{\gamma_\alpha}$, and from the construction of the Bergman tree $\mathcal{T}_n$ *with* λ *chosen sufficiently small*, we see that

$$|\arg\langle\gamma,P_\gamma\gamma_\alpha-\gamma\rangle|<\varepsilon,$$

as well as

$$\operatorname{Re}\langle\gamma,P_\gamma\gamma_\alpha-\gamma\rangle=\operatorname{Re}\langle\gamma,\gamma_\alpha-\gamma\rangle\geq cr^{-d(\gamma)}.$$

It follows that

$$\operatorname{Re}\left(\langle\gamma,P_\gamma\gamma_\alpha-\gamma\rangle^m\right)\geq c^mr^{-md(\gamma)}(1-m\varepsilon)\geq c_0r^{-md(\gamma)}.$$

This proves the first line in (8.43) for $0\leq m\leq M$.

The proof of the second line in (8.43) can easily be completed by induction as follows. By the recursion formula (8.42), we have

$$\begin{aligned}\left|D_m^\ell \mathbf{k}_\alpha^{(m,t)}(\gamma)\right|_\gamma &= \left|\sum_{o<\beta\leq\alpha} \frac{1}{\ell!}\mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma)\wedge\left\{\otimes^\ell\overline{\beta - A\beta}\right\}\right|_\gamma \\ &\leq \sum_{o<\beta\leq\alpha}\frac{1}{\ell!}\left|\mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma)\wedge\left\{\otimes^\ell\overline{\beta - A\beta}\right\}\right|_\gamma,\end{aligned}$$

which is bounded by a constant C using the induction assumption with $m-\ell<m$, together with the fact that $\beta > A\beta \geq \gamma$ if $\mathbf{k}_{A\beta}^{(m-\ell,t+\ell)}(\gamma)$ is nonzero. The boundedness of $\left|\mathcal{Q}_\gamma^{(m,t)}\left(\mathcal{D}_\gamma \mathbf{k}_\alpha^{(m,t)}\right)\right|_\gamma$ follows from (8.38), and this completes the proof of Lemma 8.11.

8.3. Carleson measures. Here we characterize Carleson measures on the holomorphic Besov space $HB_{p,m}(\mathcal{T}_n)$.

THEOREM 8.12. *Let $1<p<\infty$ and $1\leq m\leq M$. Then there are λ and θ in the construction of the Bergman tree, sufficiently small and large respectively, such that μ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure, i.e.*

$$\left(\sum_{\alpha\in\mathcal{T}_n}|f(\alpha)|^p\mu(\alpha)\right)^{\frac{1}{p}} \leq C\|f\|_{HB_{p,m}(\mathcal{T}_n)}, \tag{8.44}$$

if and only if the tree condition (3.2) holds, i.e.

$$\sum_{\beta\geq\alpha}\left(\sum_{\gamma\geq\beta}\mu(\gamma)\right)^{p'} \leq C\sum_{\beta\geq\alpha}\mu(\beta)<\infty, \qquad \alpha\in\mathcal{T}_n. \tag{8.45}$$

PROOF. We first show that (8.45) implies (8.44). To see this, note that by Definition 8.9 with $t=0$ and (8.40) with f in place of $\mathbf{A}$, we have

$$\begin{aligned}|f(\alpha)| &\leq |f(o)| + \sum_{\ell=1}^m\sum_{o<\beta\leq\alpha}\frac{1}{\ell!}\left|D_m^\ell f(A\beta)\wedge\left\{\otimes^\ell(\beta-A\beta)\right\}\right| \\ &\qquad + \left|\sum_{o<\beta\leq\alpha}\mathcal{Q}_{A\beta}^{(m,t)}\mathcal{D}_{A\beta}\mathbf{A}(\beta)\right| \\ &\leq |f(o)| + C\sum_{\ell=1}^m\sum_{o<\beta\leq\alpha}\frac{1}{\ell!}\left|D_m^\ell f(A\beta)\right|_{A\beta} + \left|\sum_{o<\beta\leq\alpha}\mathcal{Q}_{A\beta}^{(m,t)}\mathcal{D}_{A\beta}\mathbf{A}(\beta)\right| \\ &\leq |f(o)| + CIg(\alpha),\end{aligned}$$

where

$$g(\beta) = \sum_{\ell=1}^m\left|D_m^\ell f(A\beta)\right|_{A\beta} + \left|\mathcal{Q}_{A\beta}\mathcal{D}_{A\beta}f(\beta)\right|.$$

However, because the terms $\left|D_m^\ell f(A\beta)\right|_{A\beta}$ are large when $\ell<m$, the ℓ^p norm of g is not dominated by $\|f\|_{HB_{p,m}(\mathcal{T}_n)}$. Instead we must iterate (8.40) with $\mathbf{A}$ replaced

first by $D_m^\ell f$, $\ell < m$, then by $D_{m-\ell_1}^{\ell_2} D_m^{\ell_1} f$, and in general by

$$D_m^\ell f = D_{m-s_{k-1}}^{\ell_k} ... D_{m-\ell_1}^{\ell_2} D_m^{\ell_1} f, \qquad s_k \equiv \ell_1 + \ell_2 + ... + \ell_k,$$

where $\ell = (\ell_1, \ell_2, ..., \ell_k)$ is now a k-tuple, $k \leq m$. The resulting estimate is

$$|f(\alpha)| \leq \sum_{s_k < m} \left| D_{m-s_{k-1}}^{\ell_k} ... D_{m-\ell_1}^{\ell_2} D_m^{\ell_1} f(o) \right| + CIg(\alpha),$$

where g is now given by

$$g(\beta) = \sum_{s_k = m} \left| D_{m-s_{k-1}}^{\ell_k} ... D_m^{\ell_1} f\left(A^k \beta\right) \right|_{A^k \beta} \tag{8.46}$$

$$+ \sum_{s_k = m} \left| \mathcal{Q}_{A^k \beta}^{(m-s_{k-1}, s_{k-1})} \mathcal{D}_{A^k \beta} D_{m-s_{k-2}}^{\ell_{k-1}} ... D_m^{\ell_1} f \right|_{A^k \beta}. \tag{8.47}$$

Using this, together with our assumption (8.45) and Theorem 3.1, we have

$$\begin{aligned}
\left(\sum_{\alpha \in \mathcal{T}_n} |f(\alpha)|^p \mu(\alpha) \right)^{\frac{1}{p}} &\leq C \left\{ |f(o)| + \left(\sum_{\alpha \in \mathcal{T}_n} |Ig(\alpha)|^p \mu(\alpha) \right)^{\frac{1}{p}} \right\} \\
&\leq C \left\{ |f(o)| + \left(\sum_{\alpha \in \mathcal{T}_n} |g(\alpha)|^p \right)^{\frac{1}{p}} \right\} \\
&\leq C \|f\|_{HB_{p,m}(\mathcal{T}_n)},
\end{aligned}$$

which is (8.44). The final line above is proved in more detail in (8.59) below.

REMARK 8.13. The Bergman norm $\left| D_m^\ell f(A\beta) \right|_{A\beta}$ of $D_m^\ell f(A\beta)$ arises naturally in (8.46) as a pointwise bound for the expression $\left| D_m^\ell f(A\beta) \wedge \left\{ \otimes^\ell (\beta - A\beta) \right\} \right|$. The somewhat simpler scaled Euclidean norm $r^{-\ell d(A\beta)} \left| D_m^\ell f(A\beta) \right|$ does not provide a pointwise upper bound for this, and this is one reason why we choose to use the slightly more complicated Bergman norms over the scaled Euclidean norms. Another reason is the growth estimate (8.6) for functions in $HB_{p,m}(\mathcal{T}_n)$ that ensures point evaluations are continuous linear functionals on $HB_{p,m}(\mathcal{T}_n)$. The Bergman norms are also natural in view of the almost invariant seminorms $\|\cdot\|_{B_{p,m}}^*$ defined on the ball in Definition 6.3. On the other hand, it seems likely that one can develop the theory of holomorphic Besov spaces on Bergman trees using the scaled Euclidean norm together with Schur lemma techniques, just as on the ball, but we will not pursue this here.

Conversely, we show that the dual of (8.44) implies (8.45). Let $\mathbf{k}_\alpha^{(m,0)}$ be the (scalar-valued) reproducing kernel for $HB_{p,m}(\mathcal{T}_n)$. Since

$$\begin{aligned}
\sum_{\alpha \in \mathcal{T}_n} f(\alpha) \overline{g(\alpha)} \mu(\alpha) &= \sum_{\alpha \in \mathcal{T}_n} \left\langle \left\langle f, \mathbf{k}_\alpha^{(m,0)} \right\rangle \right\rangle_m^{(0)} \overline{g(\alpha)} \mu(\alpha) \\
&= \left\langle \left\langle f, \sum_{\alpha \in \mathcal{T}_n} g(\alpha) \mu(\alpha) \mathbf{k}_\alpha^{(m,0)} \right\rangle \right\rangle_m^{(0)},
\end{aligned}$$

(8.44) and Proposition 8.10 imply the dual inequality

$$\begin{aligned}
&\left\|\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,\mathbf{k}_\alpha^{(m,0)}\right\|_{HB_{p',m}(\mathcal{T}_n)} \\
&\qquad = \sup_{\|f\|_{HB_{p,m}(\mathcal{T}_n)}\le 1}\left|\left\langle\left\langle f,\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,k_\alpha\right\rangle\right\rangle_m^{(0)}\right| \\
&\qquad = \sup_{\|f\|_{HB_{p,m}(\mathcal{T}_n)}\le 1}\left|\sum_{\alpha\in\mathcal{T}_n} f(\alpha)\,\overline{g(\alpha)}\mu(\alpha)\right| \\
&\qquad \le \sup_{\|f\|_{HB_{p,m}(\mathcal{T}_n)}\le 1}\|f\|_{L^p(\mu)}\,\|g\|_{L^{p'}(\mu)} \\
&\qquad \le C\,\|g\|_{L^{p'}(\mu)}\,.
\end{aligned}$$

By Definition 8.9 with $t=0$, this implies in particular that

(8.48)

$$\begin{aligned}
\left\|D_m^m\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,\mathbf{k}_\alpha^{(m,0)}\right)\right\|_{HB_{p',0}^{(m)}(\mathcal{T}_n)} &\le \left\|\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,\mathbf{k}_\alpha^{(m,0)}\right\|_{HB_{p',m}(\mathcal{T}_n)} \\
&\le C\,\|g\|_{L^{p'}(\mu)}\,.
\end{aligned}$$

Note as always that we may assume μ has finite support.

We now restrict g to be nonnegative in (8.48). From the first part of (8.43) in Lemma 8.11, and the fact that the support of $D_m^m\mathbf{k}_\alpha^{(m,0)}$ is contained in the geodesic $[o,\alpha)$, we obtain that

$$\begin{aligned}
&\|I^*g\mu\|_{\ell^{p'}(\gamma)} \\
&\quad = \left\|\sum_{\alpha\in\mathcal{T}_n:\alpha\ge\gamma} g(\alpha)\,\mu(\alpha)\right\|_{\ell^{p'}(\gamma)} \\
&\quad \le C\left\|\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,r^{-md(\gamma)}\,\mathrm{Re}\left(D_m^m\mathbf{k}_\alpha^{(m,0)}(\gamma)\wedge\{\otimes^m\gamma\}\right)\right\|_{\ell^{p'}(\gamma)} \\
&\quad = C\left\|r^{-md(\gamma)}\,\mathrm{Re}\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,D_m^m\mathbf{k}_\alpha^{(m,0)}(\gamma)\wedge\{\otimes^m\gamma\}\right)\right\|_{\ell^{p'}(\gamma)} \\
&\quad \le C\left\|r^{-md(\gamma)}D_m^m\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,\mathbf{k}_\alpha^{(m,0)}(\gamma)\right)\right\|_{\ell^{p'}(\gamma)} \\
&\quad \le C\left\|D_m^m\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)\,\mu(\alpha)\,\mathbf{k}_\alpha^{(m,0)}(\gamma)\right)\right\|_{HB_{p',0}^{(m)}(\mathcal{T}_n)} \le C\,\|g\|_{L^{p'}(\mu)}
\end{aligned}$$

for all $g\ge 0$ by (8.48). This yields (8.45) as required upon taking $g=\chi_{S(\alpha)}$.

8.4. The holomorphic restriction map. In the special case where f arises as the restriction $f=TF=\{F(c_\alpha)\}_{\alpha\in\mathcal{T}_n}$ of a holomorphic function $F\in B_p(\mathbb{B}_n)$, $m>\frac{2n}{p}$, then $D_m^\ell f(\alpha)\approx F^{(\ell)}(c_\alpha)$ for $1\le\ell\le m$, and using Taylor's formula we will see that $\mathcal{Q}_{\alpha}^{(m,0)}\mathcal{D}_{\alpha}f$ is controlled by $F^{(m+1)}$. Similarly, $D_{m}^{\ell}f(\alpha)=F^{(\ell)}(\alpha)+\left\{D_{m}^{\ell}f(\alpha)-F^{(\ell)}(\alpha)\right\}$, and we will show that each term in this sum is also controlled by $F^{(m+1)}$. In this way we will obtain the following Besov space restriction theorem, as well as the corresponding multiplier space restriction theorem:

THEOREM 8.14. *Let $m>\frac{2n}{p}$. Then provided θ is chosen large enough in the construction of the Bergman tree $\mathcal{T}_n$, the restriction map*

$$TF=\{F(\alpha)\}_{\alpha\in\mathcal{T}_n}, \qquad \text{where } TF(\alpha)=F(c_\alpha),$$

is bounded from $B_p(\mathbb{B}_n)$ to $HB_{p,m}(\mathcal{T}_n)$, and if in addition $p<2+\frac{1}{n-1}$, then T is also bounded from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_{p,m}(\mathcal{T}_n)}$.

The proof of the first assertion in Theorem 8.14 will be given immediately below. The more difficult second assertion will be the content of Subsubsection 8.4.1, and will require the lengthy proof of Lemma 8.17, characterizing the pointwise multipliers of $HB_{p,m}(\mathcal{T}_n)$.

PROOF. We first prove a stronger assertion for Besov spaces by induction on m. To state this stronger version, we need to recast the seminorms $\|\cdot\|_{B_{p,m}(\mathbb{B}_n)}^{*}$ we introduced for Besov spaces on the ball in Section 6, in the language of the discrete tensors we are using on the Bergman tree. This is easily accomplished by observing that for $f\in H(\mathbb{B}_n)$, and $z,\alpha\in\mathbb{B}_n$, we have

$$\left|D_{\alpha}^{\ell}f(z)\right|=\left|f^{(\ell)}(z)\right|_{\alpha}. \tag{8.49}$$

Indeed,

$$\begin{aligned}
\left|D_\alpha f(z)\right|^2&=\left|f'(z)\left\{\left(1-|\alpha|^2\right)P_\alpha+\left(1-|\alpha|^2\right)^{\frac12}Q_\alpha\right\}\right|^2\\
&=\left|f'(z)\left\{r^{-d(\alpha)}P_\alpha+r^{-\frac{d(\alpha)}{2}}Q_\alpha\right\}\right|^2\\
&=\left|\sum_i\frac{\partial f}{\partial z_i}(z)\,e^i\left\{r^{-d(\alpha)}P_\alpha+r^{-\frac{d(\alpha)}{2}}Q_\alpha\right\}\right|^2\\
&=\sum_i\left|\frac{\partial f}{\partial z_i}(z)\right|^2\left\langle\mathbf{R}^{-d(\alpha)}e^i,\mathbf{R}^{-d(\alpha)}e^i\right\rangle\\
&=\left|f'(z)\right|_\alpha^2,
\end{aligned}$$

and the general case $\ell\geq1$ can be verified by expanding each of the ℓ-tensors $D_\alpha^\ell f(z)$ and $f^{(\ell)}(z)$ as a sum of the basis tensors $e^{i_1}\otimes\ldots\otimes e^{i_\ell}$, and then using definitions. Then by Lemma 6.4 we have

$$\|f\|_{B_p(\mathbb{B}_n)}\approx\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right|+\left(\sum_{\alpha\in\mathcal{T}_n}\int_{K_\alpha}\left|f^{(m)}(z)\right|_\alpha^p d\lambda_n(z)\right)^{\frac1p},$$

provided $m>\frac{2n}{p}$.

To fully exploit this realization of the Besov space norm, we now define the Besov space $B_{p,m}^{(t)}(\mathbb{B}_n)$ of t-tensors on the ball.

DEFINITION 8.15. *The space $B_{p,m}^{(t)}(\mathbb{B}_n)$ consists of all holomorphic t-tensor-valued functions $\mathbf{A}$ on the ball $\mathbb{B}_n$ such that the norm*

$$\|\mathbf{A}\|_{B_{p,m}^{(t)}(\mathbb{B}_n)} \equiv \sum_{j=0}^{m-1} \left|\nabla^j \mathbf{A}(0)\right| + \left(\sum_{\alpha\in\mathcal{T}_n} \int_{K_\alpha} \left|\mathbf{A}^{(m)}(z)\right|_\alpha^p d\lambda_n(z)\right)^{\frac{1}{p}}$$

is finite.

Note that $\mathbf{A}^{(m)}(z)$ is a $(t+m)$-tensor and $\left|\mathbf{A}^{(m)}(z)\right|_\alpha$ is its norm in the Hilbert space $\mathcal{E}_\alpha^{(t+m)}$ given by $\sqrt{\left\langle \mathbf{A}^{(m)}(z), \mathbf{A}^{(m)}(z)\right\rangle_\alpha^{(t+m)}}$. If $\mathbf{A}$ is a polynomial, then

$$\left|\mathbf{A}^{(m)}(z)\right|_\alpha \leq C_{\mathbf{A},m,t}\left(1-|z|^2\right)^{\frac{t+m}{2}}, \qquad z\in K_\alpha,$$

and we thus see that the space $B_{p,m}^{(t)}(\mathbb{B}_n)$ contains all polynomials if $p\left(\frac{t+m}{2}\right)-n-1>-1$, or $m>\frac{2n}{p}$. One can in fact show that the above norms are equivalent for different m,m' if both $t+m$ and $t+m'$ are greater than $\frac{2n}{p}$, but we will not need this fact. We will use however the trivial inequality

$$\left\|\mathbf{A}^{(\ell)}\right\|_{B_{p,m-\ell}^{(t+\ell)}(\mathbb{B}_n)} \leq \|\mathbf{A}\|_{B_{p,m}^{(t+m)}(\mathbb{B}_n)}, \qquad t+m>\frac{2n}{p}, \tag{8.50}$$

for holomorphic t-tensor valued functions, which uses only the definitions and the identity $\left(\mathbf{A}^{(\ell)}\right)^{(k)}=\mathbf{A}^{(\ell+k)}$.

We now extend the definition of the restriction map T to t-tensor-valued functions $\mathbf{A}$ on the ball by

$$T\mathbf{A}=\{\mathbf{A}(\alpha)\}_{\alpha\in\mathcal{T}_n}, \qquad \text{where } T\mathbf{A}(\alpha)=\mathbf{A}(c_\alpha).$$

The stronger assertion we will prove by induction on m is:

$$\|T\mathbf{A}\|_{HB_{p,m}^{(t)}(\mathcal{T}_n)} \leq C\|\mathbf{A}\|_{B_{p,m}^{(t)}(\mathbb{B}_n)}, \qquad t+m>\frac{2n}{p}. \tag{8.51}$$

The case $t=0$ is the required Besov space restriction theorem.

The case $m=0$ of (8.51), i.e.

$$\sum_{\alpha\in\mathcal{T}_n} |\mathbf{A}(\alpha)|_\alpha^p \leq C\sum_{\alpha\in\mathcal{T}_n}\int_{K_\alpha} |\mathbf{A}(z)|_\alpha^p\, d\lambda_n(z),$$

follows immediately from the mean value equality for holomorphic functions and tensors,

$$\mathbf{A}(\alpha)=\int_{\varphi_\alpha(K_0)} \mathbf{A}(z)\, d\lambda_n(z), \tag{8.52}$$

followed by an application of Minkowski's inequality with the norm $|\cdot|_\alpha$, and then observing that $\varphi_\alpha(K_0)\approx K_\alpha$. Inequality (8.52) is in turn a consequence of the mean value equality $\mathbf{A}(0)=\int_{B\left(0,\frac{1}{2}\right)} \mathbf{A}(z)\, d\lambda_n(z)$ and the invariance of the measure $d\lambda_n$.

Now fix $0<m,t\leq M$ with $t+m>\frac{2n}{p}$ and make the induction assumption that (8.51) holds for all smaller $m'<m$ and $0\leq t'\leq M$ satisfying $t'+m'>\frac{2n}{p}$. By Definition 8.9 we have, with $\mathbf{a}=T\mathbf{A}$,

$$\|\mathbf{a}\|_{HB_{p,m}^{(t)}(\mathcal{T}_n)}^p = |\mathbf{a}(o)|^p + \sum_{\ell=1}^{m} \left\|D_m^\ell \mathbf{a}\right\|_{HB_{p,m-\ell}^{(t+\ell)}(\mathcal{T}_n)}^p + \left\|\mathcal{Q}^{(m,t)}\mathcal{D}\mathbf{a}\right\|_{\ell^p(\mathcal{T}_n)}^p. \tag{8.53}$$

To estimate the term $\left\|D_m^\ell \mathbf{a}\right\|^p_{HB^{(t+\ell)}_{p,m-\ell}(\mathcal{T}_n)}$ in (8.53) we write

$$D_m^\ell \mathbf{a} = T\left(\mathbf{A}^{(\ell)}\right) + \left(D_m^\ell \mathbf{a} - T\left(\mathbf{A}^{(\ell)}\right)\right),$$

and estimate the terms

$$\left\|T\left(\mathbf{A}^{(\ell)}\right)\right\|^p_{HB^{(t+\ell)}_{p,m-\ell}(\mathcal{T}_n)} \tag{8.54}$$

and

$$\left\|D_m^\ell \mathbf{a} - T\left(\mathbf{A}^{(\ell)}\right)\right\|^p_{HB^{(t+\ell)}_{p,m-\ell}(\mathcal{T}_n)} \tag{8.55}$$

separately. The first term (8.54) is easy by the induction assumption:

$$\left\|T\left(\mathbf{A}^{(\ell)}\right)\right\|^p_{HB^{(t+\ell)}_{p,m-\ell}(\mathcal{T}_n)} \leq C\left\|\mathbf{A}^{(\ell)}\right\|^p_{B^{(t+\ell)}_{p,m-\ell}(\mathbb{B}_n)},$$

since $(t+\ell)+(m-\ell) = t+m > \frac{2n}{p}$, and then (8.50) yields

$$\left\|T\left(\mathbf{A}^{(\ell)}\right)\right\|^p_{HB^{(t+\ell)}_{p,m-\ell}(\mathcal{T}_n)} \leq C\left\|\mathbf{A}\right\|^p_{B^{(t)}_{p,m}(\mathbb{B}_n)}.$$

To handle the second term (8.55), let $\mathbf{a}^{(\ell)} = T\left(\mathbf{A}^{(\ell)}\right)$ denote the restriction of the holomorphic $(t+\ell)$-tensor $\mathbf{A}^{(\ell)}$ to the tree, and use the structure inequality (8.37) with $\mathbf{v}^\ell = D_m^\ell \mathbf{a} - \mathbf{a}^{(\ell)}$, to obtain

$$\begin{aligned} c_m \sum_{\ell=1}^m \left|D_m^\ell \mathbf{a} - \mathbf{a}^{(\ell)}\right|_\alpha &\leq \left|L_\alpha^{(m,t)}\left(D_m^1 \mathbf{a}, ..., D_m^m \mathbf{a}\right) - L_\alpha^{(m,t)}\left(\mathbf{a}^{(1)}, ..., \mathbf{a}^{(m)}\right)\right|_\alpha \\ &= \left|\mathcal{P}_\alpha^{(m,t)} \mathcal{D}_\alpha \mathbf{A} - \mathsf{U}_\alpha^{(m,t)}\right|_\alpha = \left|\mathsf{V}_\alpha^{(m,t)}\right|_\alpha \end{aligned}$$

where the vectors $\mathsf{U}_\alpha^{(m,t)}$ and $\mathsf{V}_\alpha^{(m,t)}$ in $\left(\mathcal{E}_\alpha^{(t)}\right)^N$ are defined by

$$\begin{aligned} \mathsf{U}_\alpha^{(m,t)} &= L_\alpha^{(m,t)}\left(\mathbf{a}^{(1)}, ..., \mathbf{a}^{(m)}\right) = L_\alpha^{(m,t)}\left(\mathbf{A}^{(1)}, ..., \mathbf{A}^{(m)}\right), \\ \mathsf{V}_\alpha^{(m,t)} &= \mathcal{D}_\alpha \mathbf{a} - \mathsf{U}_\alpha^{(m,t)} = \mathcal{D}_\alpha \mathbf{A} - \mathsf{U}_\alpha^{(m,t)}. \end{aligned}$$

By a generalization of the local oscillation inequality (6.16) in Proposition 6.5, with the scalar f replaced by a t-tensor $\mathbf{A}$, we have the estimate (recall we are identifying α with c_α),

$$\begin{aligned} \left|\mathsf{V}_\alpha^{(m,t)}\right|_\alpha &= \left|\left\{\mathbf{A}\left(\alpha^j\right) - \mathbf{A}\left(\alpha\right) - \sum_{\ell=1}^m D_m^\ell \mathbf{A}\left(\alpha\right) \wedge \otimes^\ell\left(\alpha^j - \alpha\right)\right\}_{j=1}^N\right|_\alpha \\ &\leq C\left(\int_{K_\alpha^*} \left|\mathbf{A}^{m+1}\left(z\right)\right|_\alpha^p d\lambda_n\left(z\right)\right)^{\frac{1}{p}} \leq C\left(\int_{K_\alpha^{**}} \left|\mathbf{A}^m\left(z\right)\right|_\alpha^p d\lambda_n\left(z\right)\right)^{\frac{1}{p}}. \end{aligned} \tag{8.56}$$

Altogether we can now estimate the second term (8.55) above using the embedding

$$\ell^p_{(t+\ell)}\left(\mathcal{T}_n\right) \subset HB^{(t+\ell)}_{p,m-\ell}\left(\mathcal{T}_n\right),$$

where $\ell^p_{(t)}(\mathcal{T}_n)$ is the space of t-tensor-valued functions $\mathbf{A}$ on the tree with

$$\|\mathbf{A}\|_{\ell^p_{(t)}(\mathcal{T}_n)} = \left(\sum_{\alpha \in \mathcal{T}_n} |\mathbf{A}(\alpha)|^p_\alpha \right)^{\frac{1}{p}} < \infty,$$

to obtain

$$\begin{aligned}(8.57) \qquad \left\| D^\ell_m \mathbf{a} - T\left(\mathbf{A}^{(\ell)}\right) \right\|^p_{HB^{(t+\ell)}_{p,m-\ell}(\mathcal{T}_n)} &\le C \left\| \left| D^\ell_m \mathbf{a} - T\left(\mathbf{A}^{(\ell)}\right) \right|_\alpha \right\|^p_{\ell^p(\mathcal{T}_n)} \\ &\le C \left\| \left| \mathsf{V}^{(m,t)}_\alpha \right|_\alpha \right\|^p_{\ell^p(\mathcal{T}_n)} \\ &\le C \left(\sum_{\alpha \in \mathcal{T}_n} \int_{K^*_\alpha} |\mathbf{A}^m(z)|^p_\alpha \, d\lambda_n(z) \right)^{\frac{1}{p}} \\ &\le C \|\mathbf{A}\|^p_{B^{(t+m)}_{p,m}(\mathbb{B}_n)}.\end{aligned}$$

To estimate the term $\left\| \mathcal{Q}^{(m,t)} \mathcal{D} \mathbf{A} \right\|^p_{\ell^p(\mathcal{T}_n)}$ in (8.53), we note that the minimizing property (8.38) yields

$$\begin{aligned}(8.58) \qquad \left| \mathcal{Q}^{(m,t)}_\alpha \mathcal{D}_\alpha \mathbf{A} \right|_\alpha &= \min_{\mathbf{x} \in \mathcal{E}^{(1)}_\alpha \times \ldots \times \mathcal{E}^{(m)}_\alpha} \left| \mathcal{D}_\alpha \mathbf{A} - L^{(m,t)}_\alpha \mathbf{x} \right|_\alpha \\ &\le \left| \mathcal{D}_\alpha \mathbf{A} - \mathsf{U}^{(m,t)}_\alpha \right|_\alpha = \left| \mathsf{V}^{(m,t)}_\alpha \right|_\alpha.\end{aligned}$$

Thus by (8.56) we can bound $\left| \mathcal{Q}^{(m,t)}_\alpha \mathcal{D}_\alpha \mathbf{A} \right|_\alpha$ by $C \left(\int_{K^*_\alpha} |\mathbf{A}^m(z)|^p_\alpha \, d\lambda_n(z) \right)^{\frac{1}{p}}$, and so

$$\left\| \mathcal{Q}^{(m,t)} \mathcal{D} \mathbf{A} \right\|^p_{\ell^p(\mathcal{T}_n)} \le C \left\| \left| \mathsf{V}^{(m,t)}_\alpha \right|_\alpha \right\|^p_{\ell^p(\mathcal{T}_n)} \le C \|\mathbf{A}\|^p_{B^{(t+m)}_{p,m}(\mathbb{B}_n)},$$

by (8.57). This completes the proof of the first assertion in Theorem 8.14.

8.4.1. *Multiplier restriction.* We now turn to proving that T is bounded from the ball multiplier space $M_{B_p(\mathbb{B}_n)}$ to the tree multiplier space $M_{HB_{p,m}(\mathcal{T}_n)}$ for $\frac{2n}{m} < p < 2 + \frac{1}{n-1}$. First we record a variant of the ball multiplier Theorem 4.2 using the derivatives $D^m_{c_\alpha}$ in place of ∇^m.

Lemma 8.16. *Let $\varphi \in H^\infty(\mathbb{B}_n) \cap B_p(\mathbb{B}_n)$ and $m > \frac{2n}{p}$. Then φ is a multiplier on $B_p(\mathbb{B}_n)$ if and only if*

$$\sum_{\alpha \in \mathcal{T}_n} \chi_{K_\alpha}(z) \left\{ \int_{K_\alpha} \left| D^m_{c_\alpha} \varphi(\zeta) \right|^p d\lambda_n(\zeta) \right\} d\lambda_n(z)$$

is a $B_p(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$.

Proof. Since the operators $D^m_{c_\alpha}$ satisfy the same product rule as ∇^m, and can be used in place of ∇^m in the seminorm for $B_p(\mathbb{B}_n)$ when $m > \frac{2n}{p}$ by Lemma 6.4, the proof of Theorem 4.2 applies almost verbatim. This completes the proof of Lemma 8.16.

Second, we prove the analogue of the tree multiplier Lemma 7.4 for the holomorphic Besov space $HB_{p,m}(\mathcal{T}_n)$. We only need the sufficiency statement in the sequel. In order to state the lemma, we begin by expressing the Besov space norm $HB_{p,m}(\mathcal{T}_n)$ as an ℓ^p norm of appropriate quantities. We have

$$\begin{aligned}\|f\|^p_{HB_{p,m}(\mathcal{T}_n)} &= |f(o)|^p + \sum_{\alpha\in\mathcal{T}_n} \left|\mathcal{Q}^{(m,0)}_\alpha \mathcal{D}_\alpha f\right|^p + \sum_{\ell_1=1}^{m} \left\|D^{\ell_1}_m f(\alpha)\right\|^p_{HB^{(\ell_1)}_{p,m-\ell_1}(\mathcal{T}_n)} \\ &= |f(o)|^p + \sum_{\alpha\in\mathcal{T}_n} \left|\mathcal{Q}^{(m,0)}_\alpha \mathcal{D}_\alpha f\right|^p + \sum_{\alpha\in\mathcal{T}_n} |D^m_m f(\alpha)|^p_\alpha \\ &\quad + \sum_{\ell_1=1}^{m-1} \left\|D^{\ell_1}_m f(\alpha)\right\|^p_{HB^{(\ell_1)}_{p,m-\ell_1}(\mathcal{T}_n)}.\end{aligned}$$

Now write each term $\left\|D^{\ell_1}_m f(\alpha)\right\|^p_{HB^{(\ell_1)}_{p,m-\ell_1}(\mathcal{T}_n)}$ as

$$\begin{aligned}\left\|D^{\ell_1}_m f(\alpha)\right\|^p_{HB^{(\ell_1)}_{p,m-\ell_1}(\mathcal{T}_n)} &= \left|D^{\ell_1}_m f(o)\right| + \sum_{\alpha\in\mathcal{T}_n} \left|\mathcal{Q}^{(m,m-\ell_1)}_\alpha \mathcal{D}_\alpha D^{\ell_1}_m f\right|^p \\ &\quad + \sum_{\alpha\in\mathcal{T}_n} \left|D^{m-\ell_1}_{m-\ell_1} D^{\ell_1}_m f(\alpha)\right|^p_\alpha \\ &\quad + \sum_{\ell_2=1}^{m-\ell_1-1} \left\|D^{\ell_2}_{m-\ell_1} D^{\ell_1}_m f(\alpha)\right\|^p_{HB^{(\ell_1)}_{p,m-\ell_1}(\mathcal{T}_n)},\end{aligned}$$

to get

$$\begin{aligned}\|f\|^p_{HB_{p,m}(\mathcal{T}_n)} &= |f(o)|^p + \sum_{\ell_1=1}^{m-1} \left|D^{\ell_1}_m f(o)\right| \\ &\quad + \sum_{\alpha\in\mathcal{T}_n} \left|\mathcal{Q}^{(m,0)}_\alpha \mathcal{D}_\alpha f\right|^p + \sum_{\ell_1=1}^{m-1}\sum_{\alpha\in\mathcal{T}_n} \left|\mathcal{Q}^{(m,m-\ell_1)}_\alpha \mathcal{D}_\alpha D^{\ell_1}_m f\right|^p \\ &\quad + \sum_{\alpha\in\mathcal{T}_n} |D^m_m f(\alpha)|^p_\alpha + \sum_{\ell_1=1}^{m-1}\sum_{\alpha\in\mathcal{T}_n} \left|D^{m-\ell_1}_{m-\ell_1} D^{\ell_1}_m f(\alpha)\right|^p_\alpha \\ &\quad + \sum_{\ell_1=1}^{m-1}\sum_{\ell_2=1}^{m-\ell_1-1} \left\|D^{\ell_2}_{m-\ell_1} D^{\ell_1}_m f(\alpha)\right\|^p_{HB^{(\ell_1)}_{p,m-\ell_1}(\mathcal{T}_n)}.\end{aligned}$$

Continuing in this way we arrive at the desired formula (let $s_k = \ell_1 + ... + \ell_k$):

$$\begin{aligned}\|f\|^p_{HB_{p,m}(\mathcal{T}_n)} &= \sum_{s_k<m} \left|D^{\ell_k}_{m-s_{k-1}} ... D^{\ell_2}_{m-\ell_1} D^{\ell_1}_m f(o)\right|^p \\ &\quad + \sum_{s_k=m}\sum_{\alpha\in\mathcal{T}_n} \left|\mathcal{Q}^{(m-s_{k-1},s_{k-1})}_\alpha \mathcal{D}_\alpha D^{\ell_{k-1}}_{m-s_{k-2}} ... D^{\ell_1}_m f\right|^p_\alpha \\ &\quad + \sum_{s_k=m}\sum_{\alpha\in\mathcal{T}_n} \left|D^{\ell_k}_{m-s_{k-1}} ... D^{\ell_1}_m f(\alpha)\right|^p_\alpha.\end{aligned} \tag{8.59}$$

LEMMA 8.17. *Let $m > \frac{2n}{p}$. Then $f \in M_{HB_{p,m}(\mathcal{T}_n)}$ if and only if f is bounded and $\{\omega(\alpha)\}_{\alpha\in\mathcal{T}}$ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure, where*

(8.60)

$$\omega(\alpha) \equiv \sum_{s_k=m} \left|D^{\ell_k}_{m-s_{k-1}} ... D^{\ell_1}_m f(\alpha)\right|^p_\alpha + \sum_{s_k=m} \left|\mathcal{Q}^{(m-s_{k-1},s_{k-1})}_\alpha \mathcal{D}_\alpha D^{\ell_{k-1}}_{m-s_{k-2}} ... D^{\ell_1}_m f\right|^p_\alpha.$$

For the proof of this characterization of the pointwise multipliers of $HB_{p,m}(\mathcal{T}_n)$, we will need the embeddings

$$HB_{q,2}(\mathcal{T}_n) \subset HB_{r,2}(\mathcal{T}_n), \quad 1 < q < r < \infty \tag{8.61}$$
$$HB_{p,m}(\mathcal{T}_n) = HB_{p,m'}(\mathcal{T}_n), \quad m, m' > \frac{2n}{p}.$$

The embedding $HB_{q,2}(\mathcal{T}_n) \subset HB_{r,2}(\mathcal{T}_n)$ for $q < r$ is automatic from Definition 7.14 and the embeddings of ℓ^q spaces. The equality $HB_{p,m}(\mathcal{T}_n) = HB_{p,m'}(\mathcal{T}_n)$ for $m, m' > \frac{2n}{p}$ is established in the following lemma.

LEMMA 8.18. *$HB_{p,m}(\mathcal{T}_n) \subset HB_{p,m+1}(\mathcal{T}_n)$ for $m \geq 0$, $1 < p < \infty$; and $HB_{p,m}(\mathcal{T}_n) = HB_{p,m'}(\mathcal{T}_n)$ for $m, m' > \frac{2n}{p}$.*

PROOF. We first show by induction on m that

$$\|\mathbf{A}\|_{HB^{(t)}_{p,m+1}(\mathcal{T}_n)} \leq C \|\mathbf{A}\|_{HB^{(t)}_{p,m}(\mathcal{T}_n)}, \quad 1 < p < \infty, m \geq 0,$$

for all t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$. We have from (8.37) that

$$\begin{aligned}
\|\mathbf{A}\|^p_{HB^{(t)}_{p,1}(\mathcal{T}_n)} &= |\mathbf{A}(o)|^p + \|D_1\mathbf{A}\|^p_{HB_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&\approx |\mathbf{A}(o)|^p + \left\|\mathcal{P}^{(1,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&\leq |\mathbf{A}(o)|^p + C\|\mathcal{D}\mathbf{A}\|^p_{\ell^p(\mathcal{T}_n)} \\
&\leq |\mathbf{A}(o)|^p + C\|\mathbf{A}\|^p_{\ell^p(\mathcal{T}_n)} \\
&= C\|\mathbf{A}\|^p_{HB^{(t)}_{p,0}(\mathcal{T}_n)}.
\end{aligned}$$

The significant inequality above is $\|\mathcal{D}\mathbf{A}\|^p_{\ell^p(\mathcal{T}_n)} \leq C\|\mathbf{A}\|^p_{\ell^p(\mathcal{T}_n)}$, which cannot be reversed for general functions $\mathbf{A}$. This establishes the case $m = 0$ of the induction. Now from the case $m = 0$ applied to the $(t+1)$-tensor $D_2\mathbf{A}$ we have

$$\begin{aligned}
\|\mathbf{A}\|^p_{HB^{(t)}_{p,2}(\mathcal{T}_n)} &= |\mathbf{A}(o)|^p + \|D_2\mathbf{A}\|^p_{HB_{p,1}(\mathcal{T}_n)} \\
&\qquad + \left\|D_2^2\mathbf{A}\right\|^p_{HB_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&\leq |\mathbf{A}(o)|^p + \|D_2\mathbf{A}\|^p_{HB_{p,0}(\mathcal{T}_n)} \\
&\qquad + \left\|D_2^2\mathbf{A}\right\|^p_{HB_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&\approx |\mathbf{A}(o)|^p + \left\|\mathcal{P}^{(2,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&\approx |\mathbf{A}(o)|^p + \|\mathcal{D}\mathbf{A}\|^p_{\ell^p(\mathcal{T}_n)} \\
&\approx |\mathbf{A}(o)|^p + \left\|\mathcal{P}^{(1,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,t)}\mathcal{D}\mathbf{A}\right\|^p_{\ell^p(\mathcal{T}_n)} \\
&= \|\mathbf{A}\|^p_{HB^{(t)}_{p,1}(\mathcal{T}_n)},
\end{aligned}$$

which is the case $m = 1$, and the general case is left to the interested reader.

This proof of the opposite inequality,

$$\|f\|_{HB_{p,m}(\mathcal{T}_n)} \leq C\|f\|_{HB_{p,m+1}(\mathcal{T}_n)}, \quad m > \frac{2n}{p}, \tag{8.62}$$

is similar to the proof of the analogous Theorem 2.1 on the ball - see Theorem 6.1 of [**Zhu**]. To illustrate we prove only the case $m=1$ of (8.62), the case $m=0$ being trivial. We must show that $\|f\|_{HB_{p,1}(\mathcal{T}_n)} \leq C\|f\|_{HB_{p,2}(\mathcal{T}_n)}$ provided $p>2n$. Recall that

$$\|f\|_{HB_{p,1}(\mathcal{T}_n)} \approx |f(o)| + \|D_1 f\|_{HB_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,0)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)},$$
$$\|f\|_{HB_{p,2}(\mathcal{T}_n)} \approx |f(o)| + \|D_2 f\|_{HB_{p,1}(\mathcal{T}_n)} + \left\|D_2^2 f\right\|_{HB_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,0)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)}.$$

Now we have

$$\begin{aligned}\|D_1 f\|_{HB_{p,0}(\mathcal{T}_n)} &+ \left\|\mathcal{Q}^{(1,0)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)} \\ &\approx \left\|\mathcal{P}^{(1,0)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(1,0)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)} \\ &\approx \|\mathcal{D}f\|_{\ell^p(\mathcal{T}_n)} \\ &\approx \left\|\mathcal{P}^{(2,t)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,t)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)} \\ &\approx \|D_2 f\|_{HB_{p,0}(\mathcal{T}_n)} + \left\|D_2^2 f\right\|_{HB_{p,0}(\mathcal{T}_n)} + \left\|\mathcal{Q}^{(2,0)}\mathcal{D}f\right\|_{\ell^p(\mathcal{T}_n)},\end{aligned}$$

and thus it suffices to show that $\|D_2 f\|_{HB_{p,0}(\mathcal{T}_n)} \leq C\|D_2 f\|_{HB_{p,1}(\mathcal{T}_n)}$, or

$$\begin{aligned}\sum_{\alpha\in\mathcal{T}_n} |D_2 f(\alpha)|_\alpha^p &\leq C\left(|D_2 f(o)|^p + \sum_{\alpha\in\mathcal{T}_n} |D_1 D_2 f|_\alpha^p + \sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_\alpha^{(1,0)}\mathcal{D}_\alpha D_2 f\right|_\alpha^p\right) \\ &\approx |D_2 f(o)|^p + \sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{P}_\alpha^{(1,0)}\mathcal{D}_\alpha D_2 f\right|_\alpha^p + \sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_\alpha^{(1,0)}\mathcal{D}_\alpha D_2 f\right|_\alpha^p \\ &\approx |D_2 f(o)|^p + \sum_{\alpha\in\mathcal{T}_n} |\mathcal{D}_\alpha D_2 f|_\alpha^p,\end{aligned}$$

i.e.

$$\sum_{\alpha\in\mathcal{T}_n} |D_2 f(\alpha)|_\alpha^p \leq C\left(|D_2 f(o)|^p + \sum_{\alpha\in\mathcal{T}_n} |\triangle\, D_2 f(\alpha)|_\alpha^p\right) \tag{8.63}$$

(we use the convention $\triangle\, D_2 f(o) = D_2 f(o)$).

Now using (8.2), (8.3) and the operator

$$\left(\mathbf{R}^{\pm d}\right)(\alpha) = r^{\pm d(\alpha)} P_\alpha + r^{\pm\frac{d(\alpha)}{2}} Q_\alpha$$

as in Definition 8.2, we have

$$\begin{aligned}\left(\mathbf{R}^{-d} f'\right)(\alpha) &= f'(\alpha)\left(\mathbf{R}^{-d}\right)(\alpha) \\ &= \left(\sum_{o\leq\gamma\leq\alpha} \triangle\, f'(\gamma)\right)\left(\mathbf{R}^{-d}\right)(\alpha) \\ &= \sum_{o\leq\gamma\leq\alpha} \left(\mathbf{R}^{-d}\,\triangle\, f'\right)(\gamma)\left(\mathbf{R}^{d}\right)(\gamma)\left(\mathbf{R}^{-d}\right)(\alpha),\end{aligned}$$

and thus the estimate

$$\begin{aligned} |D_2 f(\alpha)|_\alpha &= \left|\left(\mathbf{R}^{-d} D_2 f\right)(\alpha)\right| \\ &\leq \sum_{o \leq \gamma \leq \alpha} \left|\left(\mathbf{R}^{-d} \vartriangle D_2 f\right)(\gamma)\right| \left|\left(\mathbf{R}^{d}\right)(\gamma)\left(\mathbf{R}^{-d}\right)(\alpha)\right| \\ &= \sum_{o \leq \gamma \leq \alpha} \left|\left(\mathbf{R}^{d}\right)(\gamma)\left(\mathbf{R}^{-d}\right)(\alpha)\right| \left|\vartriangle D_2 f(\gamma)\right|_\gamma . \end{aligned} \tag{8.64}$$

We now claim that

$$\left|\left(\mathbf{R}^{d}\right)(\gamma)\left(\mathbf{R}^{-d}\right)(\alpha)\right| \leq C r^{-\frac{d(\alpha)-d(\gamma)}{2}}. \tag{8.65}$$

To see this we expand the product of operators as

$$\begin{aligned} &\left(\mathbf{R}^{d}\right)(\gamma)\left(\mathbf{R}^{-d}\right)(\alpha) \\ &= \left(r^{d(\gamma)} P_\gamma + r^{\frac{d(\gamma)}{2}} Q_\gamma\right)\left(r^{-d(\alpha)} P_\alpha + r^{-\frac{d(\alpha)}{2}} Q_\alpha\right) \\ &= r^{-\frac{d(\alpha)-d(\gamma)}{2}} \left\{ r^{-\frac{d(\alpha)-d(\gamma)}{2}} P_\gamma P_\alpha + r^{\frac{d(\gamma)}{2}} P_\gamma Q_\alpha + r^{-\frac{d(\alpha)}{2}} Q_\gamma P_\alpha + Q_\gamma Q_\alpha \right\}, \end{aligned}$$

which reduces the proof of the claim to

$$|P_\gamma Q_\alpha| \leq C r^{-\frac{d(\gamma)}{2}}.$$

For this it is enough to show that the adjoint $(P_\gamma Q_\alpha)^* = Q_\alpha^* P_\gamma^* = Q_\alpha P_\gamma$ has norm bounded by $C r^{-\frac{d(\gamma)}{2}}$. However, for a unit vector $\mathbf{v}$, $P_\gamma \mathbf{v} = \lambda \gamma$ where λ is a complex number of modulus at most one, and so

$$\begin{aligned} |Q_\alpha P_\gamma \mathbf{v}| &= |Q_\alpha(\lambda\gamma)| = |\lambda\gamma - P_\alpha(\lambda\gamma)| \\ &= |\lambda| \left| (\gamma - \alpha) - \frac{((\gamma-\alpha)\cdot\overline{\alpha})\,\alpha}{|\alpha|^2} \right| \\ &\leq 2|\gamma - \alpha| \leq C\left(1 - |\gamma|^2\right)^{\frac{1}{2}} = C r^{-\frac{d(\gamma)}{2}}, \end{aligned}$$

by (8.3).

Combining (8.64) and (8.65) we obtain

$$|D_2 f(\alpha)|_\alpha \leq C \sum_{o \leq \gamma \leq \alpha} r^{-\frac{d(\alpha)-d(\gamma)}{2}} \left|\vartriangle D_2 f(\gamma)\right|_\gamma .$$

We write this as

$$|D_2 f(\alpha)|_\alpha \leq C \sum_{\beta \in \mathcal{T}} K(\alpha, \beta) \left|\vartriangle D_2 f(\beta)\right|_\beta$$

where the kernel $K(\alpha,\beta)$ is given by $\chi_{[o,\alpha]}(\beta)\, r^{\frac{1}{2}[d(\beta)-d(\alpha)]}$. We now apply Schur's test, Lemma 5.17, with auxiliary function with $h(\beta) = r^{t d(\beta)}$. We have

$$\sum_{\beta \in \mathcal{T}} K(\alpha,\beta)\, h(\beta)^{p'} = \sum_{\beta \leq \alpha} r^{\left(\frac{1}{2} + p' t\right) d(\beta)} r^{-\frac{1}{2} d(\alpha)} \leq C h(\alpha)^{p'}$$

provided $\frac{1}{2}+p't>0$. We also have using the "sparse" argument of Lemma 7.3, that for $\varepsilon>0$,

$$\begin{aligned}\sum_{\beta\in\mathcal{T}}K(\alpha,\beta)h(\alpha)^{p}&=\sum_{\alpha\geq\beta}r^{\left(pt-\frac{1}{2}\right)d(\alpha)}r^{\frac{1}{2}d(\beta)}\\&\leq C_{\ell}\sum_{k=0}^{\infty}\left(r^{n+\varepsilon}\right)^{k\ell}r^{\left(pt-\frac{1}{2}\right)(d(\beta)+k\ell)}r^{\frac{1}{2}d(\beta)}\\&\leq C_{\ell}h(\beta)^{p}\end{aligned}$$

provided $n+\varepsilon+pt-\frac{1}{2}<0$, where ℓ is chosen so large in Definition 2.7 that $2^{(N_\ell)^{\frac{1}{\ell}}}<r^{n+\varepsilon}$. Now since $p>2n$, we can choose $-\frac{1}{2p'}<t<\frac{\frac{1}{2}-n-\varepsilon}{p}$ for some $\varepsilon>0$, and then Schur's test shows that

$$\begin{aligned}\left(\sum_{\alpha\in\mathcal{T}}|D_2f(\alpha)|_{\alpha}^{p}\right)^{\frac{1}{p}}&\leq C\left(\sum_{\alpha\in\mathcal{T}}\left|\sum_{\beta\in\mathcal{T}}K(\alpha,\beta)\left|\triangle\ D_2f(\beta)\right|_{\beta}\right|^{p}\right)^{\frac{1}{p}}\\&\leq C\left(\sum_{\beta\in\mathcal{T}}\left|\triangle\ D_2f(\beta)\right|_{\beta}^{p}\right)^{\frac{1}{p}},\end{aligned}$$

which is (8.63) as required. This completes the proof of the case $m=1$ of (8.62). The general case is similar and is left to the interested reader. The proof of Lemma 8.18 is complete.

Proof. (of Lemma 8.17) We first prove the sufficiency assertion of the Lemma 8.17. Let f be bounded and suppose that $\{\omega(\alpha)\}_{\alpha\in\mathcal{T}}$ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure where ω is as in (8.60). Then if $g\in HB_{p,m}(\mathcal{T}_n)$, we must show that $fg\in HB_{p,m}(\mathcal{T}_n)$ with norm control

$$\|fg\|_{HB_{p,m}(\mathcal{T}_n)}\leq C\|g\|_{HB_{p,m}(\mathcal{T}_n)},$$

where C depends on the Carleson norm $\|\omega\|_{Carleson}$ of $\{\omega(\alpha)\}_{\alpha\in\mathcal{T}}$. We have

(8.66)
$$\begin{aligned}\|fg\|_{HB_{p,m}(\mathcal{T}_n)}^{p}=\sum_{s_k<m}&\left|D_{m-s_{k-1}}^{\ell_k}...D_{m-\ell_1}^{\ell_2}D_m^{\ell_1}(fg)(o)\right|^{p}\\&+\sum_{s_k=m}\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_{\alpha}^{(m-s_{k-1},s_{k-1})}\mathcal{D}_{\alpha}D_{m-s_{k-2}}^{\ell_{k-1}}...D_m^{\ell_1}(fg)\right|_{\alpha}^{p}\\&+\sum_{s_k=m}\sum_{\alpha\in\mathcal{T}_n}\left|D_{m-s_{k-1}}^{\ell_k}...D_m^{\ell_1}(fg)(\alpha)\right|_{\alpha}^{p}.\end{aligned}$$

Consider first the case that $s_1=m$ in (8.66), so that $\ell_1=m$. We must show that

(8.67)
$$\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_{\alpha}^{(m,0)}\mathcal{D}_{\alpha}(fg)\right|_{\alpha}^{p}+\sum_{\alpha\in\mathcal{T}_n}|D_m^m(fg)(\alpha)|_{\alpha}^{p}\leq C\|\omega\|_{Carleson}^{p}\|g\|_{HB_{p,m}(\mathcal{T}_n)}^{p}.$$

To see this we write

$$\begin{aligned}\mathcal{D}_\alpha(fg) &= \{(fg)(\alpha_j)-(fg)(\alpha)\}_{j=1}^N \\ &= f(\alpha)\{\triangle\, g(\alpha_j)\}_{j=1}^N + g(\alpha)\{\triangle\, f(\alpha_j)\}_{j=1}^N \\ &\qquad + \{\triangle\, f(\alpha_j)\,\triangle\, g(\alpha_j)\}_{j=1}^N,\end{aligned}$$

and using the definitions of $D_m^\ell f(\alpha)$ and $D_m^\ell g(\alpha)$ we obtain

$$\begin{aligned}\mathcal{D}_\alpha(fg) = f(\alpha)&\left\{\sum_{\ell=1}^m \frac{1}{\ell!}D_m^\ell g(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right\}_{j=1}^N \\ &+ g(\alpha)\left\{\sum_{\ell=1}^m \frac{1}{\ell!}D_m^\ell f(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right\}_{j=1}^N \\ &+ f(\alpha)\,\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g + g(\alpha)\,\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f \\ &+ \{\triangle\, f(\alpha_j)\,\triangle\, g(\alpha_j)\}_{j=1}^N.\end{aligned}$$

Now using (8.35), the j^{th} component in the final term above is

$$\begin{aligned}&\left(\sum_{\ell=1}^m \frac{1}{\ell!}D_m^\ell f(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right)\left(\sum_{\ell=1}^m \frac{1}{\ell!}D_m^\ell g(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right) \\ &\qquad + \left(\sum_{\ell=1}^m \frac{1}{\ell!}D_m^\ell f(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right)\left(\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\right) \\ &\qquad + \left(\sum_{\ell=1}^m \frac{1}{\ell!}D_m^\ell g(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right)\left(\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right) \\ &\qquad + \left(\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right)\left(\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\right),\end{aligned}$$

where the first line is

$$\begin{aligned}&\sum_{k=1}^m\sum_{\ell=1}^m \frac{1}{k!\ell!}\left(D_m^k f(\alpha)\wedge\left\{\otimes^k(\alpha_j-\alpha)\right\}\right)\left(D_m^\ell g(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right) \\ &\quad = \sum_{k=1}^m\sum_{\ell=1}^m \frac{1}{k!\ell!}\left(D_m^k f(\alpha)\otimes D_m^\ell g(\alpha)\right)\otimes^{k+\ell}(\alpha_j-\alpha) \\ &\quad = \sum_{r=1}^m\left[\sum_{\ell=1}^r \begin{pmatrix} r \\ \ell \end{pmatrix}\left(D_m^{r-\ell} f(\alpha)\otimes D_m^\ell g(\alpha)\right)\right]\otimes^r(\alpha_j-\alpha).\end{aligned}$$

Altogether we have

$$(fg)(\alpha_j) = \sum_{r=0}^{m} \left[\sum_{\ell=0}^{r} \binom{r}{\ell} \left(D_m^{r-\ell} f(\alpha) \otimes D_m^{\ell} g(\alpha) \right) \right] \otimes^r (\alpha_j - \alpha) \tag{8.68}$$
$$+ \left(\sum_{\ell=1}^{m} \frac{1}{\ell!} D_m^{\ell} g(\alpha) \wedge \left\{ \otimes^{\ell} (\alpha_j - \alpha) \right\} \right) \left(\mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha f(\alpha_j) \right)$$
$$+ \left(\sum_{\ell=1}^{m} \frac{1}{\ell!} D_m^{\ell} f(\alpha) \wedge \left\{ \otimes^{\ell} (\alpha_j - \alpha) \right\} \right) \left(\mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha g(\alpha_j) \right)$$
$$+ \left(\mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha f(\alpha_j) \right) \left(\mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha g(\alpha_j) \right).$$

Recall the notation $|\mathbf{A}|_\alpha = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle_\alpha^{(t)}}$ introduced in (8.23). The first term on the right side of (8.68) lies in M_α, and by the fact that $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are orthogonal projections onto M_α and $M_\alpha^\perp$ respectively, we thus have

$$\left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha (fg) \right| \leq \sum_{\ell=1}^{m} \left| D_m^{\ell} f(\alpha) \right|_\alpha \left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha g \right| \tag{8.69}$$
$$+ \sum_{\ell=1}^{m} \left| D_m^{\ell} g(\alpha) \right|_\alpha \left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha f \right|$$
$$+ \left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha f \right| \left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha g \right|.$$

As in the proof of the corresponding multiplier characterization on the ball, Theorem 4.2, we set

$$q_\ell = \frac{m}{m-\ell}, q'_\ell = \frac{m}{\ell}, \qquad 1 \leq \ell \leq m-1, \tag{8.70}$$

and continue by estimating the ℓ^p norm of the first terms on the right side of (8.69) by

$$\sum_{\alpha \in \mathcal{T}_n} \left| D_m^{\ell} f(\alpha) \right|_\alpha^p \left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha g \right|^p \leq \left\{ \sum_{\alpha \in \mathcal{T}_n} \left| D_m^{\ell} f(\alpha) \right|_\alpha^{pq'_\ell} \right\}^{\frac{1}{q'_\ell}}$$
$$\times \left\{ \sum_{\alpha \in \mathcal{T}_n} \left| \mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha g \right|_\alpha^{pq_\ell} \right\}^{\frac{1}{q_\ell}}$$
$$\leq \|f\|_{HB_{pq'_\ell,\ell}(\mathcal{T}_n)}^p \|g\|_{HB_{p,m}(\mathcal{T}_n)}^p$$
$$\leq \|f\|_{HB_{pq'_\ell,m}(\mathcal{T}_n)}^p \|g\|_{HB_{p,m}(\mathcal{T}_n)}^p$$
$$\leq \|f\|_{HB_{p,m}(\mathcal{T}_n)}^p \|g\|_{HB_{p,m}(\mathcal{T}_n)}^p,$$

where we have used first that

$$\left\{ \sum_{\alpha \in \mathcal{T}_n} \left| D_m^{\ell} f(\alpha) \right|_\alpha^{pq'_\ell} \right\}^{\frac{1}{pq'_\ell}} \leq C \|f\|_{HB_{pq'_\ell,\ell}(\mathcal{T}_n)}$$

for $\ell > \frac{2n\ell}{pm} = \frac{2n}{pq'_\ell}$ and $m \geq \ell$ (the case $m = \ell$ is by definition, and the case $m > \ell$ follows easily), and then (8.61). The ℓ^p norms of the remaining terms in (8.69) are handled similarly and thus we have obtained

$$\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_{\alpha}^{(m,0)}\mathcal{D}_{\alpha}(fg)\right|^{p}\leq C\|f\|_{HB_{p,m}(\mathcal{T}_n)}^{p}\|g\|_{HB_{p,m}(\mathcal{T}_n)}^{p} \tag{8.71}$$
$$\leq C\|\omega\|_{Carleson}^{p}\|g\|_{HB_{p,m}(\mathcal{T}_n)}^{p},$$

which is the first half of (8.67).

The other half of (8.67) requires that

$$\sum_{\alpha\in\mathcal{T}_n}\left|D_m^m(fg)(\alpha)\right|_{\alpha}^{p}\leq C\|\omega\|_{Carleson}^{p}\|g\|_{HB_{p,m}(\mathcal{T}_n)}^{p}. \tag{8.72}$$

To prove this we write

(8.73)

$$D_m^m(fg)(\alpha)=\sum_{\ell=0}^{m}\binom{m}{\ell}\left(D_m^{m-\ell}f(\alpha)\otimes D_m^{\ell}g(\alpha)\right)+\left\{D_m^m(fg)(\alpha)-\sum_{\ell=0}^{m}\binom{m}{\ell}\left(D_m^{m-\ell}f(\alpha)\otimes D_m^{\ell}g(\alpha)\right)\right\},$$

where the first term is what one expects from Liebniz' rule, and the second term is the error. With q_ℓ and q'_ℓ as in (8.70), we estimate the first term on the right side of (8.73) by

$$\begin{aligned}\sum_{\alpha\in\mathcal{T}_n}\left|D_m^{m-\ell}f(\alpha)\otimes D_m^{\ell}g(\alpha)\right|_{\alpha}^{p} &\leq\sum_{\alpha\in\mathcal{T}_n}\left|D_m^{m-\ell}f(\alpha)\right|_{\alpha}^{p}\left|D_m^{\ell}g(\alpha)\right|_{\alpha}^{p}\\ &\leq\left\{\sum_{\alpha\in\mathcal{T}_n}\left|D_m^{m-\ell}f(\alpha)\right|_{\alpha}^{pq_\ell}\right\}^{\frac{1}{q_\ell}}\left\{\sum_{\alpha\in\mathcal{T}_n}\left|D_m^{\ell}g(\alpha)\right|_{\alpha}^{pq'_\ell}\right\}^{\frac{1}{q'_\ell}}\\ &\leq\|f\|_{HB_{pq_\ell,m-\ell}(\mathcal{T}_n)}^{p}\|g\|_{HB_{pq'_\ell,\ell}(\mathcal{T}_n)}^{p}\\ &\leq\|f\|_{HB_{p,m}(\mathcal{T}_n)}^{p}\|g\|_{HB_{p,m}(\mathcal{T}_n)}^{p},\end{aligned} \tag{8.74}$$

again since $m-\ell>\frac{2n(m-\ell)}{pm}=\frac{2n}{pq_\ell}$ and $\ell>\frac{2n\ell}{pm}=\frac{2n}{pq'_\ell}$. To estimate the second term on the right side of (8.73), we use (8.34) to obtain

$$\left|D_m^m(fg)(\alpha)-\sum_{\ell=0}^{m}\binom{m}{\ell}\left(D_m^{m-\ell}f(\alpha)\otimes D_m^{\ell}g(\alpha)\right)\right|_{\alpha}$$
$$\leq C\left|L_{\alpha}^{(m,0)}\left\{D_m^r(fg)(\alpha)-\sum_{\ell=0}^{r}\binom{r}{\ell}\left(D_m^{r-\ell}f(\alpha)\otimes D_m^{\ell}g(\alpha)\right)\right\}_{r=1}^{m}\right|.$$

We now compute that $L_\alpha^{(m,0)}\left\{D_m^r(fg)(\alpha)\right\}_{r=1}^m=\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha(fg)$ and that the j^{th} component of $L_\alpha^{(m,0)}\left\{\sum_{\ell=0}^r\begin{pmatrix} r\\ \ell\end{pmatrix}\left(D_m^{r-\ell}f(\alpha)\otimes D_m^\ell g(\alpha)\right)\right\}_{r=1}^m$ is

$$\begin{aligned}
&\sum_{r=1}^m\sum_{\ell=0}^r\frac{1}{(r-\ell)!\ell!}\left(D_m^{r-\ell}f(\alpha)\otimes D_m^\ell g(\alpha)\right)\otimes^r(\alpha_j-\alpha)\\
&\quad=\sum_{r=1}^m\sum_{\ell=0}^r\frac{1}{(r-\ell)!\ell!}\left(D_m^{r-\ell}f(\alpha)\otimes^{r-\ell}(\alpha_j-\alpha)\right)\left(D_m^\ell g(\alpha)\otimes^\ell(\alpha_j-\alpha)\right)\\
&\quad=\left(\sum_{\ell=0}^m\frac{1}{\ell!}D_m^\ell f(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right)\left(\sum_{\ell=0}^m\frac{1}{\ell!}D_m^\ell g(\alpha)\wedge\left\{\otimes^\ell(\alpha_j-\alpha)\right\}\right)\\
&\quad\quad-f(\alpha)g(\alpha)\\
&\quad=\left(f(\alpha)+\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right)\left(g(\alpha)+\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\right)-f(\alpha)g(\alpha).
\end{aligned}$$

Thus the j^{th} component of

$$L_\alpha^{(m,0)}\left\{D_m^r(fg)(\alpha)-\sum_{\ell=0}^r\begin{pmatrix} r\\ \ell\end{pmatrix}\left(D_m^{r-\ell}f(\alpha)\otimes D_m^\ell g(\alpha)\right)\right\}_{r=1}^m \tag{8.75}$$

is

$$\begin{aligned}
&\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha(fg)(\alpha_j)-g(\alpha)\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)-f(\alpha)\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\\
&\qquad\qquad-\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\mathcal{P}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\\
&\quad=\triangle(fg)(\alpha_j)-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha(fg)(\alpha_j)-g(\alpha)\left(\triangle f(\alpha_j)-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right)\\
&\qquad\qquad-f(\alpha)\left(\triangle g(\alpha_j)-\mathcal{Q}_\alpha^{(m0)}\mathcal{D}_\alpha g(\alpha_j)\right)\\
&\qquad\qquad-\left(\triangle f(\alpha_j)-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right)\left(\triangle g(\alpha_j)-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\right),
\end{aligned}$$

and now, noting that all products not involving a projection $\mathcal{Q}_\alpha^{(m,0)}$ cancel, we obtain that the j^{th} component of (8.75) is

$$\begin{aligned}
&\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha(fg)(\alpha_j)+g(\alpha)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)+f(\alpha)\mathcal{Q}_\alpha^{(m,0)}\triangle g(\alpha_j)\\
&\qquad\qquad+\triangle f(\alpha_j)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)+\triangle g(\alpha_j)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\\
&\qquad\qquad-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j),
\end{aligned}$$

or simply

$$\begin{aligned}
&f(\alpha_j)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)+g(\alpha_j)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\\
&\qquad\qquad-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha(fg)(\alpha_j)-\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\\
&\quad=I(\alpha_j)+II(\alpha_j)+III(\alpha_j)+IV(\alpha_j).
\end{aligned}$$

To handle the sum

$$\sum_{\alpha\in\mathcal{T}_n}|I(\alpha_j)|^p=\sum_{\alpha\in\mathcal{T}_n}|f(\alpha_j)|^p\left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g(\alpha_j)\right|^p,$$

we simply use the boundedness of f together with the definition of $\|g\|^p_{HB_{p,m}(\mathcal{T}_n)}$. To handle the sum

$$\sum_{\alpha\in\mathcal{T}_n} |II(\alpha_j)|^p = \sum_{\alpha\in\mathcal{T}_n} |g(\alpha_j)|^p \left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right|^p,$$

we use the inequality

$$\left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f(\alpha_j)\right|^p \leq \omega(\alpha)$$

together with our assumption that $\{\omega(\alpha)\}_{\alpha\in\mathcal{T}_n}$ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure. The sum $\sum_{\alpha\in\mathcal{T}_n} |III(\alpha_j)|^p$ is controlled by (8.71), and the final sum is easy:

$$\begin{aligned}\sum_{\alpha\in\mathcal{T}_n} |IV(\alpha_j)|^p &\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha f\right|^{2p}\right)^{\frac{1}{2}}\left(\sum_{\alpha\in\mathcal{T}_n}\left|\mathcal{Q}_\alpha^{(m,0)}\mathcal{D}_\alpha g\right|^{2p}\right)^{\frac{1}{2}}\\ &\leq C\|f\|^p_{HB_{2p,m}(\mathcal{T}_n)}\|g\|^p_{HB_{2p,m}(\mathcal{T}_n)}\\ &\leq C\|f\|^p_{HB_{p,m}(\mathcal{T}_n)}\|g\|^p_{HB_{p,m}(\mathcal{T}_n)}.\end{aligned}$$

This completes the proof of (8.67), which is the case $s_1 = 1$ in (8.66).

The remaining cases $s_k = \ell_1 + ... + \ell_k = m$ with $k > 1$ are handled similarly, using repeated application of Hölder's inequality on products of tensors as in (8.74), and using the structural inequality (8.37) to estimate the error in the Liebniz formula, as in (8.73). The details are routine but long, and are left to the interested reader. The normalizing terms

$$\sum_{s_k<m}\left|D^{\ell_k}_{m-s_{k-1}}...D^{\ell_2}_{m-\ell_1}D^{\ell_1}_m(fg)(o)\right|^p$$

in (8.66) are handled easily, just as the corresponding terms

$$\sum_{k=0}^{m-1}\left|\nabla^k(\varphi f)(0)\right|$$

in the proof of Theorem 4.2 on the ball. This completes the proof of the sufficiency assertion in Lemma 8.17.

For the necessity, a standard argument using the boundedness of the adjoint of the multiplication operator shows that $\|f\|_\infty \leq C\|f\|_{M_{HB_{p,m}(\mathcal{T}_n)}}$. The arguments above can then be reversed to show that $\{\omega(\alpha)\}_{\alpha\in\mathcal{T}_n}$ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure. This completes the proof of Lemma 8.17.

Now we return to the proof of the multiplier restriction in Theorem 8.14. Let $F \in M_{B_p(\mathbb{B}_n)}$ and set

$$d\mu(z) = \sum_{\alpha\in\mathcal{T}_n}\chi_{K_\alpha}(z)\left\{\int_{K_\alpha^*}\left|D^m_{c_\alpha}F(\zeta)\right|^p d\lambda_n(\zeta)\right\}d\lambda_n(z).$$

Since $p > \frac{2n}{m}$ by hypothesis, Lemma 8.16 shows that μ is a $B_p(\mathbb{B}_n)$-Carleson measure. Define the discretization of μ in the usual way by

$$\mu(\alpha) = \int_{K_\alpha^*}\left|D^m_{c_\alpha}F(\zeta)\right|^p d\lambda_n(\zeta).$$

Since $p < 2 + \frac{1}{n-1}$, Theorem 3.1 shows that $\{\mu(\alpha)\}_{\alpha \in \mathcal{T}_n}$ satisfies the tree condition (8.45). Set

$$\omega(\alpha) = \sum_{s_k = m} \left| D^{\ell_k}_{m-s_{k-1}} ... D^{\ell_1}_m f(\alpha) \right|^p_\alpha + \sum_{s_k = m} \left| \mathcal{Q}^{(m-s_{k-1}, s_{k-1})}_\alpha \mathcal{D}_\alpha D^{\ell_{k-1}}_{m-s_{k-2}} ... D^{\ell_1}_m f \right|^p_\alpha,$$

as in (8.60) in the proof of Lemma 8.17. It will follow from Lemma 8.17 that the restriction $f = TF$ lies in $M_{HB_{p,m}(\mathcal{T}_n)}$ if we can show that $\{\omega(\alpha)\}_{\alpha \in \mathcal{T}_n}$ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure.

To this end, we invoke the following local version of (8.51):

$$\begin{aligned} &\sum_{s_k = m} \left| D^{\ell_k}_{m-s_{k-1}} ... D^{\ell_1}_m \mathbf{A}(\alpha) \right|^p_\alpha + \sum_{s_k = m} \left| \mathcal{Q}^{(m-s_{k-1}, t+s_{k-1})}_\alpha \mathcal{D}_\alpha D^{\ell_{k-1}}_{m-s_{k-2}} ... D^{\ell_1}_m \mathbf{A} \right|^p_\alpha \\ &\quad \leq C \int_{K^{**}_\alpha} \left| \mathbf{A}^{(m)}(z) \right|^p_\alpha d\lambda_n(z), \end{aligned}$$

for all holomorphic t-tensor-valued functions $\mathbf{A}$ on the Bergman tree. This can be proved by induction similar to the proof of (8.51), and we omit the details. From this we obtain that

$$\begin{aligned} I^* \omega(\alpha) &= \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \omega(\beta) \\ &\leq C \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \int_{K^{**}_\beta} \left| D^m_{c_\beta} F(z) \right|^p d\lambda_n(z) \\ &= C I^* \mu(\alpha). \end{aligned}$$

It now follows that $\{(\mu + \omega)(\alpha)\}_{\alpha \in \mathcal{T}_n}$ satisfies the tree condition (8.45), hence is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure by Theorem 8.12. This finally yields that $\{\omega(\alpha)\}_{\alpha \in \mathcal{T}_n}$ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure with norm bounded by that of the $B_p(\mathbb{B}_n)$-Carleson measure μ. At long last, this completes the proof of Theorem 8.14.

Embeddings and isomorphisms. We first observe that $\mathcal{T}_n$ is neither a zero set for $B_p(\mathbb{B}_n)$, nor for $M_{B_p(\mathbb{B}_n)}$, and hence the restriction map T is one-to-one from $B_p(\mathbb{B}_n)$ to $HB_{p,m}(\mathcal{T}_n)$, as well as from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_{p,m}(\mathcal{T}_n)}$. Indeed, if a holomorphic function f in the ball vanishes on $\mathcal{T}_n$, then the admissible limits f^* of f on $\partial\mathbb{B}_n$ are zero whenever they exist, and thus f vanishes identically on the ball if it is in the Nevanlinna class ([**Rud**]: Theorem 5.6.4).

Second, we observe that Lemma 8.18 shows that $\ell^p(\mathcal{T}_n) = HB_{p,0}(\mathcal{T}_n)$ embeds continuously into $HB_{p,m}(\mathcal{T}_n)$. As a consequence, the restriction map T *cannot* be onto from $B_p(\mathbb{B}_n)$ to $HB_{p,m}(\mathcal{T}_n)$. Indeed, if T is onto $HB_{p,m}(\mathcal{T}_n)$, and $\alpha \in \mathcal{T}_n$, then there is $F \in B_p(\mathbb{B}_n)$ such that $F(\beta) = 0$ for all $\beta \in \mathcal{T}_n \setminus \{\alpha\}$, and $F(\alpha) = 1$. Thus $F(z)$ is not identically zero, and hence neither is $G(z) = F(z)(z_1 - \alpha_1)$ where $\alpha = (\alpha_1, ..., \alpha_n)$. But $G \in B_p(\mathbb{B}_n)$ by Theorem 4.2, and this implies that $\mathcal{T}_n$ is a zero set for $B_p(\mathbb{B}_n)$, a contradiction. The same argument also shows that T *cannot* be onto from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_{p,m}(\mathcal{T}_n)}$.

Conjecture. *We conjecture that the above restriction maps have closed range.*

8.5. The modified Bergman tree. In this subsection, we construct a modified Bergman tree $\mathcal{T}_n$ that satisfies the structural inequality (8.37) for all $0 \leq m+t \leq M$, where M is chosen so large that $M > \frac{2n}{p}$ for all $1 < p < \infty$, e.g. $M = 2n$ will do. Our construction will also have the property that given a sequence $Z = \{z_j\}_{j=1}^{\infty}$ in the ball satisfying the separation condition in (5.1),

$$\beta(z_i, 0) \le C\beta(z_i, z_j), \quad 1 \le i \ne j < \infty,$$

we can arrange to have $Z \subset \mathcal{T}_n$. This property is crucial for the arguments in Section 5 on interpolating sequences. We now recall (8.37): for $0 \le m + t \le M$, we have

$$c_m \sum_{\ell=1}^{m} \left|\mathbf{v}^\ell\right|_\alpha \le \left|L_\alpha^{(m,t)}\left(\mathbf{v}^1, ..., \mathbf{v}^m\right)\right|_\alpha \le C_m \sum_{\ell=1}^{m} \left|\mathbf{v}^\ell\right|_\alpha, \tag{8.76}$$

where

$$L_\alpha^{(m,t)}\mathbf{x} = \left\{\sum_{\ell=1}^{m} \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell\left(\alpha^j - \alpha\right)\right\}\right\}_{j=1}^{N} \in \left(\mathcal{E}_\alpha^{(t)}\right)^N,$$

and $\mathbf{x} = \left(\mathbf{v}^1, ..., \mathbf{v}^m\right) \in \mathcal{E}_\alpha^{(t+1)} \times \mathcal{E}_\alpha^{(t+2)} \times ... \times \mathcal{E}_\alpha^{(t+m)}$. The main point here is that the constants c_m and C_m in (8.76) are independent of $\alpha \in \mathcal{T}_n$. Indeed, it is not hard to see that, given a construction of the Bergman tree $\mathcal{T}_n = \{\alpha\}_{\alpha\in\mathcal{T}_n}$, we can perturb the centers $\gamma = c_\gamma$ slightly so that the expression $\left|L_\alpha^{(m,t)}\mathbf{x}\right|_\alpha$ vanishes only when $\mathbf{x}$ vanishes. However, we need a uniform version, and to see this we will use the equivalence of norms on a finite dimensional vector space together with unitary maps and the affine maps $\psi_\alpha(z) = \alpha + \varphi_\alpha'(0) z$. These latter maps have the property that ψ_α takes K_0 to K_α approximately, and thus we can initially fix our attention on the root kube K_0. The argument is however complicated by the fact that while we localize our perturbations to a sufficiently small portion of K_0 that the affine maps ψ_α are a good approximation to the corresponding automorphism φ_α, we must also perturb a large enough number of points in that portion in order that (8.76) holds. For convenience in notation, we will prove only the case $t = 0 \le m \le M$ as given in (8.34) above,

$$c_m \sum_{\ell=1}^{m} \left|\mathbf{v}^\ell\right|_\alpha \le \left|L_\alpha^{(m,0)}\left(\mathbf{v}^1, ..., \mathbf{v}^m\right)\right| \le C_m \sum_{\ell=1}^{m} \left|\mathbf{v}^\ell\right|_\alpha, \tag{8.77}$$

where

$$L_\alpha^{(m,0)}\mathbf{x} = \left\{\sum_{\ell=1}^{m} \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell\left(\alpha^j - \alpha\right)\right\}\right\}_{j=1}^{N}, \quad \mathbf{x} = \left(\mathbf{v}^1, ..., \mathbf{v}^m\right).$$

It suffices to take $m = M$.

To begin, we will assume that a perturbation has already been performed on the first $(N = 1)$ generation of centers c_j^1 in the construction of the Bergman tree $\mathcal{T}_n$, as given in section 2 above, so that $\left\{c_j^1\right\}_j$ is not contained in the zero set of any nontrivial complex polynomial of degree at most M. This will require that some of the centers c_j^1 are displaced a small distance away from the sphere $\mathcal{S}_{\frac{3}{2}\theta}$ on which they initially resided in the construction. We now construct a *fixed* collection of points $\mathcal{E}$ with the above zero set property, and then use unitary and affine maps to transplant these points as replacements for certain of the remaining centers c_j^N.

Let Π denote the *real* $(2n-1)$-dimensional vector space perpendicular to $e_1 = (1, 0, .., 0) \in \mathbb{B}_n$, i.e.

$$\Pi = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 = 0\}.$$

Let D denote the Euclidean ball of radius $\frac{1}{2}$ centered at the origin in Π,

$$D=\left\{z\in\Pi:|z|\leq\frac{1}{2}\right\},$$

and let $\mathcal{E}=\{z_j\}_{j=1}^J$ be a maximal ρ-separated subset of D in the Bergman metric, i.e.

$$\begin{aligned}\beta\left(z_i,z_j\right)&\geq\rho, &&\text{for } i\neq j,\\ \beta\left(z,\mathcal{E}\right)&<\rho, &&\text{for } z\in D,\end{aligned}$$

where $0<\rho<\lambda$ will be chosen later. Note that we can arrange to have J as large as we wish by taking ρ sufficiently small. We also suppose that θ is large enough that $D\subset K_0$.

We now perturb the points z_j slightly by moving them to points z_j' so that

$$\beta\left(z_j,z_j'\right)<\varepsilon\ll\rho,$$

and such that the set of points $\mathcal{E}=\left\{z_j'\right\}_{j=1}^J$ is *not* contained in the zero set of any nontrivial polynomial F on $\mathbb{C}^n$ of degree at most M. This can be done provided J is large enough, which in turn follows from choosing ρ small enough (the zero set property will necessarily force some of the points z_j' to lie outside the space Π). Now define $\mathcal{V}^{(M)}\left(\mathbb{C}^n\right)$ to be the vector space of polynomials on $\mathbb{C}^n$ of degree at most M that vanish at the origin. Then for all $\zeta\in\mathbb{C}^n$ the expression

$$\|F\|_\zeta=\left(\sum_{j=1}^J\left|F\left(z_j'-\zeta\right)\right|^2\right)^{\frac{1}{2}}$$

is a norm on $\mathcal{V}^{(M)}\left(\mathbb{C}^n\right)$. Another norm on $\mathcal{V}^{(M)}\left(\mathbb{C}^n\right)$ is given by $|\mathbf{x}|_\alpha$, where $\mathbf{x}=\left(\mathbf{v}^1,...,\mathbf{v}^M\right)$ is the unique element such that $F\left(z\right)=\sum_{\ell=1}^M\frac{1}{\ell!}\mathbf{v}^\ell\wedge\left\{\otimes^\ell z\right\}$. Since the vector space $\mathcal{V}^{(M)}\left(\mathbb{C}^n\right)$ is finite dimensional, these norms are all equivalent, and uniformly so for ζ in any compact subset K of $\mathbb{C}^n$. Such a K will be fixed below.

We now transport these points and norms to K_γ, where γ is a child of α, by induction on N where $\alpha=c_j^N$ is in the N^{th} generation of the construction in Section 2. So let $N\geq1$ and $\alpha=c_j^N$ so that $d\left(\alpha\right)=N$. We will perturb certain of the children of α as follows. Let α^* be the unique point on the sphere $\mathcal{S}_{\left(d(\alpha)+\frac{1}{2}\right)\theta}$ (where the children of α currently reside) such that

$$P_{d(\alpha)\theta}\alpha^*=z_\alpha=P_{d(\alpha)\theta}\alpha$$

for $\alpha\in\mathcal{T}_n$ (see the construction of the Bergman tree prior to Lemma 2.8). We now pick a unitary map U_α that takes e_1 to $\frac{\alpha}{|\alpha|}$ and use the map $\psi_{\alpha^*}U_\alpha$, where ψ_{α^*} is the affine map defined above, to transport points and norms to $\cup_{\gamma\in\mathcal{C}(\alpha)}K_\gamma$.

The points $\mathcal{E}=\left\{z_j'\right\}_{j=1}^J$ are taken by the affine map $\psi_{\alpha^*}U_\alpha$ to a set of points

$$\mathcal{E}_\alpha=\left\{\alpha^{j\prime}\right\}_{j=1}^J=\left\{\psi_{\alpha^*}U_\alpha z_j'\right\}_{j=1}^J,$$

whose Bergman distance from α^* is at most a constant C. Indeed, to see this we note that

$$\beta\left(\alpha^{j\prime},\alpha^*\right)=\beta\left(\varphi_{\alpha^*}\left(\alpha^{j\prime}\right),0\right)$$

since the automorphisms preserve the Bergman distance, and thus it suffices to prove

$$\beta\left(\varphi_{\alpha^*}\psi_{\alpha^*}U_\alpha z'_j, 0\right) \leq C. \tag{8.78}$$

However, a calculation shows that

$$\varphi_a\psi_a(z) = \frac{z}{1+\overline{a}\cdot z}, \tag{8.79}$$

which in particular yields $|\varphi_a\psi_a(z)| \leq \frac{1}{2}$ if $|z| \leq \frac{1}{3}$, and thus (8.78) as required.

To calculate (8.79), we obtain from (2.2) and (2.3) that

$$\begin{aligned}
\varphi_a(z) &= \frac{a - P_a z - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a z}{1-\overline{a}\cdot z},\\
\psi_a(z) &= a - \left(1-|a|^2\right) P_a z - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a z\\
&= (1-\overline{a}\cdot z)\,\varphi_a(z) + |a|^2 P_a z.
\end{aligned}$$

Thus with $w = \psi_a(z)$, we obtain

$$\varphi_a(w) = (1-\overline{a}\cdot w)^{-1}\left\{\psi_a(w) - |a|^2 P_a w\right\}$$

where

$$\begin{aligned}
\psi_a(w) - |a|^2 P_a w &= a - \left(1-|a|^2\right) P_a\psi_a(z)\\
&\qquad - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\psi_a(z) - |a|^2 P_a\psi_a(z)\\
&= a - P_a\left[a - \left(1-|a|^2\right) P_a z - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a z\right]\\
&\qquad - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\left[a - \left(1-|a|^2\right) P_a z - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a z\right]\\
&= a - \left[a - \left(1-|a|^2\right) P_a z\right]\\
&\qquad - \left(1-|a|^2\right)^{\frac{1}{2}}\left[-\left(1-|a|^2\right)^{\frac{1}{2}} Q_a z\right]\\
&= \left(1-|a|^2\right)[P_a z + Q_a z]\\
&= \left(1-|a|^2\right) z
\end{aligned}$$

and

$$\begin{aligned}
1-\overline{a}\cdot w &= 1-\overline{a}\cdot\left[a - \left(1-|a|^2\right) P_a z - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a z\right]\\
&= 1-|a|^2 + \left(1-|a|^2\right)\overline{a}\cdot z\\
&= \left(1-|a|^2\right)(1+\overline{a}\cdot z).
\end{aligned}$$

Combining these equalities yields (8.79).

Now project the set of points $\mathcal{E}_\alpha$ onto the sphere $\mathcal{S}_{(N+1)\theta}$ to obtain the set $P_{(N+1)\theta}\mathcal{E}_\alpha = \left\{P_{(N+1)\theta}\alpha^{j\prime}\right\}_{j=1}^J$. We note that the sets of points $P_{(N+1)\theta}\mathcal{E}_\alpha$ and $P_{(N+1)\theta}\mathcal{E}_\gamma$ are well separated from each other for $\alpha \neq \gamma$, $N = d(\alpha) = d(\gamma)$ if θ

is chosen large enough. We now redefine the points $\left\{z_j^{N+1}\right\}_{j=1}^J$ and unit qubes $\left\{Q_j^{N+1}\right\}_{j=1}^J$ in $\mathcal{S}_{(N+1)\theta}$ satisfying (2.15) in Subsubsection 2.2.1. We start with the points $E_{N+1} = \cup_{\alpha\in\mathcal{T}_n:d(\alpha)=N}P_{(N+1)\theta}\mathcal{E}_\alpha$. They are $(1-\sqrt{r}\varepsilon)\,\rho$-separated where $r = e^{2\theta}$ as in (8.2). Now extend the collection E_{N+1} to E'_{N+1} by adding those original points z_j^{N+1} that are at distance at least λ from the set E_{N+1}. The resulting collection $E'_{N+1} = \{x'_i\}$ satisfies

$$d\left(x'_i, x'_j\right) \geq \left(1-\sqrt{r}\varepsilon\right)\rho, \qquad i \neq j,$$
$$d\left(x, E'_N\right) < \left(1+\sqrt{r}\varepsilon\right)\lambda, \qquad x \in \mathcal{S}_{(d+1)\theta},$$

Now we proceed with the construction of the qubes Q_j^{N+1} as in Lemma 2.6, and then construct the new kubes K_j^{N+1} with new centers as in Subsubsection 2.2.1. Finally, we repeat this construction inductively for $N \geq 1$ to obtain a perturbed Bergman tree.

We have

$$\psi_{\alpha^*}U_\alpha z = \alpha^* + \varphi'_{\alpha^*}(0)\,U_\alpha z = \alpha^* + T_\alpha z$$

where $T_\alpha = \varphi'_{\alpha^*}(0)\,U_\alpha$ is linear, and so for $F(z) = \sum_{\ell=1}^M \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell z\right\}$ and $\mathbf{x} = \left(\mathbf{v}^1, ..., \mathbf{v}^M\right)$,

$$\begin{aligned}
\left|L_\alpha^{(M,0)}\mathbf{x}\right|^2 &= \left|\left\{\sum_{\ell=1}^M \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell\left(\alpha^j-\alpha\right)\right\}\right\}_{j=1}^N\right|^2 \\
&\geq \sum_{j=1}^J\left|\sum_{\ell=1}^M \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell\left(\alpha^{j\prime}-\alpha\right)\right\}\right|^2 \\
&= \sum_{j=1}^J\left|\sum_{\ell=1}^M \frac{1}{\ell!}\mathbf{v}^\ell \wedge \left\{\otimes^\ell T_\alpha T_\alpha^{-1}\left(\alpha^{j\prime}-\alpha\right)\right\}\right|^2 \\
&= \sum_{j=1}^J\left|\sum_{\ell=1}^M \frac{1}{\ell!}T_\alpha\mathbf{v}^\ell \wedge \left\{\otimes^\ell\left[z'_j - T_\alpha^{-1}\left(\alpha-\alpha^*\right)\right]\right\}\right|^2 \\
&= \|F\|_\zeta^2 \approx \sum_{\ell=1}^M\left|T_\alpha\mathbf{v}^\ell\right|^2 \approx \sum_{\ell=1}^M\left|\mathbf{v}^\ell\right|_\alpha^2,
\end{aligned} \tag{8.80}$$

since the perturbed children $\left\{\alpha^{j\prime}\right\}_{j=1}^J$ are a subset of $\mathcal{C}(\alpha)$, and we also have that $\zeta = T_\alpha^{-1}(\alpha-\alpha^*)$ lies uniformly in a sufficiently large compact set K, independent of $\alpha \in \mathcal{T}_n$. Thus we have proved the left-hand inequality of (8.77) for the new centers. The right hand inequality is trivial (and not used in this paper anyway).

Finally, we can adapt this construction so that $\mathcal{T}_n$ contains a given sequence $Z = \{z_j\}_{j=1}^\infty$ in the ball satisfying the separation condition

$$\beta(z_i, 0) \leq C\beta(z_i, z_j), \qquad 1 \leq i \neq j < \infty.$$

To see this, let $\mathcal{U}_\alpha = \cup_{j=1}^J K_{\alpha^{j\prime}}$ be the union of the kubes $K_{\alpha^{j\prime}}$ corresponding to a set of perturbed points $\mathcal{E}_\alpha$ constructed above. Then the separation condition implies the points z_i are so well separated that no more than one of them occurs in any $\mathcal{U}_\alpha$, and in fact those $\mathcal{U}_\alpha$ that contain a point z_i are themselves pairwise disjoint and well separated. Thus it suffices to choose the model set $\mathcal{E} = \left\{z'_j\right\}_{j=1}^{J}$ to have the somewhat stronger property that, even after the removal of a fixed number C of points z'_i, the resulting set $\mathcal{E}'$ is still not contained in the zero set of any nontrivial complex polynomial of degree at most M. Then if a point z_i from Z lies in the kube K_α, we simply replace α by z_i, and if necessary, modify at most C of the neighbouring points so as not to lie too close to z_i.

DEFINITION 8.19. *We define $HB_p(\mathcal{T}_n)$ to be any of the spaces $HB_{p,m}(\mathcal{T}_n)$ with $m > \frac{2n}{p}$. Lemma 8.18 shows that these spaces are identical, and the above construction of the Bergman tree $\mathcal{T}_n$ shows that we can use the same tree $\mathcal{T}_n$ for all Besov spaces $HB_{p,M}(\mathcal{T}_n)$ with $p > 1$ if we choose $M = 2n$.*

9. Completing the multiplier interpolation loop

We can now complete the proof of the loop of implications for $M_{B_p(\mathbb{B}_n)}$ interpolation on the ball for all $1 < p < 2 + \frac{1}{n-1}$ upon choosing $M = 2n$ in the above definition. As we will see, the following three properties of $HB_p(\mathcal{T}_n)$ essentially suffice to prove that $M_{B_p(\mathbb{B}_n)}$ interpolation implies the tree condition (3.2):

(1) The restriction map is bounded from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_p(\mathcal{T}_n)}$.
(2) The reproducing kernels $k_\alpha^{(m,0)}$ of $HB_{p,m}(\mathcal{T}_n) = HB_p(\mathcal{T}_n)$, $m > \frac{2n}{p}$, satisfy the positivity property (8.43).
(3) Carleson measures for $HB_p(\mathcal{T}_n)$ are characterized by the tree condition (3.2).

Indeed, property 1 will show that $M_{B_p(\mathbb{B}_n)}$ interpolation on the ball implies $M_{HB_p(\mathcal{T}_n)}$ interpolation on the Bergman tree. Then property 2 will show that the atomic measure μ associated with the interpolation sequence is a Carleson measure for $HB_p(\mathcal{T}_n)$. Finally, property 3 will then show that μ satisfies the tree condition. This will complete the multiplier interpolation loop since we have already shown in Section 5, that if μ satisfies the tree condition, then $M_{B_p(\mathbb{B}_n)}$ interpolation holds on the ball.

Before giving the details, we point out that property 1 follows from Theorem 8.14 if $m > \frac{2n}{p}$ and the structural constant θ is large enough; property 2 follows from Lemma 8.11 if in addition the structural constant λ is small enough; and finally, property 3 follows from Theorem 8.12 if both λ is small enough and θ is large enough.

We now give the details. If $\{z_j\}_{j=1}^\infty \subset \mathbb{B}_n$ interpolates $M_{B_p(\mathbb{B}_n)}$, i.e.

(9.1) The map $f \to \{f(z_j)\}_{j=1}^\infty$ takes $M_{B_p(\mathbb{B}_n)}$ boundedly into and onto ℓ^∞,

and if we construct the Bergman tree $\mathcal{T}_n$ so that $\{c_\alpha\}_{\alpha\in\mathcal{T}_n}$ contains $\{z_j\}_{j=1}^\infty$, say with $z_j = c_{\alpha_j}$, then it follows easily from Theorem 8.14 that $\{\alpha_j\}_{j=1}^\infty$ interpolates $M_{HB_p(\mathcal{T}_n)}$, i.e.

(9.2) The map $f \to \{f(\alpha_j)\}_{j=1}^\infty$ takes $M_{HB_p(\mathcal{T}_n)}$ boundedly into and onto ℓ^∞.

Indeed, to see that (9.2) holds, suppose that $\{\xi_j\}_{j=1}^\infty \in \ell^\infty$. Using (9.1) we can find $\varphi \in M_{B_p(\mathbb{B}_n)}$ satisfying

$$\varphi(z_j) = \xi_j, \qquad 1 \le j < \infty,$$
$$\|\varphi\|_{M_{B_p(\mathbb{B}_n)}} \le C \left\| \{\xi_j\}_{j=1}^\infty \right\|_\infty .$$

Now define f on the tree $\mathcal{T}_n$ by

$$f(\alpha) = \varphi(c_\alpha), \qquad \alpha \in \mathcal{T}_n.$$

Then we have

$$f(\alpha_j) = \varphi\left(c_{\alpha_j}\right) = \varphi(z_j) = \xi_j$$

and Theorem 8.14 shows that

$$\|f\|_{M_{HB_p(\mathcal{T}_n)}} \leq C \|\varphi\|_{M_{B_p(\mathbb{B}_n)}},$$

thus establishing (9.2).

We can now use soft arguments, together with the positivity property (8.43) of the reproducing kernels $k_\alpha^{(m,0)}$, with $m > \frac{2n}{p}$ in Lemma 8.11, to show that the measure

$$\mu = \sum_{j=1}^{\infty} \left\| k_{\alpha_j}^{(m,0)} \right\|_{HB_{p'}(\mathcal{T}_n)}^{-p} \delta_{\alpha_j}$$

is a $HB_p(\mathcal{T}_n)$-Carleson measure. Theorem 8.12 then shows that μ satisfies the tree condition (3.2). Finally then, to obtain that

$$\nu = \sum_{j=1}^{\infty} \left(\log \frac{1}{1 - |\alpha_j|^2} \right)^{1-p} \delta_{\alpha_j}$$

satisfies the tree condition (3.2), we use $\|k_\alpha\|_{HB_{p'}(\mathcal{T}_n)}^{p'} \approx \sum_{\gamma \in [o,\alpha]} 1 = d(\alpha) \approx \log \frac{1}{1-|\alpha|^2}$, by (8.43).

9.1. Soft arguments. We now give the above-mentioned soft arguments in detail. For convenience in notation, we abbreviate $HB_{p'}(\mathcal{T}_n)$ by $HB_{p'}$ and $k_{\alpha_j}^{(m,0)}$ by k_{α_j}. From (9.2) we obtain in the usual way that $\left\{k_{\alpha_j}\right\}_{j=1}^{\infty}$ is an unconditional basic sequence in $HB_{p'}$:

$$\left\| \sum_{j=1}^{\infty} b_j k_{\alpha_j} \right\|_{HB_{p'}} \leq C \left\| \sum_{j=1}^{\infty} a_j k_{\alpha_j} \right\|_{HB_{p'}}, \qquad \text{whenever } |b_j| \leq |a_j|. \tag{9.3}$$

We will now use (8.43),

$$r^{-md(\gamma)} \operatorname{Re}\left(D_m^m k_\alpha^{(m,0)}(\gamma) \wedge \{\otimes^m \gamma\} \right) \approx 1$$

$$\left| D_m^\ell k_\alpha^{(m,t)}(\gamma) \right|_\gamma + \left| \mathcal{Q}_\gamma^{(m,t)} \left(\mathcal{D}_\gamma k_\alpha^{(m,t)} \right) \right|_\gamma \leq \begin{cases} C & \text{for} \quad \gamma \leq \alpha \\ 0 & \qquad \text{otherwise} \end{cases},$$

and (9.3), along with a modification of the technique of Böe's "curious" Lemma 3.1 in [**Boe**], to obtain the following norm equivalence:

$$\left\| \sum_{j=1}^{\infty} a_j \frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}} \right\|_{HB_{p'}} \approx \left(\sum_{j=1}^{\infty} |a_j|^{p'} \right)^{\frac{1}{p'}}. \tag{9.4}$$

We use the ℓ^p variant (8.59) of Definition 8.9 for the scalar function f, namely

$$\begin{aligned}
\|f\|_{HB_p(\mathcal{T}_n)}^p &= \|f\|_{HB_{p,m}(\mathcal{T}_n)}^p \\
&= \sum_{s_k<m} \left| D_{m-s_{k-1}}^{\ell_k} ... D_{m-\ell_1}^{\ell_2} D_m^{\ell_1} f(o) \right|^p \\
&\qquad + \sum_{s_k=m} \sum_{\alpha\in\mathcal{T}_n} \left| \mathcal{Q}_\alpha^{(m-s_{k-1},s_{k-1})} \mathcal{D}_\alpha D_{m-s_{k-2}}^{\ell_{k-1}} ... D_m^{\ell_1} f \right|_\alpha^p \\
&\qquad + \sum_{s_k=m} \sum_{\alpha\in\mathcal{T}_n} \left| D_{m-s_{k-1}}^{\ell_k} ... D_m^{\ell_1} f(\alpha) \right|_\alpha^p \\
&= \left\| \check{D} f \right\|_{\ell^p(\mathcal{T}_n)}^p ,
\end{aligned}$$

where $\check{D}f$ is defined on the tree in the obvious way and the ℓ^p norm involves the metric $|\cdot|_\alpha$ as usual, together with the Rademacher functions $r_j(t)$ in conjunction with (9.3) to obtain

$$\begin{aligned}
\left\| \sum_{j=1}^\infty a_j \frac{k_{\alpha_j}}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \right\|_{HB_{p'}}^{p'} &\approx \int_0^1 \left\| \sum_{j=1}^\infty a_j r_j(t) \frac{k_{\alpha_j}}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \right\|_{HB_{p'}}^{p'} dt \\
&\approx \int_0^1 \left\| \sum_{j=1}^\infty \frac{a_j r_j(t)}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \check{D} k_{\alpha_j}(\gamma) \right\|_{\ell^{p'}(\gamma)}^{p'} dt.
\end{aligned}$$

Now Khinchine's inequality holds for finite-dimensional vector spaces in place of scalars. Indeed, if $\mathbf{a}_j = (a_j(i))_{i=1}^N \in \mathbb{C}^N$, then

$$\begin{aligned}
\left(\int_0^1 \left| \sum_{j=1}^\infty r_j(t) \mathbf{a}_j \right|^q dt \right)^{\frac{1}{q}} &\approx \left(\int_0^1 \left\{ \sum_{i=1}^N \left| \sum_{j=1}^\infty r_j(t) a_j(i) \right|^q \right\} dt \right)^{\frac{1}{q}} \\
&\approx \sum_{i=1}^N \left(\int_0^1 \left| \sum_{j=1}^\infty r_j(t) a_j(i) \right|^q dt \right)^{\frac{1}{q}} \\
&\approx \sum_{i=1}^N \left(\sum_{j=1}^\infty |a_j(i)|^2 \right)^{\frac{1}{2}} \\
&\approx \left(\sum_{j=1}^\infty |\mathbf{a}_j|^2 \right)^{\frac{1}{2}} .
\end{aligned}$$

Thus with N sufficiently large, we have that

$$\left\| \sum_{j=1}^\infty a_j \frac{k_{\alpha_j}}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \right\|_{HB_{p'}}^{p'}$$

is dominated by a constant multiple of

$$\left\| \left\{ \sum_{j=1}^{\infty} \left| \frac{|a_j|}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \check{D} k_{\alpha_j} (\gamma) \right|_{\gamma}^{2} \right\}^{\frac{1}{2}} \right\|_{\ell^{p'}(\gamma)}^{p'} .$$

Using (8.43), we can then continue the above with

$$\left\| \sum_{j=1}^{\infty} a_j \frac{k_{\alpha_j}}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \right\|_{HB_{p'}}^{p'}$$

$$\leq C \left\| \left\{ \sum_{j=1}^{\infty} \left| \frac{|a_j|}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} r^{-md(\gamma)} \operatorname{Re} \left(D_m^m k_\alpha^{(m,0)} (\gamma) \wedge \{ \otimes^m \gamma \} \right) \right|^2 \right\}^{\frac{1}{2}} \right\|_{\ell^{p'}(\gamma)}^{p'} .$$

Let

$$\begin{aligned} A_j(\gamma) &= \frac{|a_j|}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} r^{-md(\gamma)} \operatorname{Re} \left(D_m^m k_\alpha (\gamma) \wedge \{ \otimes^m \gamma \} \right) \\ &= \operatorname{Re} \left\{ \frac{|a_j|}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} r^{-md(\gamma)} D_m^m k_\alpha (\gamma) \right\} \wedge \{ \otimes^m \gamma \} . \end{aligned}$$

We now use $\|\cdot\|_{\ell^2} \leq \sqrt{\|\cdot\|_{\ell^1} \|\cdot\|_{\ell^\infty}}$ and the nonnegativity of $A_j(\gamma)$ to obtain

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_j \frac{k_{\alpha_j}}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} \right\|_{HB_{p'}}^{p'} &\leq C \left\| \left\{ \sum_{j=1}^{\infty} A_j(\gamma)^2 \right\}^{\frac{1}{2}} \right\|_{\ell^{p'}(\gamma)}^{p'} \\ &\leq C \left\| \left\{ \left[\sum_{j=1}^{\infty} A_j(\gamma) \right] \left[\sup_{1 \leq j < \infty} A_j(\gamma) \right] \right\}^{\frac{1}{2}} \right\|_{\ell^{p'}(\gamma)}^{p'} \\ &\leq C \left\| \sum_{j=1}^{\infty} A_j(\gamma) \right\|_{\ell^{p'}(\gamma)}^{\frac{p'}{2}} \left\| \sup_{1 \leq j < \infty} A_j(\gamma) \right\|_{\ell^{p'}(\gamma)}^{\frac{p'}{2}} . \end{aligned}$$

Now set $K = \sum_{j=1}^{\infty} \frac{|a_j|}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} k_{\alpha_j}$ so that

$$\begin{aligned} \sum_{j=1}^{\infty} A_j(\gamma) &= \operatorname{Re} \left\{ \sum_{j=1}^{\infty} \frac{|a_j|}{\left\| k_{\alpha_j} \right\|_{HB_{p'}}} r^{-md(\gamma)} D_m^m k_{\alpha_j} (\gamma) \right\} \wedge \{ \otimes^m \gamma \} \\ &= \operatorname{Re} \left\{ r^{-md(\gamma)} D_m^m K(\gamma) \right\} \wedge \{ \otimes^m \gamma \} \end{aligned}$$

Using the inequality

$$\left| r^{-md(\gamma)} D_m^m K(\gamma) \right|^2 \leq \left| \check{D} K \right|_{\gamma}^{2} , \tag{9.5}$$

and (8.59) again, we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}A_j(\gamma)\right\|_{\ell^{p'}(\gamma)} &= \left\|\mathrm{Re}\left\{r^{-md(\gamma)}D_m^m K(\gamma)\right\}\wedge\{\otimes^m\gamma\}\right\|_{\ell^{p'}(\gamma)}\\ &\le C\left\|\check{D}K\right\|_{\ell^{p'}(\gamma)}\\ &\le C\left\|K\right\|_{HB_{p'}}\\ &= C\left\|\sum_{j=1}^{\infty}|a_j|\frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}}\right\|_{HB_{p'}}.\end{aligned}$$

Another application of the unconditional basic sequence property (9.3), shows that this is dominated by a constant multiple of

$$\left\|\sum_{j=1}^{\infty}a_j\frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}}\right\|_{HB_{p'}}.$$

Altogether we now have

$$\left\|\sum_{j=1}^{\infty}a_j\frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}}\right\|_{HB_{p'}}^{p'} \le C\left\|\sum_{j=1}^{\infty}a_j\frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}}\right\|_{HB_{p'}}^{\frac{p'}{2}}\left\|\sup_{1\le j<\infty}A_j(\gamma)\right\|_{\ell^{p'}(\gamma)}^{\frac{p'}{2}},$$

which yields

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}a_j\frac{k_{\alpha_j}}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}}\right\|_{HB_{p'}} &\le C\left\|\sup_{1\le j<\infty}A_j(\gamma)\right\|_{\ell^{p'}(\gamma)}\\ &\le C\left\|\left(\sum_{j=1}^{\infty}A_j(\gamma)^{p'}\right)^{\frac{1}{p'}}\right\|_{\ell^{p'}(\gamma)}\\ &= C\left\{\sum_{j=1}^{\infty}\left(\frac{|a_j|}{\left\|k_{\alpha_j}\right\|_{HB_{p'}}}\right)^{p'}\left\|\mathrm{Re}\left\{r^{-md(\gamma)}D_m^m k_{\alpha_j}(\gamma)\right\}\wedge\{\otimes^m\gamma\}\right\|_{\ell^{p'}(\gamma)}^{p'}\right\}^{\frac{1}{p'}}\\ &\le C\left\{\sum_{j=1}^{\infty}|a_j|^{p'}\right\}^{\frac{1}{p'}},\end{aligned}$$

since by (9.5) and (8.59) once more,

$$\begin{aligned}\left\|\mathrm{Re}\left\{r^{-md(\gamma)}D_m^m k_{\alpha_j}(\gamma)\right\}\wedge\{\otimes^m\gamma\}\right\|_{\ell^{p'}(\gamma)} &\le C\left\|\check{D}k_{\alpha_j}(\gamma)\right\|_{\ell^{p'}(\gamma)}\\ &\le C\left\|k_{\alpha_j}\right\|_{HB_{p'}}.\end{aligned}$$

This completes the proof of the inequality $\lesssim$ in (9.4), and the opposite inequality is standard.

From the inequality $\lesssim$ in (9.4), we obtain in the usual way that the measure

$$\mu = \sum_{j=1}^{\infty} \left\| k_{\alpha_j} \right\|_{HB_{p'}}^{-p} \delta_{\alpha_j}$$

is a $HB_p(\mathcal{T}_n)$-Carleson measure, and as shown above, this completes the loop.

10. Appendix

Here we use a stopping time argument to directly prove the following lemma, rather than by appealing to Theorem 3.1.

LEMMA 10.1. *Suppose that μ is a measure on a tree $\mathcal{T}$ satisfying the tree condition (1.10), i.e.*

$$\sum_{\beta \in \mathcal{T}:\beta \geq \alpha} I^*\mu(\beta)^{p'} \leq C^{p'} I^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}.$$

If $I^*\omega \leq I^*\mu$ on $\mathcal{T}$, then ω also satisfies the tree condition (1.10), and with tree condition norm at most $C_{p'}$ times that of μ.

PROOF. (direct proof without Theorem 3.1) It suffices to prove the case when $\alpha = o$, the root of the tree $\mathcal{T}$, i.e.

$$\sum_{\beta \in \mathcal{T}} I^*\omega(\beta)^{p'} \leq C_{p'} C^{p'} I^*\omega(o).$$

Let $G_0 = \{o\}$. Let $G(o)$ consist of all minimal tree elements $\beta > o$ satisfying

$$\frac{I^*\omega(\beta)}{I^*\mu(\beta)} > 2\frac{I^*\omega(o)}{I^*\mu(o)}.$$

We refer to the elements in $G_1 = G(o)$ as first generation elements. For each first generation element $\alpha \in G_1$, let $G(\alpha)$ consist of all minimal tree elements $\beta > \alpha$ satisfying

$$\frac{I^*\omega(\beta)}{I^*\mu(\beta)} > 2\frac{I^*\omega(\alpha)}{I^*\mu(\alpha)}.$$

We refer to the elements in $G_2 = \cup_{\alpha \in G_1} G(\alpha)$ as second generation elements. Continuing in this way, we define generations G_k for $k \geq 1$ (actually k is at most $\log_2\left(\frac{I^*\omega(o)}{I^*\mu(o)}\right)$) with the property

$$\frac{I^*\omega(\beta)}{I^*\mu(\beta)} > 2\frac{I^*\omega(\alpha)}{I^*\mu(\alpha)}, \qquad \text{for } \alpha \in G_k, \beta \in G_{k+1}, \beta > \alpha, k \geq 0.$$

Let $\mathcal{G}$ be the subset $\mathcal{T}$ of whose elements are $\cup_{k\geq 0} G_k$. For each element $\alpha \in \mathcal{G}$, let $H(\alpha) = \cup_{\gamma \in G(\alpha)} [\alpha, \gamma)$ be the union of all geodesics in $\mathcal{T}$ (open at the far end) that start at α and end at an element γ of $G(\alpha)$. This yields a pairwise disjoint decomposition of the tree $\mathcal{T}$ given by $\mathcal{T} = \cup_{\alpha \in \mathcal{G}} H(\alpha)$.

We have from the defining scheme of $\mathcal{G}$ that

$$\begin{aligned}\sum_{\beta\in\mathcal{T}} I^*\omega(\beta)^{p'} &= \sum_{\alpha\in\mathcal{G}}\sum_{\beta\in H(\alpha)}\left(\frac{I^*\omega(\beta)}{I^*\mu(\beta)}\right)^{p'} I^*\mu(\beta)^{p'}\\ &\leq 2^{p'}\sum_{\alpha\in\mathcal{G}}\left(\frac{I^*\omega(\alpha)}{I^*\mu(\alpha)}\right)^{p'}\sum_{\beta\in H(\alpha)} I^*\mu(\beta)^{p'}\\ &\leq 2^{p'}\sum_{m\geq 0}2^{-mp'}\sum_{\alpha\in\mathcal{G}_m}\sum_{\beta\in H(\alpha)} I^*\mu(\beta)^{p'},\end{aligned}$$

where

$$\mathcal{G}_m = \left\{\alpha\in\mathcal{G} : 2^{-m-1}\leq\frac{I^*\omega(\alpha)}{I^*\mu(\alpha)}\leq 2^{-m}\right\}.$$

It is here in asserting that $m\geq 0$ that we use the hypothesis $I^*\omega(\alpha)\leq I^*\mu(\alpha)$. If we let $\mathcal{M}_m$ be the minimal elements in $\mathcal{G}_m$, then we have

$$\begin{aligned}2^{-mp'}\sum_{\alpha\in\mathcal{G}_m}\sum_{\beta\in H(\alpha)} I^*\mu(\beta)^{p'} &\leq 2^{-mp'}\sum_{\gamma\in\mathcal{M}_m}\sum_{\alpha\in\mathcal{G}_m:\alpha\geq\gamma}\sum_{\beta\in H(\alpha)} I^*\mu(\beta)^{p'}\\ &\leq 2^{-mp'}\sum_{\gamma\in\mathcal{M}_m}\sum_{\beta\geq\gamma} I^*\mu(\beta)^{p'}\\ &\leq 2^{-mp'}\sum_{\gamma\in\mathcal{M}_m} C^{p'} I^*\mu(\gamma).\end{aligned}$$

Now using that $\mathcal{M}_m\subset\mathcal{G}_m$, we continue with

$$\begin{aligned}C^{p'}2^{-m\left(p'-1\right)}\sum_{\gamma\in\mathcal{M}_m}2^{-m}I^*\mu(\gamma) &\leq C^{p'}2^{-m\left(p'-1\right)}\sum_{\gamma\in\mathcal{M}_m}\left(2\frac{I^*\omega(\gamma)}{I^*\mu(\gamma)}\right) I^*\mu(\gamma)\\ &= 2C^{p'}2^{-m\left(p'-1\right)}\sum_{\gamma\in\mathcal{M}_m} I^*\omega(\gamma)\\ &\leq 2C^{p'}2^{-m\left(p'-1\right)} I^*\omega(o).\end{aligned}$$

Adding these estimates up for $m\geq 0$ yields

$$\begin{aligned}\sum_{\beta\in\mathcal{T}} I^*\omega(\beta)^{p'} &\leq 2^{p'}\sum_{m\geq 0}2C^{p'}2^{-m\left(p'-1\right)} I^*\omega(o)\\ &\leq C_{p'}C^{p'} I^*\omega(o).\end{aligned}$$

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