FUNCTIONAL ANALYSIS AND APPLICATIONS

1. Introduction

We begin by solving a Volterra integral equation using a fixed point theorem, and from this we motivate the definition of Banach space and bounded linear operator.

Next, we specialize to Hilbert spaces and after proving some elementary results on duality, we use the Lax-Milgram theorem and the Fredholm alternative to derive existence and uniqueness of weak solutions to Dirichlet problems for second order elliptic linear partial differential equations on open subsets of Euclidean space.

Then we derive the completeness theorems for Banach spaces, including the uniform boundedness principle, the open mapping theorem and the closed graph theorem, and give applications to the nonconvergence of Fourier series of continuous functions, the nonconverse of the Riemann-Lebesgue lemma, and finally to interpolating sequences for the Hardy space of holomorphic functions on the unit disk.

We then continue with convexity theorems, such as the Hahn-Banach and Banach-Alaoglu theorems, and investigate the relation between the existence of invariant finitely additive measures and paradoxical decompositions, including the Banach-Tarski paradox. Further directions and applications will follow as time permits.

References are given at the end.

2. Schechter's Example ([3])

The pair of functions $\{\cos x, \sin x\}$ is a fundamental solution set on the real line $\mathbb{R}$ for the homogeneous second order equation

$$y''(x) + y(x) = 0, \qquad x \in \mathbb{R},$$

and the general solution is given by

$$y_{\text{hom}}(x) = y_{\text{hom}}(0)\cos x + y'_{\text{hom}}(0)\sin x, \qquad x \in \mathbb{R}. \tag{2.1}$$

We now wish to solve the more general equation

$$y''(x) + y(x) = \sigma(x)\, y(x),$$

where σ is a continuous function on $\mathbb{R}$. First we solve the inhomogenoeous equation

$$y''(x) + y(x) = f(x)$$

by writing it as a system in $\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$\begin{aligned} \mathbf{y}' &= \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix} \\ &\equiv A\mathbf{y} + \mathbf{f}. \end{aligned}$$

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Then the Wronskian matrix

$$W(x) = \begin{bmatrix} \cos x & \sin x \\ \cos' x & \sin' x \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

satisfies

$$W' = AW \text{ and } \left(W^{-1}\right)' = -W^{-1}A.$$

Thus

$$\begin{aligned} \left(W^{-1}\mathbf{y}\right)' &= W^{-1}\mathbf{y}' + \left(W^{-1}\right)'\mathbf{y} \\ &= W^{-1}\mathbf{y}' - W^{-1}A\mathbf{y} = W^{-1}\mathbf{f} \end{aligned}$$

implies

$$\mathbf{y} = W \int W^{-1}\mathbf{f}$$

and so a particular solution $y_{part}(x)$ is derived from

$$\begin{aligned} \begin{bmatrix} y_{part}(x) \\ y'_{part}(x) \end{bmatrix} &= \int_0^x W(x) W^{-1}(t) \mathbf{f}(t)\, dt \\ &= \int_0^x \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} dt \\ &= \int_0^x \begin{bmatrix} * & \sin x \cos t - \cos x \sin t \\ * & * \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \int_0^x \sin(x-t) f(t)\, dt \\ * \end{bmatrix}. \end{aligned} \tag{2.2}$$

Now we see from (2.1) and (2.2) that the solution to the initial value problem

$$\begin{cases} y'' + y &= \sigma y \\ y(0) &= 1 \\ y'(0) &= 0 \end{cases}$$

satisfies the integral equation

$$y(x) = \cos x + \int_0^x \sin(x-t)\sigma(t) y(t)\, dt, \qquad x \in \mathbb{R},$$

and vice versa. If we write $u(x) = \cos x$ and

$$Lh(x) = \int_0^x \sin(x-t)\sigma(t) h(t)\, dt,$$

we can rewrite this equation as

$$y = u + Ly, \tag{2.3}$$

an example of a Volterra integral equation.

2.1. **Volterra Equations.** To solve the Volterra equation (2.3) for $x \in [-N, N]$, we start with a guess $y_0 = y_0(x)$ where y_0 is any continuous function on $[-N, N]$, and plug it into the right side of (2.3), defining

$$\begin{aligned} y_1 &= y_1(x) = u(x) + Ly_0(x) \\ &= \cos x + \int_0^x \sin(x-t)\sigma(t) y_0(t)\, dt, \qquad x \in [-N, N]. \end{aligned}$$

If it happens that $y_1 = y_0$ (highly unlikely!) we are done. Otherwise set $y_2 = u + Ly_1$ and inductively

$$y_n = u + Ly_{n-1} \text{ on } [-N, N], \qquad n = 1, 2, 3, ... \tag{2.4}$$

We hope that this sequence of functions $\{y_n\}_{n=1}^{\infty}$ converges in some sense. Since uniform convergence yields a continuous limit, we define

$$\|h\| = \max_{|x| \leq N} |h(x)|$$

and hope that $\|y_m - y_n\| \to 0$ as $m, n \to \infty$ (the Cauchy criterion for uniform convergence).

Now we compute inductively that

$$\begin{aligned} y_n &= u + Ly_{n-1} \\ &= u + L(u + Ly_{n-2}) \\ &\vdots \\ &= u + Lu + ... + L^{n-1}u + L^n y_0. \end{aligned} \tag{2.5}$$

Thus we have for $n > m$,

$$\begin{aligned} \|y_m - y_n\| &= \left\|L^m u + ... + L^{n-1}u + L^n y_0 - L^m y_0\right\| \\ &\leq \|L^m u\| + ... + \left\|L^{n-1}u\right\| + \|L^n y_0\| + \|L^m y_0\|, \end{aligned} \tag{2.6}$$

and in particular this will tend to zero as $m, n \to \infty$ provided we have the "absolute convergence of orbit series":

$$\sum_{n=0}^{\infty} \|L^n v\| < \infty \text{ for every continuous } v \text{ on } [-N, N]. \tag{2.7}$$

Indeed, if (2.7) holds, then $\{y_n\}_{n=1}^{\infty}$ satisfies the Cauchy criterion for uniform convergence and hence there is a continuous function $y = y(x)$ on $[-N, N]$ such that $y_n \to y$ uniformly on $[-N, N]$. We now claim that y satisfies (2.3) on $[-N, N]$. For this we use the inequality

$$|Lv(x)| = \left|\int_0^x \sin(x - t)\, \sigma(t)\, v(t)\, dt\right| \leq \|\sigma\| \|v\| |x|, \tag{2.8}$$

from which follows

$$\|Lv\| \leq (N \|\sigma\|) \|v\| = C \|v\| \tag{2.9}$$

for all continuous v on $[-N, N]$. If we now let $n \to \infty$ in the equation (2.4) we obtain

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} (u + Ly_{n-1}) = u + Ly$$

since by (2.9),

$$\|Ly - Ly_{n-1}\| = \|L(y - y_{n-1})\| \leq C \|y - y_{n-1}\| \to 0, \qquad \text{as } n \to \infty.$$

Finally we establish the "absolute convergence of orbit series" in (2.7). By (2.8) we have

$$\left|\int_0^x \sin(x - t)\, \sigma(t)\, Lv(t)\, dt\right| \leq \int_0^x |\sin(x - t)\, \sigma(t)| \{\|\sigma\| \|v\| |t|\}\, dt \leq \|\sigma\|^2 \|v\| \frac{|x|^2}{2},$$

and continuing by induction we obtain

$$\begin{aligned} |L^n v(x)| &= \left| \int_0^x \sin(x-t)\,\sigma(t)\,L^{n-1}v(t)\,dt \right| \le \|\sigma\|^n \|v\| \frac{|x|^n}{n!}, \\ \|L^n v\| &\le \|\sigma\|^n \|v\| \frac{N^n}{n!}, \end{aligned}$$

from which (2.7) follows immediately:

$$\sum_{n=0}^{\infty} \|L^n v\| \le \sum_{n=0}^{\infty} \|\sigma\|^n \|v\| \frac{N^n}{n!} = e^{N\|\sigma\|} \|v\| < \infty.$$

2.2. **Banach spaces and bounded linear operators.** We now examine the above argument and extract the essential properties needed of the set X of continuous functions on $[-N,N]$, and of the mapping $h \to Lh$. First, in (2.4) we use that X is a vector space, and in (2.6), we use a nonnegative function $\|\cdot\|$ defined on X that satisfies the triangle inequality among other things. Our use of the Cauchy criterion on the sequence of approximations $\{y_n\}$ requires "completeness" in the metric $d(f,g) = \|f-g\|$ induced by $\|\cdot\|$. These considerations motivate the following definition of a Banach space.

Definition 1. *A complex vector space X is a* normed linear space *if there is a map $\|\cdot\| : X \to [0,\infty)$ satisfying $\|x+y\| \le \|x\| + \|y\|$, $\|\lambda x\| = |\lambda| \|x\|$ and $\|x\| = 0 \Longleftrightarrow x = 0$, for all $x, y \in X$ and $\lambda \in \mathbb{C}$. Then $d(x,y) = \|x-y\|$ defines a metric on X, and X is a* Banach space *if (X,d) is a complete metric space.*

There are versions of these definitions and those below when the scalar field is the real field $\mathbb{R}$ instead of the complex field $\mathbb{C}$. Normally there is little difference in the interaction of the concepts, and we will usually use the complex scalar field $\mathbb{C}$ - but will explicitly mention the scalar field $\mathbb{R}$ when it matters. We will denote by $C(K)$ the Banach space of continuous functions on a compact topological space K, equipped with the supremum norm $\|f\|_\infty = \sup_{x\in K} |f(x)|$. Now we examine the properties used of the map L from the Banach space $C([-N,N])$ into itself. First, in (2.5) we used that L is linear, and then in (2.9) we used that L takes bounded sets in $C([-N,N])$ to bounded sets. This motivates the following definition of a bounded linear operator between normed linear spaces.

Definition 2. *A map L from one normed linear space X to another Y is* linear *if $L(\lambda x + y) = \lambda Lx + Ly$ for all $x, y \in X$ and $\lambda \in \mathbb{C}$, and* bounded *if there is a nonnegative constant C such that $\|Lx\|_Y \le C \|x\|_X$ for all $x \in X$.*

The proof of the next result is easy and is left to the reader.

Lemma 1. *Let $L : X \to Y$ be linear where X, Y are normed linear spaces. Then L is bounded $\Longleftrightarrow$ L is continuous on X $\Longleftrightarrow$ L is continuous at 0.*

Remark 1. *If Y is the scalar field, then in addition we have that L is continuous $\Longleftrightarrow$ the null space of L is closed. Indeed, if the null space $\mathcal{N}$ of L is closed and $x_0 \notin \mathcal{N}$, then there is a ball $B(x_0,r) \subset X \backslash \mathcal{N}$. Now if $L(B(0,r))$ is unbounded, then it must be all of the scalar field $\mathbb{C}$, and so $L(B(x_0,r)) = \mathbb{C}$ as well, contradicting the fact that $L(B(x_0,r)) \subset L(X \setminus \mathcal{N})$, where the latter set doesn't include 0. Thus $L(B(0,r))$ is bounded as required. However, this equivalence of continuity and closed null space fails for general $Y = X$ as evidenced by the space X of polynomials on $[0,1]$ with the supremum norm, and $L : X \to X$ by $LP = P'$ for $P \in X$.*

Our arguments above prove the following general theorem in a Banach space.

Theorem 1. *Let X be a Banach space and L be a bounded linear operator from X to itself. If $\sum_{n=0}^{\infty} \|L^n x\| < \infty$ for all $x \in X$, then the equation*

$$x = y + Lx$$

has a unique solution $x \in X$ for every $y \in X$.

Proof: To see uniqueness, let $x_1 = y + Lx_1$ and $x_2 = y + Lx_2$. Then

$$x = x_1 - x_2 = Lx_1 - Lx_2 = Lx = L^2 x = ... = L^n x$$

for all $n \geq 1$ implies that $\|x_1 - x_2\| = \|L^n x\| \to 0$ as $n \to \infty$, which implies $x_1 = x_2$. The existence is proved as above using the approximating sequence $\{x_n\}_{n=0}^{\infty}$ defined inductively by $x_n = y + Lx_{n-1}$, x_0 arbitrary in X.

The contraction mapping theorem is a special case.

Theorem 2. *Let X be a Banach space and L be a bounded linear operator from X to itself. If L is a contraction, i.e. there is a constant $0 \leq \gamma < 1$ such that*

$$\|Lx\| \leq \gamma \|x\|, \qquad x \in X,$$

then L has a unique fixed point *x, i.e. $x = Lx$.*

Proof: We can apply the previous theorem with $y = 0$ since $\sum_{n=0}^{\infty} \|L^n x\| \leq \sum_{n=0}^{\infty} \gamma^n \|x\| = \frac{1}{1-\gamma} \|x\| < \infty$.

3. Hilbert spaces ([2])

There is a class of special Banach spaces that enjoy many of the properties of the familiar Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, namely the *Hilbert spaces*, whose norms arise from an inner product.

Definition 3. *A complex vector space H is an* inner product space *if there is a map $\langle \cdot, \cdot \rangle$ from $H \times H$ to $\mathbb{C}$ satisfying for all $x, y \in H$ and $\lambda \in \mathbb{C}$,*

$$\begin{aligned} \langle x, y \rangle &= \overline{\langle y, x \rangle}, \\ \langle x + z, y \rangle &= \langle x, y \rangle + \langle z, y \rangle, \\ \langle \lambda x, y \rangle &= \lambda \langle x, y \rangle, \\ \langle x, x \rangle &\geq 0 \text{ and } \langle x, x \rangle = 0 \Longleftrightarrow x = 0. \end{aligned}$$

Then $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm on H (see below) and if this makes H into a Banach space, i.e. the metric $d(x, y) = \|x - y\|$ is complete, then we say H is a Hilbert space*.*

A simple example of a Hilbert space is real or complex Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual inner product. More generally, the space $\ell^2(\mathbb{N})$ of square summable sequences $a = \{a_n\}_{n=1}^{\infty}$ with inner product $\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$ is a Hilbert space. Both of these examples are included as special cases of the Hilbert space $L^2(\mu)$ where μ is a positive measure on a measure space X and the inner product is $\langle f, g \rangle = \int_X f \overline{g} d\mu$.

Lemma 2. *Let H be an inner product space and define $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in H$. Then $\|\cdot\|$ is a norm on H and for all $x, y \in H$,*

$$\begin{aligned} |\langle x, y\rangle| &\leq \|x\| \|y\|, \\ \|y\| &\leq \|\lambda x + y\| \text{ for all } \lambda \in \mathbb{C} \text{ iff } \langle x, y\rangle = 0, \\ \|x+y\|^2 + \|x-y\|^2 &= 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

Proof: For $x, y \in H$ and $\lambda \in \mathbb{C}$,

$$0 \leq \|\lambda x + y\|^2 = |\lambda|^2 \|x\|^2 + 2\operatorname{Re}(\lambda \langle x, y\rangle) + \|y\|^2. \tag{3.1}$$

Thus $\langle x, y\rangle = 0$ implies $\|y\| \leq \|\lambda x + y\|$ for all $\lambda \in \mathbb{C}$. Conversely, if $x \neq 0$ we minimize the right side of (3.1) with $\lambda = -\frac{\overline{\langle x,y\rangle}}{\|x\|^2}$ to get

$$0 \leq \|\lambda x + y\|^2 = -\frac{|\langle x, y\rangle|^2}{\|x\|^2} + \|y\|^2.$$

This shows that $\|y\| \leq \|\lambda x + y\|$ fails for some λ if $\langle x, y\rangle \neq 0$, and also proves the Cauchy-Schwarz inequality $|\langle x, y\rangle| \leq \|x\| \|y\|$. With $\lambda = 1$ in (3.1) we now have

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

which shows $\|\cdot\|$ satisfies the triangle inequality, and $\|\cdot\|$ is now easily seen to be a norm. Finally, the parallelogram law follows from expanding the inner products on the left side.

The next easy theorem lies at the heart of the great success of Hilbert spaces in analysis.

Theorem 3. *Suppose E is a nonempty closed convex subset of a Hilbert space H. Then E contains a unique element x of minimal norm, i.e. $\|x\| = \inf_{y \in E} \|y\|$.*

Proof: Let $d = \inf_{y \in E} \|y\|$, which is finite since E is *nonempty*. Pick $\{x_n\}_{n=1}^\infty \subset E$ with $\|x_n\| \to d$ as $n \to \infty$. Since E is *convex*, $\frac{x_m + x_n}{2} \in E$ and so has norm at least d. The parallelogram law now yields

$$\begin{aligned} \left\|\frac{x_m - x_n}{2}\right\|^2 &= \frac{\|x_m\|^2 + \|x_n\|^2}{2} - \left\|\frac{x_m + x_n}{2}\right\|^2 \\ &\leq \frac{\|x_m\|^2 + \|x_n\|^2}{2} - d^2 \\ &\to \frac{d^2 + d^2}{2} - d^2 = 0 \end{aligned}$$

as $m, n \to \infty$. Thus $\{x_n\}_{n=1}^\infty$ is Cauchy and since H is complete and E *closed*, $x = \lim_{n\to\infty} x_n \in E$. Since $\|\cdot\|$ is continuous, we have $\|x\| = d$. If $x' \in E$ also satisfies $\|x'\| = d$, then using the parallelogram law as above yields $\left\|\frac{x-x'}{2}\right\|^2 = \frac{\|x\|^2 + \|x'\|^2}{2} - \left\|\frac{x+x'}{2}\right\|^2 \leq 0$, hence $x = x'$.

Let H be a Hilbert space. We say that x and y in H are perpendicular, written $x \perp y$, if $\langle x, y \rangle = 0$. We say subsets E and F of H are perpendicular, written $E \perp F$, if $\langle x, y \rangle = 0$ for all $x \in E$ and $y \in F$. Finally, we define

$$E^{\perp} = \{ y \in H : \langle x, y \rangle = 0 \text{ for all } x \in E \} .$$

The next theorem uses Theorem 3 to establish an orthogonal decomposition of H relative to any closed subspace M of a Hilbert space H.

Theorem 4. *Suppose that M is a closed subspace of a Hilbert space H. Then*

$$H = M \oplus M^{\perp},$$

which means that M and $M^{\perp}$ are closed subspaces of H whose intersection is the smallest subspace $\{0\}$, and whose span is the largest subspace H.

Proof: $M^{\perp}$ is a subspace since $\langle x, y \rangle$ is linear in x, and is closed by the Cauchy-Schwarz inequality. The fact that $\langle x, x \rangle = 0 \Longleftrightarrow x = 0$ gives $M \cap M^{\perp} = \{0\}$. Finally, to show $M + M^{\perp} = H$, let $x \in H$ and set $E = x - M$, a nonempty closed convex set. Thus there is a unique element $m^{\perp} \in x - M$ of minimal norm having the form $x - m$ with $m \in M$. Thus for all $z \in M$ and $\lambda \in \mathbb{C}$,

$$\left\| m^{\perp} \right\| \leq \left\| m^{\perp} + \lambda z \right\|$$

and Lemma 2 implies that $\left\langle z, m^{\perp} \right\rangle = 0$ for all $z \in M$, which yields $m^{\perp} \in M^{\perp}$. Thus $x = m + m^{\perp} \in M + M^{\perp}$.

Corollary 1. $\left(M^{\perp} \right)^{\perp} = M$.

Proof: $M \subset \left(M^{\perp} \right)^{\perp}$ is obvious, and since $M \oplus M^{\perp} = H = M^{\perp} \oplus \left(M^{\perp} \right)^{\perp}$, we cannot have that M is a proper subset of $\left(M^{\perp} \right)^{\perp}$.

3.1. **Duality.** Given any normed linear space X we define X^* to be the vector space of all continuous linear *functionals* on X, i.e. continuous linear maps $\Lambda : X \to \mathbb{C}$ (or into $\mathbb{R}$ if the scalar field is real). By Lemma 1 a linear functional is continuous on X if and only if it is continuous at the origin, or equivalently bounded. If we set

$$\|\Lambda\|^* = \sup_{\|x\| \leq 1} |\Lambda x| , \tag{3.2}$$

then it is easily verified that $\|\cdot\|^*$ is a norm on X^*, and since the scalar field is complete, so is the metric on X^* induced from $\|\cdot\|^*$. Thus X^* is a Banach space (even if X is not).

Remark 2. *Note that $\|\Lambda\|^*$ is the smallest nonnegative constant C which exhibits the boundedness of Λ on X in the inequality $|\Lambda x| \leq C \|x\|$.*

Now we specialize this definition to a Hilbert space H. An example of a continuous linear functional on H is the linear functional Λ_y associated with $y \in H$ given by

$$\Lambda_y x = \langle x, y \rangle , \qquad x \in H. \tag{3.3}$$

The boundedness of Λ_y follows from the Cauchy-Schwarz inequality $|\Lambda_y x| \leq \|y\| \|x\|$. In fact, this together with the choice $x = \frac{y}{\|y\|}$ in (3.2) yields $\|\Lambda_y\|^* = \|y\|$. It turns out that there are no other continuous linear functionals on H and this is the first major consequence of Theorem 4, and hence also of Theorem 3.

Theorem 5. *(Riesz representation) Let H be a Hilbert space. Every $\Lambda \in H^*$ is of the form Λ_y for some $y \in H$. Moreover, there is a conjugate linear isometry from H to H^* given by $y \to \Lambda_y$ where Λ_y is as in (3.3).*

Proof: We've already shown that $\Lambda_y \in H^*$ with $\|\Lambda_y\|^* = \|y\|$, and since $\Lambda_{\lambda y} = \overline{\lambda}\Lambda_y$ we have that the map $y \to \Lambda_y$ is a conjugate linear isometry from H into H^*. To see that this map is onto, take $\Lambda \neq 0$ in H^* and let $\mathcal{N} = \{x \in H : \Lambda x = 0\} = \Lambda^{-1}\{0\}$ be the null space of Λ. Since $\mathcal{N}$ is a proper closed subspace of H, Theorem 4 shows that $\mathcal{N}^\perp \neq \{0\}$. Take $z \neq 0$ in $\mathcal{N}^\perp$ and note that

$$(\Lambda x) z - (\Lambda z) x \in \mathcal{N} \text{ for all } x \in H.$$

Thus

$$0 = \langle (\Lambda x) z - (\Lambda z) x, z \rangle = (\Lambda x) \|z\|^2 - (\Lambda z) \langle x, z \rangle$$

yields

$$\Lambda x = \frac{(\Lambda z) \langle x, z \rangle}{\|z\|^2} = \left\langle x, \frac{\overline{\Lambda z}}{\|z\|^2} z \right\rangle = \Lambda_y x, \qquad x \in H,$$

with $y = \frac{\overline{\Lambda z}}{\|z\|^2} z$.

4. Weak solutions to the Dirichlet problem ([1])

The Riesz representation theorem turns out to be adequate for dealing with the existence of weak solutions to the Dirichlet problem

$$\left\{ \begin{array}{rcll} \triangle u - u & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. , \tag{4.1}$$

where Ω is an open set in $\mathbb{R}^n$, $f \in L^2(\Omega)$ and $\triangle$ is Laplace's operator $\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$. We first must define what is meant by a solution u to this problem. Given $f, g \in L^1_{loc}(\Omega)$ and $1 \leq j \leq n$, we say that g is the *weak j^{th} partial derivative* of f in Ω provided

$$\int_\Omega g\varphi dx = -\int_\Omega f\left(\frac{\partial \varphi}{\partial x_j}\right) dx, \qquad \text{for all } \varphi \in C^1_{com}(\Omega).$$

Here $C^1_{com}(\Omega)$ denotes the normed linear space of all continuously differentiable functions φ with compact support in Ω, and norm given by $\|\varphi\|_{C^1_{com}(\Omega)} = \sup_{x\in\Omega} |\varphi(x)| + \sup_{x\in\Omega} |\nabla \varphi(x)|$. When g is the weak j^{th} partial derivative of f in Ω we write $g = \frac{\partial f}{\partial x_j}$. This should cause no confusion since it is easily verified that this weak definition is an extension of the classical definition of partial derivative for a continously differentiable function f (use integration by parts).

Definition 4. *Let $W^{1,2}(\Omega)$ consist of those (complex-valued) functions $f \in L^2(\Omega)$ with $\nabla f \in L^2(\Omega)$ in the weak sense. Define an inner product on $W^{1,2}(\Omega)$ by*

$$\langle f, g \rangle = \int_\Omega f\overline{g} dx + \int_\Omega \nabla f \cdot \overline{\nabla g} dx. \tag{4.2}$$

Theorem 6. *$W^{1,2}(\Omega)$ is a Hilbert space with the inner product (4.2).*

Proof: We prove completeness. If $\{f_k\}_{k=1}^\infty$ is Cauchy in $W^{1,2}(\Omega)$, then $\{f_k\}_{k=1}^\infty$ and $\{\nabla f_k\}_{k=1}^\infty$ are Cauchy in $L^2(\Omega)$ and $\oplus^n L^2(\Omega)$. Thus there are $f, g_1, ..., g_n \in$

$L^2(\Omega)$ such that $f_k \to f$ and $\frac{\partial f_k}{\partial x_j} \to g_j$ in $L^2(\Omega)$ for $1 \le j \le n$. We must now show that $g_j = \frac{\partial f}{\partial x_j}$ in the weak sense. Letting $k \to \infty$ in the equation

$$\int_\Omega f_k \left(\frac{\partial \varphi}{\partial x_j}\right) dx = -\int_\Omega \frac{\partial f_k}{\partial x_j} \varphi dx, \qquad \varphi \in C^1_{com}(\Omega),$$

yields $\int_\Omega f \left(\frac{\partial \varphi}{\partial x_j}\right) dx = -\int_\Omega g_j \varphi dx$ for all $\varphi \in C^1_{com}(\Omega)$ as required.

We seek a solution $u \in W^{1,2}(\Omega)$ to (4.1) in the following sense. In order to capture the notion that a function u in $W^{1,2}(\Omega)$ vanishes on the boundary $\partial\Omega$ of Ω, we define $W^{1,2}_0(\Omega)$ to be the closure in $W^{1,2}(\Omega)$ of the space $C^1_{com}(\Omega)$. Clearly $W^{1,2}_0(\Omega)$ is a Hilbert space with the inner product (4.2). We now interpret the boundary condition in (4.1) as meaning that u should lie in the Hilbert space $W^{1,2}_0(\Omega)$. Next we interpret a solution u to the partial differential equation in (4.1) in the following weak sense:

$$(4.3) \qquad -\int_\Omega \sum_{j=1}^n \left(\partial_j \overline{\psi}\right)\left(\partial_j u\right) - \int_\Omega u\overline{\psi} = \int_\Omega f\overline{\psi}, \qquad \psi \in C^1_{com}(\Omega).$$

Note that if $u \in C^2(\Omega)$, $f \in C(\Omega)$, and $\triangle u(x) - u(x) = f(x)$ for all $x \in \Omega$, then integration by parts yields (4.3), so that this notion of weak solution is an extension of the classical notion. Moreover, equation (4.3) holds for all $\psi \in W^{1,2}_0(\Omega)$ by a simple density argument.

Let H be the Hilbert space $W^{1,2}_0(\Omega)$ and observe that the left side of (4.3) is $-\langle u, \psi\rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product for $W^{1,2}_0(\Omega)$ given in (4.2). Thus a weak solution to the Dirichlet problem (4.1) is simply an element $u \in H$ satisfying (after taking conjugates in (4.3))

$$(4.4) \qquad \langle \psi, u\rangle = -\int_\Omega \psi \overline{f}, \qquad \psi \in H.$$

However,

$$\|\psi\|^2_H = \|\psi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)} \ge \|\psi\|^2_{L^2(\Omega)}$$

implies that the linear functional Λ on H given by $\Lambda\psi = -\int_\Omega \psi\overline{f}$ for all $\psi \in H$ is bounded:

$$|\Lambda\psi| = \left|\int_\Omega \psi\overline{f}\right| \le \|f\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \|\psi\|_H .$$

The Riesz representation theorem (Theorem 5) now yields a unique element $u \in H$ such that $\Lambda\psi = \langle \psi, u\rangle$ for all $\psi \in H$. Thus u is a unique solution to (4.4), and hence also a unique weak solution to the Dirichlet problem (4.1).

Remark 3. *We can also consider real-valued solutions u to (4.1) when f is real-valued. If we consider the Hilbert space $W^{1,2}_0(\Omega)$ with scalar field $\mathbb{R}$ in place of $\mathbb{C}$, then the above arguments apply verbatim to show that there is a unique real-valued weak solution u to (4.1) whenever $f \in L^2(\Omega)$ is real-valued.*

However, to handle more general elliptic operators in divergence form we will need an extension of Theorem 5 due to Lax and Milgram, as well as a Fredholm alternative together with a maximum principle.

4.1. **Lax-Milgram theorem.**

Theorem 7. *(Lax-Milgram) Let H be a Hilbert space and suppose $B(x,y)$ is a sesquilinear form on $H \times H$ that is both bounded, i.e. $|B(x,y)| \leq C\|x\|\|y\|$, and coercive, i.e. $B(x,x) \geq \delta\|x\|^2$. Then for every $\Lambda \in H^*$ there is $y \in H$ such that*

$$\Lambda x = B(x,y), \qquad \text{for all } x \in H. \tag{4.5}$$

Moreover, the map that sends $\Lambda \in H^*$ to the unique $y \in H$ satisfying (4.5) is a bounded conjugate linear operator from H^* to H.

The case $B(x,y) = \langle x,y \rangle$ is the Riesz representation theorem.

Proof: Given $y \in H$, the boundedness of B shows that $B(\cdot,y) \in H^*$, and so the Riesz representation theorem shows that there is a unique element $Ty \in H$ such that $B(x,y) = \langle x, Ty \rangle$ for all $x \in H$. It is easy to see that T is a linear map from H to H that is in fact bounded since

$$\|Ty\| = \|\Lambda_{Ty}\|^* = \sup_{\|x\| \leq 1} |\Lambda_{Ty} x| = \sup_{\|x\| \leq 1} |B(x,y)| \leq C\|y\|.$$

From the fact that B is coercive we obtain

$$\delta\|y\|^2 \leq B(y,y) = \langle y, Ty \rangle \leq \|y\|\|Ty\|,$$

and altogether we have

$$\delta\|y\| \leq \|Ty\| \leq C\|y\|, \qquad y \in H. \tag{4.6}$$

Now (4.6) easily shows that T is one-to-one and that its range $\mathcal{R}_T$ is closed. It now follows that T maps H onto H since if not, then Theorem 4 shows that $(\mathcal{R}_T)^\perp \neq \{0\}$, and the existence of $z \neq 0$ in $(\mathcal{R}_T)^\perp$ contradicts the coercivity of B:

$$B(z,z) = \langle z, Tz \rangle = 0 \text{ since } Tz \in \mathcal{R}_T.$$

Thus T^{-1} exists and is a bounded linear map from H to H.

Now given $\Lambda \in H^*$, the Riesz representation theorem yields $w \in H$ such that $\Lambda = \Lambda_w$, and we have with $y = T^{-1}w$,

$$\Lambda x = \Lambda_w x = \langle x, w \rangle = \left\langle x, TT^{-1}w \right\rangle = B\left(x, T^{-1}w\right) = B(x,y), \qquad x \in H.$$

4.2. **The Fredholm alternative.** The second ingredient in our treatment of the Dirichlet problem is a Fredholm alternative for compact operators on the Sobolev space $W_0^{1,2}(\Omega)$. We begin with a discussion of compact operators on Banach spaces, and later specialize to Sobolev spaces.

A linear operator T mapping one Banach space X to another Y is said to be *compact* if TB is precompact in Y where B is the unit ball in X - precompact means the closure is compact. Thus if $\{x_n\}_{n=1}^\infty$ is any bounded sequence in X, the sequence $\{Tx_n\}_{n=1}^\infty$ has a convergent subsequence in Y.

Examples of compact operators include all bounded linear operators $T : X \to Y$ into a finite dimensional space Y, as well as bounded linear operators T with finite dimensional range $\mathcal{R}_T$:

Lemma 3. *If F is a finite dimensional subspace of a normed linear space Y, then F is closed in Y, and the restriction of the Y topology to F coincides with the topology induced by any linear isomorphism of Y with $\mathbb{C}^n$.*

Proof: Let $f:\mathbb{C}^n\to F$ be a linear isomorphism. Since $f(z_1,...,z_n)=z_1 f(\mathbf{e}_1)+...+z_n f(\mathbf{e}_n)$ and vector space operations are continuous in Y, it follows that f is continuous. Thus $f\left(\mathbb{S}^{n-1}\right)$ is compact and disjoint from 0, and there is $r>0$ such that $B(0,r)\cap f\left(\mathbb{S}^{n-1}\right)=\phi$ where $\mathbb{S}^{n-1}=\partial\mathbb{B}_n$ and $\mathbb{B}_n$ is the unit ball in $\mathbb{C}^n$. Now $0\in E=f^{-1}(B(0,r)\cap F)$ is convex, hence connected, and it follows that $E\subset\mathbb{B}_n$. From this we obtain that each component of $f^{-1}:F\to\mathbb{C}^n$ is a bounded linear functional on F (with norm at most $\frac{1}{r}$), and so f^{-1} is continuous by Lemma 1. This proves that the restriction of the Y topology to F coincides with the topology induced by the isomorphism f.

It remains to prove that F is closed in Y. Pick $y\in\overline{F}$. Then $y\in 2\frac{\|y\|}{r}B(0,r)=tB(0,r)$ and so

$$y\in\overline{F\cap tB(0,r)}=\overline{t(F\cap B(0,r))}\subset\overline{f(t\mathbb{B}_n)}\subset\overline{f\left(t\overline{\mathbb{B}_n}\right)}=f\left(t\overline{\mathbb{B}_n}\right)$$

since f continuous and $t\overline{\mathbb{B}_n}$ compact imply that $f\left(t\overline{\mathbb{B}_n}\right)$ is compact and hence closed in Y.

It can be shown that $T:X\to Y$ is compact if and (provided Y is a Hilbert space) only if there is a sequence of bounded linear operators $T_n:X\to Y$ with $\dim\mathcal{R}_{T_n}<\infty$ such that $T_n\to T$ in *operator norm*, i.e. $\|T-T_n\|\equiv\sup_{\|x\|_X\leq 1}\|(T-T_n)x\|_Y$ tends to 0 as $n\to\infty$, but we will not use this.

Theorem 8. *(Fredholm alternative) Suppose that* $T:H\to H$ *is a compact operator on a Hilbert space* H*. Then*

1. either *the equation* $(I-T)x=0$ *has a* nonzero *solution* $x\in H$,
2. or *the equation* $(I-T)x=y$ *has a* unique *solution* $x\in H$ *for each* $y\in H$. *In this case, the inverse linear operator* $(I-T)^{-1}$ *is bounded on* H.

We will defer the proof of the Fredholm alternative until the end of this section dealing with the Dirichlet problem.

Now we give an explicit example of a compact operator T on the Hilbert space $H=W^{1,2}(\Omega)$ with $\dim\mathcal{R}_T=\infty$, and which will be the key to solving the Dirichlet problem. For this we need the the Sobolev embedding theorem which shows that functions f in $W^{1,2}(\Omega)$ are better than square integrable, namely $f\in L^{2^*}(\Omega)$ for $\frac{1}{2^*}=\frac{1}{2}-\frac{1}{n}$, at least when $n\geq 3$ and Ω is a Lipschitz domain. The difficulties with the boundary disappear for the space $W_0^{1,2}(\Omega)$ and we have the following embedding.

Theorem 9. *(Sobolev embedding) Let* Ω *be an open subset of* $\mathbb{R}^n$*. If* $u\in W_0^{1,2}(\Omega)$*, then* $u\in L^{2^*}(\Omega)$ *where* $\frac{1}{2^*}=\frac{1}{2}-\frac{1}{n}$ *when* $n\geq 3$*, and for all* $2^*<\infty$ *when* $n=2$*. Moreover we have*

$$\|u\|_{L^{2^*}(\Omega)}\leq C_n\|\nabla u\|_{L^2(\Omega)},\qquad u\in W_0^{1,2}(\Omega).$$

Proof: For any $f\in W_0^{1,2}(\Omega)$, extended to vanish outside Ω, that satisfies the inequalities

$$|f(x)|\leq\int_{t_j=-\infty}^{\infty}|\partial_j f(\widehat{x}_j)|\,dt_j,\qquad 1\leq j\leq n,\tag{4.7}$$

where $\widehat{x}_j = (x_1, ..., x_{j-1}, t_j, x_{j+1}, ..., x_n)$ (for example $f \in C^1_{com}(\Omega)$), we have

$$|f(x)|^{\frac{n}{n-1}} \leq \left\{ \prod_{j=1}^{n} \int_{t_j=-\infty}^{\infty} |\partial_j f(\widehat{x}_j)| \, dt_j \right\}^{\frac{1}{n-1}} = \prod_{j=1}^{n} \left\{ \int_{t_j=-\infty}^{\infty} |\partial_j f(\widehat{x}_j)| \, dt_j \right\}^{\frac{1}{n-1}}.$$

Now integrate over x_1 in $\mathbb{R}$ and use Hölder's inequality

$$\|h_1 ... h_{n-1}\|_{L^1} \leq \prod_{j=1}^{n-1} \|h_j\|_{L^{n-1}} \tag{4.8}$$

to obtain that

$$\begin{aligned}
&\int_{x_1=-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}} \, dx_1 \\
\leq &\left\{ \int_{t_1=-\infty}^{\infty} |\partial_1 f(\widehat{x}_1)| \, dt_1 \right\}^{\frac{1}{n-1}} \int_{x_1=-\infty}^{\infty} \prod_{j=2}^{n} \left\{ \int_{t_j=-\infty}^{\infty} |\partial_j f(\widehat{x}_j)| \, dt_j \right\}^{\frac{1}{n-1}} dx_1 \\
\leq &\left\{ \int_{t_1=-\infty}^{\infty} |\partial_1 f(\widehat{x}_1)| \, dt_1 \right\}^{\frac{1}{n-1}} \prod_{j=2}^{n} \left\{ \int_{x_1=-\infty}^{\infty} \int_{t_j=-\infty}^{\infty} |\partial_j f(\widehat{x}_j)| \, dt_j dx_1 \right\}^{\frac{1}{n-1}}.
\end{aligned}$$

Integrating successively in this way over $x_2, ..., x_n$ and applying (4.8) after each integration leads to

$$\int_{\Omega} |f(x)|^{\frac{n}{n-1}} \, dx \leq \prod_{j=1}^{n} \left\{ \int_{\Omega} |\partial_j f(x)| \, dx \right\}^{\frac{1}{n-1}}.$$

Raising the inequality to the power $\frac{n-1}{n}$ and then applying the geometric/arithmetic mean inequality we get

$$\left(\int_{\Omega} |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \frac{1}{n} \int_{\Omega} \sum_{j=1}^{n} |\partial_j f| \leq \frac{1}{\sqrt{n}} \int_{\Omega} |\nabla f|.$$

In the case $n \geq 3$, we can replace f with $|u|^\gamma$ where $u \in C^1_{com}(\Omega)$ is real and $\gamma > 1$. Indeed, $|u|^\gamma$ is absolutely continuous in x_j and the pointwise derivative $\partial_j |u|^\gamma$ satisfies $|\partial_j |u|^\gamma| = \gamma |u|^{\gamma-1} |\partial_j u|$ a.e. since equality holds if either $u \neq 0$ or $\partial_j u = 0$. For this we note that the set $\{u = 0 \text{ and } \partial_j u \neq 0\}$ is an F_σ set, hence measurable, with at most countably many points on each line parallel to the j^{th} direction. Integration by parts shows that $\partial_j |u|^\gamma$ is the j^{th} weak partial derivative of $|u|^\gamma$, that (4.7) holds and that $|\nabla |u|^\gamma| = \gamma |u|^{\gamma-1} |\nabla u|$ a.e. With $\gamma = 2\frac{n-1}{n-2}$ we have

$$\left(\int_{\Omega} |u|^{2\frac{n}{n-2}} \right)^{\frac{n-1}{n}} \leq \frac{\gamma}{\sqrt{n}} \int_{\Omega} |u|^{\gamma-1} |\nabla u| \leq \frac{\gamma}{\sqrt{n}} \left(\int_{\Omega} |u|^{2(\gamma-1)} \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{1}{2}},$$

and since $2(\gamma - 1) = 2\frac{n}{n-2} = 2^*$, we conclude that

$$\left(\int_{\Omega} |u|^{2\frac{n}{n-2}} \right)^{\frac{n-1}{n}-\frac{1}{2}} \leq C_n \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{1}{2}}, \qquad u \in C^1_{com}(\Omega),$$

as required. The case $n = 2$ is similar and is left as an exercise.

Finally, to extend this inequality to arbitrary $u \in W_0^{1,2}(\Omega)$, choose a sequence $\{u_k\} \subset C^1_{com}(\Omega)$ that converges to u in $W^{1,2}(\Omega)$. Then we obtain that

$$\|u_m - u_k\|_{L^{2^*}(\Omega)} \leq C_n \|u_m - u_k\|_{W^{1,2}(\Omega)} \to 0 \text{ as } m, k \to \infty,$$

so that $\{u_k\}$ converges in $L^{2^*}(\Omega)$ to a function which must be u. Then

$$\|u\|_{L^{2^*}(\Omega)} \leq \liminf_{k\to\infty} \|u_k\|_{L^{2^*}(\Omega)} \leq C_n \liminf_{k\to\infty} \|\nabla u_k\|_{L^2(\Omega)} = C_n \|\nabla u\|_{L^2(\Omega)}.$$

When $q < 2^*$ and Ω is bounded, the Lebesgue space $L^q(\Omega)$ is strictly larger than $L^{2^*}(\Omega)$, and the natural embedding of $W_0^{1,2}(\Omega)$ into $L^q(\Omega)$ via the identity map turns out to be not only continuous, but also *compact*.

Theorem 10. *(compact embedding) If Ω is a bounded open subset of $\mathbb{R}^n$, then $W_0^{1,2}(\Omega)$ embeds compactly in $L^q(\Omega)$ for all $q < 2^* = 2\frac{n}{n-2}$.*

Proof: Recall that a set E in a metric space is compact if and only if every sequence has a convergent subsequence. We will also use a standard characterization of compactness in a *complete* metric space: E is compact if and only if E is closed and *totally bounded* (for every $\varepsilon > 0$, $E \subset \cup_{j=1}^N B_j$ for some *finite* collection of balls $\{B_j\}_{j=1}^N$ of radius ε). Also, an interpolation inequality for Lebesgue spaces (which follows from Hölder's inequality) together with the Sobolev embedding theorem above shows that

(4.9)

$$\|u\|_{L^q(\Omega)} \leq \|u\|^{\theta}_{L^1(\Omega)} \|u\|^{1-\theta}_{L^{2^*}(\Omega)} \leq \|u\|^{\theta}_{L^1(\Omega)} \left(C_n \|u\|_{W_0^{1,2}(\Omega)}\right)^{1-\theta}, \qquad \frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{2^*}.$$

Thus it is enough to prove the compactness of the embedding of $W_0^{1,2}(\Omega)$ into $L^1(\Omega)$. Indeed, if $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in $W_0^{1,2}(\Omega)$ that converges in $L^1(\Omega)$, then (4.9) shows that $\{u_k\}_{k=1}^{\infty}$ is Cauchy in $L^q(\Omega)$, hence convergent there as well.

So let A be a bounded set in $W_0^{1,2}(\Omega)$. We must show that $\overline{A}$ is compact in $L^1(\Omega)$, and for this we may assume without loss of generality that $A \subset C^1_{com}(\Omega)$ and $\|u\|_{W_0^{1,2}(\Omega)} \leq 1$ for all $u \in A$. Let ϕ be a smooth nonnegative function supported in the unit ball of $\mathbb{R}^n$ having integral 1, and set $\phi_h(x) = h^{-n}\phi\left(\frac{x}{h}\right)$ for $h > 0$. Then $u_h = u * \phi_h$ is smooth and the following elementary estimates hold:

$$\begin{aligned} |u_h(x)| &\leq \int_{|z|\leq h} |u(x-z)|\, \phi\left(\frac{z}{h}\right) h^{-n} dz \leq h^{-n} \|\phi\|_\infty \|u\|_{L^1(\Omega)}, \\ |\nabla u_h(x)| &\leq \int_{|z|\leq h} |u(x-z)| \left|h^{-1}\nabla\phi\left(\frac{z}{h}\right)\right| h^{-n} dz \leq h^{-n-1} \|\nabla\phi\|_\infty \|u\|_{L^1(\Omega)}. \end{aligned}$$

This shows that for every $h > 0$ the set $A_h = \{u_h : u \in A\}$ is a bounded equicontinuous subset of $C\left(\overline{\Omega}\right)$. By Arzela's theorem, A_h is precompact in $C\left(\overline{\Omega}\right)$, and thus precompact in $L^1(\Omega)$ since the embedding of $C\left(\overline{\Omega}\right)$ into $L^1(\Omega)$ is obviously continuous.

Next we observe that writing $z = |z|\,\omega$,

$$\begin{aligned} \|u-u_h\|_{L^1(\Omega)} &= \int_\Omega \left| u(x) - \int_{|z|\le 1} \phi(z)\, u(x-hz)\, dz \right| dx \\ &\le \int_\Omega \int_{|z|\le 1} \phi(z)\, |u(x) - u(x-hz)|\, dz dx \\ &\le \int_\Omega \int_{|z|\le 1} \phi(z) \int_0^{h|z|} \left| \frac{\partial}{\partial r} u(x - r\omega) \right| dr dz dx \\ &\le h \int_\Omega |\nabla u| \le h\, |\Omega|^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}} \le h\, |\Omega|^{\frac{1}{2}} . \end{aligned} \tag{4.10}$$

Since A_h is totally bounded in $L^1(\Omega)$ for all $h > 0$ (since it is precompact), (4.10) shows that A is totally bounded in $L^1(\Omega)$ as well, and thus precompact as required.

4.3. **Elliptic operators.** Now let us consider more generally an elliptic second order partial differential operator L in divergence form with bounded measurable coefficients. More precisely, let Ω be a bounded open subset of $\mathbb{R}^n$, and let $a^{ij}(x), d(x)$ be bounded measurable real-valued functions on Ω. Formally the operator L is given by

$$L = \sum_{i,j=1}^n \partial_i a^{ij}(x)\, \partial_j + d(x),$$

and L is said to be elliptic if its principal symbol (obtained by replacing the partial derivative ∂_k by the dual variable ξ_k in the highest order derivatives) is a positive definite quadratic form in the dual variable ξ, uniformly for $x \in \Omega$, i.e.

$$\lambda\, |\xi|^2 \le \sum_{i,j=1}^n a^{ij}(x)\, \xi_i \xi_j \le \Lambda\, |\xi|^2, \qquad \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega. \tag{4.11}$$

Note that since all variables are now real, we no longer use the conjugation symbol.

It does not make sense to apply the operator L to even a smooth function u in the pointwise sense since the derivatives $\partial_i a^{ij}(x)$ need not exist. However, the weak sense used above for the operator $L = \triangle - 1$ works equally well here. Accordingly, let $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ be the real-valued Hilbert spaces with inner product (4.2) (except that we no longer need the conjugation symbol in the second variable), and consider real-valued data $g, f^i \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$. We say that $u \in W^{1,2}(\Omega)$ is a weak solution to the Dirichlet problem (note that we have added additional inhomogeneous terms of the form $\partial_i f^i$)

$$\left\{ \begin{array}{ccc} Lu & = & g + \sum_{i=1}^n \partial_i f^i & \text{in } \Omega \\ u & = & \varphi & \text{on } \partial\Omega \end{array} \right. , \tag{4.12}$$

provided that $u - \varphi \in W_0^{1,2}(\Omega)$ and

$$-\int_\Omega \sum_{i,j=1}^n a^{ij} (\partial_i \psi)(\partial_j u)\, dx + \int_\Omega d u \psi dx = \int_\Omega \left\{ g\psi - \sum_{i=1}^n f^i \partial_i \psi \right\} dx, \tag{4.13}$$

for all $\psi \in W_0^{1,2}(\Omega)$. Note that the integrals in (4.13) are all absolutely convergent under the given hypotheses on a^{ij}, d, g, f^i, u and ψ. Moreover, a classical solution u is also a weak solution in this sense.

Theorem 11. *With Ω, L, g, f^i and φ as above, the Dirichlet problem (4.12) has a unique solution $u \in W^{1,2}(\Omega)$ provided $d(x) \leq 0$.*

The proof of this theorem is technically complicated compared to the case $L = \triangle - 1$ by the fact that the Riesz representation theorem is no longer directly applicable. Here is a step-by-step outline of the argument.

- First, we reduce to the case of zero boundary data, $\varphi = 0$.
- Second, we cast the integral formula (4.13) in the form of an equation $B(\psi, u) = \Lambda\psi$ where B is a bounded bilinear form on $H = W_0^{1,2}(\Omega)$, and $\Lambda \in H^*$.
- Third, we are done by the Lax-Milgram theorem if the form B is coercive.
- Fourth, we consider a related equation $Lu - \sigma u = f$ for $\sigma > 0$ so large that the related bilinear form is coercive. This related problem is then solved as above using the Lax-Milgram theorem, and then the compactness of the Sobolev embedding together with the Fredholm alternative show that the original problem (4.12) is solvable if and only if the corresponding homogeneous problem (when φ, g and f^i all vanish) has only the trivial solution $u = 0$.
- Finally, a maximum principle is proved that yields uniqueness of solutions to (4.12) when $d(x) \leq 0$, and thus that the homogeneous problem has only the trivial solution.

First, we reduce (4.12) to the case $\varphi = 0$ simply by letting $w = u - \varphi$. Then $u \in W^{1,2}(\Omega)$ solves (4.12) if and only if $w \in W_0^{1,2}(\Omega)$ solves

$$Lw = Lu - L\varphi = g - d\varphi + \sum_{i=1}^n \partial_i \left\{ f^i - \sum_{j=1}^n a^{ij}\partial_j\varphi \right\} = \widetilde{g} + \sum_{i=1}^n \partial_i \widetilde{f^i}$$

in Ω in the weak sense. Since $\widetilde{g}, \widetilde{f^i} \in L^2(\Omega)$, we see that it is enough to solve (4.12) when $\varphi = 0$.

Second, motivated by the right side of (4.13), we define a linear functional Λ on $H = W_0^{1,2}(\Omega)$ by

$$\Lambda\psi = -\int_\Omega \left\{ g\psi - \sum_{i=1}^n f^i \partial_i \psi \right\} dx.$$

Since $|\Lambda\psi| \leq \left\| \left(g, f^1, ..., f^n\right) \right\|_{L^2(\Omega)} \|\psi\|_H$, we have $\psi \in H^*$. Motivated by the left side of (4.13), we also define a bilinear form $B(u, \psi)$ on H by

$$B(u, \psi) = \int_\Omega \sum_{i,j=1}^n a^{ij} (\partial_i \psi)(\partial_j u)\, dx - \int_\Omega du\psi dx$$

The boundedness of a^{ij} and d show that B is bounded, and in the opposite direction we compute using (4.11) that

$$\begin{aligned} B(\psi, \psi) &= \int_\Omega \sum_{i,j=1}^n a^{ij} (\partial_i \psi)(\partial_j \psi)\, dx - \int_\Omega d\psi^2 dx \\ &\geq \lambda \int_\Omega |\nabla u|^2 - \int_\Omega d\psi^2 dx \\ &\geq \lambda \int_\Omega |\nabla u|^2 - \|d\|_{L^\infty(\Omega)} \int_\Omega \psi^2. \end{aligned} \tag{4.14}$$

The middle inequality here shows that if $d(x) \leq -c < 0$ for some positive constant c, then

$$\|\psi\|_H^2 = \int_\Omega \left(\psi^2 + |\nabla u|^2\right) \leq \frac{1}{\min\{\lambda, c\}} B(\psi, \psi),$$

which is the coercivity of B. Then the Lax-Milgram theorem yields a unique $u \in H$ such that $\Lambda\psi = B(\psi, u)$ for all $\psi \in H$, which is equivalent to (4.13), and hence solves (4.12).

In the event we do not have $d(x) \leq -c < 0$ for some positive constant c, we consider the operators L_σ given formally by

$$L_\sigma u = Lu - \sigma u = \sum_{i,j=1}^n \partial_i a^{ij} \partial_j u + (d - \sigma) u.$$

For $\sigma > \|d\|_{L^\infty(\Omega)}$, (4.14) shows that the related bilinear form $B_\sigma(u, \psi) = B(u, \psi) + \sigma \int_\Omega u\psi$ is coercive. Now define a map $I : H \to H^*$ by $(Iu)\psi = \int_\Omega u\psi$. We claim that I is a compact linear operator. Indeed, I is the composition of the compact embedding of H into $L^2(\Omega)$ (Theorem 10) followed by the continuous map from $L^2(\Omega)$ to H^* that sends $u \in L^2(\Omega)$ to the linear functional $\psi \to \int_\Omega u\psi$, $\psi \in H$.

Now fix $\sigma > \|d\|_{L^\infty(\Omega)}$ so that B_σ is bounded and coercive on H. Using the definitions of Λ and B we can write (4.13) as $B(\psi, u) = \Lambda\psi$ for all $\psi \in H$, which we abbreviate as the equation

$$Lu = -\Lambda$$

in the *weak sense*. We next note that this equation is equivalent to

$$L_\sigma u + \sigma I u = -\Lambda \tag{4.15}$$

in the weak sense. Since B_σ is bounded and coercive on H, the last part of the Lax-Milgram theorem (Theorem 7) shows that $L_\sigma^{-1} : H^* \to H$ is bounded, and so (4.15) is equivalent to the equation

$$u + \sigma L_\sigma^{-1} I u = -L_\sigma^{-1} \Lambda$$

in H. However, the operator $T = -\sigma L_\sigma^{-1} I$ is compact (since I is compact and L_σ^{-1} is continuous) and hence the Fredholm alternative (Theorem 8) shows that

- *either* $(I - T)u = 0$ has a nonzero solution $u \in H$,
- *or* $(I - T)u = w$ has a uniquely determined solution $u \in H$ for each $w \in H$.

It remains now to show that the first of the Fredholm alternatives fails. But

$$0 = (I - T)u = u + \sigma L_\sigma^{-1} I u$$

in H holds if and only if

$$0 = L_\sigma u + \sigma I u = Lu$$

in the weak sense, and the corollary to the maximum principle below shows that the only solution $u \in H$ to the equation $Lu = 0$ in the weak sense is the zero solution provided $d(x) \leq 0$. With this we will have completed the proof of Theorem 11.

4.3.1. *The maximum principle.* We have already captured the notion that a function $u \in W^{1,2}(\Omega)$ vanishes on the boundary of Ω by declaring that $u = 0$ on $\partial\Omega$ if and only if $u \in W_0^{1,2}(\Omega)$, the closure in $W^{1,2}(\Omega)$ of C^1 functions that have compact support in Ω. We now wish to extend this definition in the obvious way to give meaning to expressions such as $u \geq v$ on $\partial\Omega$ and $\sup_{\partial\Omega} u$. But first we need to know that $u_+ = \max\{u, 0\} \in W^{1,2}(\Omega)$ if $u \in W^{1,2}(\Omega)$. For this we use the following four lemmas.

Lemma 4. *If $u \in L^1_{loc}(\Omega)$, then $u * \phi_\varepsilon \to u$ in $L^1_{loc}(\Omega)$, i.e. $\int_K |u * \phi_\varepsilon - u| \to 0$ as $\varepsilon \to 0$ for every compact $K \subset \Omega$.*

Proof: Suppose first that $f \in C^0_{com}(\Omega)$. Then for $\delta > 0$,

$$\begin{aligned} \int |f * \phi_\delta - f| &= \int \left| \int \{f(x-y) - f(y)\} \phi_\delta(y)\, dy \right| dx \\ &\leq \sup_x \sup_{|y| \leq \delta} |f(x-y) - f(y)| \end{aligned}$$

tends to 0 as $\delta \to 0$ by uniform continuity of f. Now given $u \in L^1_{loc}(\Omega)$, K compact in Ω and $\varepsilon > 0$, choose $K \subset \Omega' \Subset \Omega$ and use that $C^0_{com}(\Omega')$ is dense in $L^1(\Omega')$ to find $f \in C^0_{com}(\Omega')$ such that $\int_{\Omega'} |u - f| < \frac{\varepsilon}{3}$. Then we also have for $\delta < dist(K, \partial\Omega')$,

$$\int_K |(u-f) * \phi_\delta| = \int_K \left| \int_{\Omega'} \phi_\delta(x-y) \{u(y) - f(y)\}\, dy \right| dx \leq \int_{\Omega'} |u(y) - f(y)|\, dy < \frac{\varepsilon}{3},$$

and so we conclude that

$$\begin{aligned} \int_K |u * \phi_\delta - u| &\leq \int_K |(u-f) * \phi_\delta| + \int |f * \phi_\delta - f| + \int_K |(f-u)| \\ &\leq \frac{\varepsilon}{3} + \sup_x \sup_{|y| \leq \delta} |f(x-y) - f(y)| + \frac{\varepsilon}{3} < \varepsilon \end{aligned}$$

for $\delta > 0$ sufficiently small.

Lemma 5. *Let $u, v \in L^1_{loc}(\Omega)$ and $1 \leq j \leq n$. Then $v = \frac{\partial u}{\partial x_j}$ if and only if there is a sequence of smooth functions $\{u_m\}$ converging to u in $L^1_{loc}(\Omega)$ whose derivatives $\frac{\partial u_m}{\partial x_j}$ converge to v in $L^1_{loc}(\Omega)$.*

Proof: We have $u * \phi_{\frac{1}{m}} \to u$ in $L^1_{loc}(\Omega)$ by the previous lemma. If $v = \frac{\partial u}{\partial x_j}$ then $v * \phi_{\frac{1}{m}} = u * \frac{\partial}{\partial x_j} \phi_{\frac{1}{m}}$ by definition, and a difference quotient argument shows that this is $\frac{\partial}{\partial x_j}\left(u * \phi_{\frac{1}{m}}\right)$, which then converges to v in $L^1_{loc}(\Omega)$. The converse is easy.

Lemma 6. *Let $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$ and suppose $u \in W^{1,2}(\Omega)$. Then $f \circ u \in W^{1,2}(\Omega)$ and $\nabla(f \circ u) = (f' \circ u) \nabla u$.*

Proof: By the previous lemma there is a sequence $\{u_m\}$ of smooth functions such that $u_m \to u$ and $\nabla u_m \to \nabla u$ in $L^1_{loc}(\Omega)$. Then for $\Omega' \Subset \Omega$,

$$\begin{aligned} \int_{\Omega'} |f(u_m) - f(u)| &\leq \|f'\|_\infty \int_{\Omega'} |u_m - u| \to 0 \qquad \text{as } m \to \infty; \\ \int_{\Omega'} |f'(u_m) \nabla u_m - f'(u) \nabla u| &\leq \|f'\|_\infty \int_{\Omega'} |\nabla u_m - \nabla u| + \int_{\Omega'} |f'(u_m) - f'(u)| |\nabla u| \end{aligned}$$

also tends to 0 as $m \to \infty$ upon applying the dominated convergence theorem to the last integral using the continuity of f' and assuming, as we may by passing to a subsequence, that $u_m \to u$ a.e. in Ω'. This shows that $f \circ u_m \to f \circ u$ and $\nabla (f \circ u_m) = (f' \circ u_m) \nabla u_m \to (f' \circ u) \nabla u$ in $L^1_{loc}(\Omega)$, and we're done.

Lemma 7. *Let* $u \in W^{1,2}(\Omega)$. *Then* $u_+ \in W^{1,2}(\Omega)$ *and* $\nabla u_+ = \begin{cases} \nabla u & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u \le 0 \end{cases}$.

Proof: For $\varepsilon > 0$ define $f_\varepsilon(u) = \begin{cases} \sqrt{u^2+\varepsilon^2} - \varepsilon & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u \le 0 \end{cases}$. Then by the previous lemma we have for any $\varphi \in C^1_{com}(\Omega)$,

$$\int_\Omega f_\varepsilon(u) \nabla \varphi = -\int_{\{u>0\}} \varphi \frac{u \nabla u}{\sqrt{u^2+\varepsilon^2}},$$

and letting $\varepsilon \to 0$ yields $\int_\Omega u_+ \nabla \varphi = -\int_{\{u>0\}} \varphi \nabla u$ as required.

Definition 5. *Let* $u, v \in W^{1,2}(\Omega)$. *We define* $u \le v$ *on* $\partial\Omega$ *if*

$$(u-v)_+ \in W^{1,2}_0(\Omega),$$

and define

$$\sup_{\partial\Omega} u = \inf \{\alpha \in \mathbb{R} : u \le \alpha \text{ on } \partial\Omega\}.$$

Theorem 12. *Suppose* Ω *and* L *are as above with* $d(x) \le 0$ *in* Ω. *If* $u \in W^{1,2}(\Omega)$ *satisfies* $Lu = 0$, *then*

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u|.$$

Proof: From $Lu = 0$ and $d(x) \le 0$ we obtain

$$\int_\Omega \sum_{i,j=1}^n a^{ij} (\partial_i v)(\partial_j u) = \int_\Omega duv \le 0$$

for all $v \in W^{1,2}_0(\Omega)$ such that $uv \ge 0$ in Ω. Now let $\beta = \sup_{\partial\Omega} u_+ \ge 0$ and set

$$v = \max\{u-\beta, 0\} = (u-\beta)_+ = (u_+ - \beta)_+.$$

By definition $u_+ - \beta \le 0$ on $\partial\Omega$ and so $v = (u_+ - \beta)_+ \in W^{1,2}_0(\Omega)$. We also have $uv \ge 0$ since $u > \beta \ge 0$ where $v > 0$, and it follows using (4.11) that

$$\lambda \int_\Omega |\nabla v|^2 \le \int_\Omega \sum_{i,j=1}^n a^{ij} (\partial_i v)(\partial_j v) = \int_\Omega \sum_{i,j=1}^n a^{ij} (\partial_i v)(\partial_j u) \le 0,$$

which implies that v is constant. Indeed, the above lemmas imply $\partial_j v = \partial_j u$ a.e. on the set where $v > 0$, and $\partial_j v = 0$ a.e. on the set where $v \le 0$. Moreover $\nabla (v * \phi_h)(x) = (\nabla v * \phi_h)(x) = 0$ if $dist(x, \partial\Omega) > h$. Thus $v * \phi_h(x)$ is constant for $dist(x, \partial\Omega) > h$ and since $v * \phi_h \to v$ in $L^1_{loc}(\Omega)$, v is constant in Ω.

Since $v \in W^{1,2}_0(\Omega)$, v is the zero constant which yields $u_+ \le \beta = \sup_{\partial\Omega} u_+$ in Ω. If we apply the same argument to $-u$ we get $(-u)_+ \le \sup_{\partial\Omega} (-u)_+$ in Ω, and combining these two inequalities with $|u| = u_+ + (-u)_+$ gives $\sup_\Omega |u| \le \sup_{\partial\Omega} |u|$.

Corollary 2. *Suppose* Ω *and* L *are as above with* $d(x) \le 0$ *in* Ω. *If* $u \in W^{1,2}_0(\Omega)$ *satisfies* $Lu = 0$, *then* $u = 0$ *in* Ω.

4.4. **Proof of the Fredholm alternative.** First we recast a slight strengthening of Theorem 8 in terms of the null space $\mathcal{N}_S$ and range $\mathcal{R}_S$ of a linear operator S.

Theorem 13. *(Fredholm alternative) Suppose that $T : H \to H$ is a compact operator on a Hilbert space H, and set $S = I - T$. Then $\mathcal{N}_S = \{0\}$* if and only if *$\mathcal{R}_S = H$. If $\mathcal{R}_S = H$, then S has a bounded linear inverse $S^{-1} : H \to H$.*

Proof: We give the proof in four steps.

First, if $\mathcal{N}_S = \{0\}$, there is a constant C such that

$$\|x\| \leq C \|Sx\|, \qquad x \in H. \tag{4.16}$$

Indeed, if not then there is a sequence $\{x_n\}_{n=1}^{\infty} \subset H$ with $\|Sx_n\| = 1$ and $\|x_n\| \nearrow \infty$. Then $z_n = \frac{x_n}{\|x_n\|}$ is in the unit ball of H and $\|Sz_n\| \searrow 0$. Since T is compact, there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that Tz_{n_k} converges in H, to say w. But then $z_{n_k} = Sz_{n_k} + Tz_{n_k}$ converges to $0 + w = w$ and since $Sw = \lim_{k\to\infty} Sz_{n_k} = 0$, the assumption $\mathcal{N}_S = \{0\}$ yields $w = 0$. This contradicts $\|z_{n_k}\| = 1$ for all k, and completes the proof of (4.16).

Second, still assuming $\mathcal{N}_S = \{0\}$, we obtain from (4.16) and the boundedness of S that $\|x\| \leq C \|Sx\| \leq C' \|x\|$ for all $x \in H$. This easily yields that $\mathcal{R}_S$ is closed, and moreover that S takes closed sets to closed sets.

Third, still assuming $\mathcal{N}_S = \{0\}$, we claim that $\mathcal{R}_S = H$. Let $V_k = S^k H$. Then V_k is closed by induction using the previous step, and $V_{k+1} \subset V_k$ for all k. We must have $V_k = V_{k+1}$ for some k since otherwise there is $y_k \in V_k \diagdown V_{k+1}$ with $\|y_k\| = 1$ and $y_k \perp V_{k+1}$. But then we have for $n > m$,

$$\|Ty_n - Ty_m\| = \|(Sy_m - Sy_n + y_n) - y_m\| \geq \|y_m\| = 1$$

by Lemma 2 since $Sy_m - Sy_n + y_n \in V_{m+1}$ and $y_m \perp V_{m+1}$. Thus $\{Ty_n\}_{n=1}^{\infty}$ has no convergent subsequence, contradicting T compact.

So $V_k = V_{k+1}$ for some k. Then for $y \in H$ we have $S^k y = S^{k+1} x$ for some $x \in H$. Thus $S^k (y - Sx) = 0$ implies $y = Sx$ upon iterating (4.16):

$$\begin{aligned} \|y - Sx\| &\leq C \|S(y - Sx)\| \leq C^2 \left\|S^2 (y - Sx)\right\| \\ &\leq ... \leq C^k \left\|S^k (y - Sx)\right\| = 0. \end{aligned}$$

This shows that S is onto, and then (4.16) shows that $S^{-1} : H \to H$ exists and is bounded.

Fourth, we claim that $\mathcal{R}_S = H$ implies $\mathcal{N}_S = \{0\}$. This time let $V_k = S^{-k}(\{0\})$. Then V_k is closed by the continuity of S, and $V_k \subset V_{k+1}$ for all k. An argument analogous to that above shows that there is m such that $V_n = V_m$ for all $n \geq m$. Given $y \in V_m$, an induction using $\mathcal{R}_S = H$ shows that $\mathcal{R}_{S^m} = H$, and so there is $x \in H$ such that $y = S^m x$. Thus $S^{2m} x = S^m y = 0$ by the definition of $y \in V_m$. So $x \in V_{2m} = V_m$ implies that $y = S^m x = 0$ as well. Thus $V_m = \{0\}$ and hence so also the smaller space $V_1 = S^{-1}(\{0\})$. This completes the proof that $\mathcal{N}_S = \{0\}$.

5. Completeness theorems ([2])

The uniform boundedness principle, the open mapping theorem and the closed graph theorem all depend on the following result of Baire.

Theorem 14. *If X is either **(1)** a complete metric space or **(2)** a locally compact Hausdorff space, then the intersection of countably many open dense subsets of X is dense in X.*

Proof: Let $\{V_k\}_{k=1}^{\infty}$ be a sequence of open dense subsets of X, and let B_0 be any nonempty open subset of X. Define sets B_n inductively by choosing B_n open and nonempty with $\overline{B_n} \subset V_n \cap B_{n-1}$ and in addition,

$$\begin{aligned} &diam\,(B_n) < \frac{1}{n} \text{ in case (1)}, \\ &\overline{B_n} \text{ is compact in case (2)}. \end{aligned}$$

Let $K = \cap_{n=1}^{\infty} \overline{B_n}$. Then in case (1), if we choose points $x_n \in B_n$, the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy and converges in K since K is closed. Thus $K \neq \phi$. In case (2), $K \neq \phi$ since the sets $\overline{B_n}$ are compact and decreasing, hence satisfy the finite intersection property. Thus in both cases $\phi \neq K \subset B_0 \cap (\cap_{k=1}^{\infty} V_k)$, and this shows that $\cap_{k=1}^{\infty} V_k$ is dense in X.

Remark 4. *A subset V of X is open and dense if and only if $X \diagdown V$ is closed with empty interior. Thus the conclusion of Baire's theorem can be restated as "every countable union of closed sets with empty interior in X has empty interior in X".*

Definition 6. *Let E be a subset of a topological space X. We say that E is* nowhere dense *if $\overline{E}$ has empty interior, that E is of the* first category *if it is a countable union of nowhere dense sets, and that E is of the* second category *if it is not of the first category.*

5.1. **The uniform boundedness principle.**

Theorem 15. *(Banach-Steinhaus uniform boundedness principle) Let X, Y be Banach spaces and Γ a set of bounded linear maps from X to Y. Let*

$$B = \left\{ x \in X : \sup_{\Lambda \in \Gamma} \|\Lambda x\|_Y < \infty \right\}.$$

If B is of the second category in X, then $B = X$ and Γ is equicontinuous, i.e.

$$\sup_{\Lambda \in \Gamma} \|\Lambda\| < \infty,$$

where $\|\Lambda\| = \sup_{\|x\| \leq 1} \|\Lambda x\|_Y$.

Proof: Let $E = \cap_{\Lambda \in \Gamma} \Lambda^{-1} \left(\overline{B_Y\left(0, \frac{1}{2}\right)} \right)$ where $B_Y(0, r)$ is the ball of radius r about the origin in Y. Then E is closed by the continuity of the maps Λ. If $x \in B$, then there is $n \in \mathbb{N}$ such that $\Lambda x \in n B_Y\left(0, \frac{1}{2}\right)$ for all $\Lambda \in \Gamma$. Thus $B = \cup_{n=1}^{\infty} nE$ and since B is of the second category in X, so is nE for some $n \in \mathbb{N}$. Since $x \to nx$ is a homeomorphism of X, we have that E is of the second category in X. Thus E has an interior point x and there is $r > 0$ so that $x - E \supset B_X(0, r)$. Then we conclude

$$\Lambda\left(B_X(0, r)\right) \subset \Lambda x - \Lambda E \subset \overline{B_Y\left(0, \frac{1}{2}\right)} - \overline{B_Y\left(0, \frac{1}{2}\right)} \subset \overline{B_Y(0, 1)},$$

which implies $\|\Lambda\| \leq \frac{1}{r}$ for all $\Lambda \in \Gamma$.

5.1.1. *Nonconvergence of Fourier series of continuous functions.* Recall that $\left\{e^{int}\right\}_{n \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$, i.e.

$$\left\langle e^{imt}, e^{int} \right\rangle = \int_0^{2\pi} e^{imt} \overline{e^{int}} \frac{dt}{2\pi} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}.$$

Given $\mathcal{U} = \{u_\alpha\}_{\alpha\in A}$ an orthonormal set in a Hilbert space H, and $x \in H$, define the *Fourier coefficients* of x (relative to $\mathcal{U}$) by

$$\widehat{x}(\alpha) = \langle x, u_\alpha \rangle, \qquad \alpha \in A.$$

Theorem 16. *Let $\mathcal{U} = \{u_\alpha\}_{\alpha\in A}$ be an orthonormal set in a Hilbert space H, and suppose $\{a_1, ..., \alpha_N\}$ is a finite subset of A. Then*

(1) *$x = \sum_{n=1}^N c_n u_{\alpha_n}$ implies that $c_n = \widehat{x}(\alpha_n)$ and $\|x\|^2 = \sum_{n=1}^N |\widehat{x}(\alpha_n)|^2$.*

(2) *$x \in H$ implies*

$$\left\| x - \sum_{n=1}^N \widehat{x}(\alpha_n) u_{\alpha_n} \right\| \le \left\| x - \sum_{n=1}^N \lambda_n u_{\alpha_n} \right\|$$

for all scalars $\lambda_1, ...\lambda_N$, and morever, equality holds if and only if $\lambda_n = \widehat{x}(\alpha_n)$ for $1 \le n \le N$. The vector $\sum_{n=1}^N \widehat{x}(\alpha_n) u_{\alpha_n}$ is the orthogonal projection of x onto the linear space spanned by $\{u_{\alpha_n}\}_{n=1}^N$.

Proof: Statement (1) is a straightforward computation using orthonormality, and (2) is equivalent, after squaring and expanding, to the inequality

$$\|x\|^2 - \sum_{n=1}^N |\widehat{x}(\alpha_n)|^2 \le \|x\|^2 - 2\,\mathrm{Re} \sum_{n=1}^N \widehat{x}(\alpha_n) \overline{\lambda_n} + \sum_{n=1}^N |\lambda_n|^2,$$

which in turn follows from $\left|\sum_{n=1}^N \widehat{x}(\alpha_n) \overline{\lambda_n}\right| \le \sqrt{\sum_{n=1}^N |\widehat{x}(\alpha_n)|^2} \sqrt{\sum_{n=1}^N |\lambda_n|^2}$.

Theorem 17. *(Bessel's inequality) If $\mathcal{U} = \{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in a Hilbert space H, then $\sum_{\alpha\in A} |\widehat{x}(\alpha)|^2 \le \|x\|^2$ for all $x \in H$.*

Theorem 18. *(Riesz-Fischer) If $\mathcal{U} = \{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in a Hilbert space H and $\varphi \in \ell^2(A)$, then there is $x \in H$ such that $\widehat{x} = \varphi$.*

Proof: There is $E = \{\alpha_n\}_{n=1}^\infty \subset A$ such that $\varphi(\alpha) = 0$ for $\alpha \in A \diagdown E$. Then $x_N = \sum_{n=1}^N \varphi(\alpha_n) u_{\alpha_n}$ is Cauchy in H, hence convergent to some $x \in H$, and continuity now yields $\widehat{x} = \varphi$.

Theorem 19. *Suppose $\mathcal{U} = \{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in a Hilbert space H. Equality holds in Bessel's inequality, i.e.*

$$\|x\| = \left\{ \sum_{\alpha\in A} |\widehat{x}(\alpha)|^2 \right\}^{\frac{1}{2}} = \|\widehat{x}\|_{\ell^2(A)}, \qquad x \in H,$$

if and only if

$$Span\mathcal{U} \equiv \left\{ \sum_{\alpha\in F} c_\alpha u_\alpha : c_\alpha \text{ scalar, } F \text{ a finite subset of } A \right\}$$

is dense in H.

Remark 5. *The orthornormal set $\mathcal{U} = \{e^{int}\}_{n\in\mathbb{Z}}$ is dense in $H = L^2(\mathbb{T})$ by the Stone-Weierstrass theorem, and thus the map $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ given by*

$$\mathcal{F}f(n) = \widehat{f}(n) = \langle f, e^{int} \rangle = \int_0^{2\pi} f(t) e^{-int} \frac{dt}{2\pi}, \qquad n \in \mathbb{Z},$$

is a Hilbert space isomorphism of $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$. Note that an inner product $\langle\cdot,\cdot\rangle$ on an inner product space X can always be recovered from its norm $\|\cdot\|$ by polarization:

$$4\operatorname{Re}\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2, \qquad x,y\in X.$$

Now consider the symmetric partial sums $S_n f$ of the Fourier series of $f\in L^2(\mathbb{T})$:

$$\begin{aligned}
S_n f(x) &= \sum_{k=-n}^{n}\widehat{f}(k)\,e^{ikx} = \sum_{k=-n}^{n}\int_0^{2\pi} f(t)\,e^{-ikt}\frac{dt}{2\pi}e^{ikx} \\
&= \int_0^{2\pi} f(t)\left\{\sum_{k=-n}^{n} e^{ik(x-t)}\right\}\frac{dt}{2\pi} \\
&= \int_0^{2\pi} f(t)\,\mathcal{D}_n(x-t)\,\frac{dt}{2\pi} = f*\mathcal{D}_n(x),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}_n(\theta) &= \sum_{k=-n}^{n} e^{ik\theta} = \frac{\left(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}\right)\sum_{k=-n}^{n} e^{ik\theta}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}} \\
&= \frac{e^{i\left(n+\frac{1}{2}\right)\theta}-e^{-i\left(n+\frac{1}{2}\right)\theta}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}} = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}
\end{aligned}$$

satisfies

$$\begin{aligned}
\int_0^{2\pi}\left|\mathcal{D}_n(\theta)\right|\frac{d\theta}{2\pi} &> 2\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)\theta\right|}{\left|\frac{\theta}{2}\right|}\frac{d\theta}{2\pi} \\
&= \frac{2}{\pi}\int_0^{\left(n+\frac{1}{2}\right)\pi}\left|\sin\theta\right|\frac{d\theta}{\theta} \\
&> \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}\left|\sin\theta\right|d\theta \\
&= \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k},
\end{aligned}$$

and so tends to ∞ as $n\to\infty$.

From the Hilbert space theory above, we obtain that $S_n f$ converges to f in $L^2(\mathbb{T})$ for all $f\in L^2(\mathbb{T})$:

$$\|S_n f - f\|^2 = \sum_{|k|>n}\left|\widehat{f}(k)\right|^2 \to 0 \text{ as } n\to\infty, \qquad f\in L^2(\mathbb{T}).$$

For $f\in C(\mathbb{T})$ we ask if we have pointwise convergence of $S_n f$ to f on $\mathbb{T}$. However, the property $\sup_{n\geq 1}\|\mathcal{D}_n\|_{L^1(\mathbb{T})}=\infty$ of the Dirichlet kernel $\mathcal{D}_n$, when combined with the uniform boundedness principle, implies that there are continuous functions $f\in C(\mathbb{T})$ whose Fourier series $\sum_{k=-\infty}^{\infty}\widehat{f}(k)\,e^{ikx}$ fail to converge at some points x in $\mathbb{T}$. In fact there is a dense G_δ subset Γ of $C(\mathbb{T})$ (a set is a G_δ subset of X if it is a countable intersection of open subsets of X) such that $\{x\in\mathbb{T} : S_n f(x) \text{ fails to converge at } x\}$ contains a dense G_δ subset of $\mathbb{T}$ for every $f\in\Gamma$.

To see this, set $\Lambda_n f = S_n f(0) = \int_0^{2\pi} f(t)\mathcal{D}_n(t)\frac{dt}{2\pi}$. Then $\Lambda_n \in C(\mathbb{T})^*$ and $\|\Lambda_n\|^* = \int_0^{2\pi} |\mathcal{D}_n(t)| \frac{dt}{2\pi} \nearrow \infty$ as $n \to \infty$. By the uniform boundedness principle we have

$$\sup_{n\geq 1} |\Lambda_n f| = \sup_{n\geq 1} |S_n f(0)| = \infty$$

for f in some dense G_δ subset of $C(\mathbb{T})$ since $\sup_{n\geq 1} |\Lambda_n f|$ is a lower semicontinuous function of f. In particular there is a continuous function f on $\mathbb{T}$ whose Fourier series fails to converge at 0.

Now choose $\{x_i\}_{i=1}^\infty$ dense in $\mathbb{T}$, and by applying the above argument with x_i in place of 0, choose E_i to be a dense G_δ subset of $C(\mathbb{T})$ such that

$$\sup_{n\geq 1} |S_n f(x_i)| = \infty, \qquad f \in E_i,\ i \geq 1.$$

By Baire's theorem, $E = \cap_{i=1}^n E_i$ is also a dense G_δ subset of $C(\mathbb{T})$. Thus for every $f \in E$ we have $\sup_{n\geq 1} |S_n f(x_i)| = \infty$ for all $i \geq 1$. Now we note that $\sup_{n\geq 1} |S_n f(x)|$ is a lower semicontinuous function of x (since it is a supremum of continuous functions), and thus the set

$$\left\{x \in \mathbb{T} : \sup_{n\geq 1} |S_n f(x)| = \infty\right\}$$

is a G_δ subset of $\mathbb{T}$ for every $f \in C(\mathbb{T})$. Combining these observations yields that there is a dense G_δ subset E of $C(\mathbb{T})$ such that for every $f \in E$, the set of x where the Fourier series of f fails to converge contains a dense G_δ subset of $\mathbb{T}$.

Remark 6. *In a complete metric space X without isolated points, every dense G_δ subset is uncountable. Indeed, if $E = \{x_k\}_{k=1}^\infty = \cap_{n=1}^\infty V_n$, V_n open, is a* countable *dense G_δ subset of X, then $W_n = V_n \diagdown \{x_k\}_{k=1}^n$ is still a dense open subset of X, but $\cap_{n=1}^\infty W_n = \phi$, contradicting Baire's theorem.*

Remark 7. *A famous theorem of L. Carleson shows that for* every $f \in L^2(\mathbb{T})$, $\lim_{n\to\infty} S_n f(x) = f(x)$ *for a.e.* $x \in \mathbb{T}$.

5.2. **The open mapping theorem.** A map $f : X \to Y$ where X, Y are topological spaces is *open* if $f(G)$ is open in Y for every G open in X. A famous "open mapping theorem" is that a holomorphic function f on a connected open subset Ω of the complex plane is open if it is not constant. If we consider continuous linear maps $\Lambda : X \to Y$ where X, Y are Banach spaces, then Λ is open if it is onto. Note that for a linear map $\Lambda : X \to Y$ from one normed linear space X to another Y, Λ is open if and only if $\Lambda(B_X(0,1)) \supset B_Y(0,r)$ for some $r > 0$.

Theorem 20. *(Open mapping theorem) Suppose X, Y are Banach spaces and $\Lambda : X \to Y$ is bounded and onto. Then Λ is an open map.*

Remark 8. *More generally, if $\Lambda : X \to Y$ is a bounded linear operator from a Banach space X to a normed linear space Y, and if ΛX is of the second category in Y, then Λ is open and onto Y, and Y is a Banach space. The proof is essentially the same as that given below.*

Proof: Since Λ is onto we have $Y = \cup_{k=1}^\infty \overline{\Lambda\left(kB_X\left(0,\frac{1}{4}\right)\right)}$, and thus by Baire's theorem, one of the sets $\overline{\Lambda\left(kB_X\left(0,\frac{1}{4}\right)\right)} = k\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}$ must have nonempty interior, and hence so must $\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}$, say

$$B_Y\left(y_0,r\right)\subset\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}.$$

Then we have

$$\begin{aligned}
\overline{\Lambda\left(B_X\left(0,\frac{1}{2}\right)\right)} &\supset \overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)-\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)} \\
&\supset \overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}-\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)} \\
&\supset B_Y\left(y_0,r\right)-B_Y\left(y_0,r\right) \\
&\supset B_Y\left(0,r\right).
\end{aligned} \tag{5.1}$$

It remains only to prove that $\overline{\Lambda\left(B_X\left(0,\frac{1}{2}\right)\right)}\subset\Lambda\left(B_X\left(0,1\right)\right)$. For this, fix $y_1\in\overline{\Lambda\left(B_X\left(0,\frac{1}{2}\right)\right)}$. Now the argument above shows that $\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}$ contains an open ball $B_Y\left(0,r_1\right)$ about the origin as well. There is $x_1\in B_X\left(0,\frac{1}{2}\right)$ such that $\Lambda x_1\in\Lambda\left(B_X\left(0,\frac{1}{2}\right)\right)$ satisfies $\left\|\Lambda x_1-y_1\right\|_Y<r_1$. Then we have

$$\Lambda x_1\in B_Y\left(y_1,r_1\right)\subset\left\{y_1-\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}\right\}.$$

Now define

$$y_2=y_1-\Lambda x_1\in\overline{\Lambda\left(B_X\left(0,\frac{1}{4}\right)\right)}.$$

We can repeat this procedure inductively to obtain sequences $\{x_n\}_{n=1}^\infty\subset X$ and $\{y_n\}_{n=1}^\infty\subset Y$ satisfying

$$\begin{aligned}
x_n &\in B_X\left(0,\frac{1}{2^n}\right), \\
y_n &\in \overline{\Lambda\left(B_X\left(0,\frac{1}{2^n}\right)\right)}, \\
y_{n+1} &= y_n-\Lambda x_n,
\end{aligned}$$

for all $n\geq 1$. Then $x=\lim_{m\to\infty}\sum_{n=1}^m x_n\in B_X\left(0,1\right)$ since $\|x\|\leq\sum_{n=1}^\infty\|x_n\|<\sum_{n=1}^\infty\frac{1}{2^n}=1$, and since $\|y_n\|\leq\|\Lambda\|\,2^{-n}$,

$$\Lambda x=\lim_{m\to\infty}\sum_{n=1}^m\Lambda x_n=\lim_{m\to\infty}\sum_{n=1}^m\left(y_n-y_{n+1}\right)=y_1-\lim_{m\to\infty}y_{m+1}=y_1.$$

5.2.1. *Fourier coefficients of integrable functions.* If $f\in L^1\left(\mathbb{T}\right)$, then $\left|\widehat{f}\left(n\right)\right|=\left|\int_0^{2\pi}f\left(t\right)e^{-int}\frac{dt}{2\pi}\right|\leq\|f\|_{L^1(\mathbb{T})}$ for all $n\in\mathbb{Z}$, i.e. $\mathcal{F}=\widehat{}$ is a bounded linear map from $L^1\left(\mathbb{T}\right)$ to $\ell^\infty\left(\mathbb{Z}\right)$ of norm 1 ($\widehat{1}=\delta_0$). More is true because of the density of trigonometric polynomials $\sum_{n=-N}^N c_ne^{inx}$ in $L^1\left(\mathbb{T}\right)$, namely the Riemann-Lebesgue lemma:

$$\lim_{n\to\infty}\left|\widehat{f}\left(n\right)\right|=0,\qquad f\in L^1\left(\mathbb{T}\right).$$

To prove this, simply let $\varepsilon > 0$ be given and choose $P(x) = \sum_{n=-N}^{N} c_n e^{inx}$ such $\|f - P\|_{L^1(\mathbb{T})} < \varepsilon$. Since $\widehat{P}(n) = 0$ for $|n| > N$, we have

$$\left|\widehat{f}(n)\right| = \left|\widehat{f - P}(n)\right| \leq \|f - P\|_{L^1(\mathbb{T})} < \varepsilon$$

for $|n| > N$. Thus $\mathcal{F} : L^1(\mathbb{T}) \to \ell_0^\infty(\mathbb{Z})$ with norm 1 where $\ell_0^\infty(\mathbb{Z})$ is the closed subspace of $\ell^\infty(\mathbb{Z})$ consisting of those sequences with limit zero at $\pm\infty$. The following application of the open mapping theorem shows that not every such sequence arises as the Fourier transform of an integrable function on $\mathbb{T}$.

Theorem 21. *The Fourier transform $\mathcal{F} : L^1(\mathbb{T}) \to \ell_0^\infty(\mathbb{Z})$ is bounded and one-to-one, but* not *onto.*

Proof: To see that $\mathcal{F}$ is one-to-one, suppose that $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then if $P(x) = \sum_{n=-N}^{N} c_n e^{inx}$ is a trigonometric polynomial,

$$\int_0^{2\pi} f(t) P(t) \, dt = \sum_{n=-N}^{N} c_n \int_0^{2\pi} f(t) e^{int} dt = 0, \tag{5.2}$$

and since trigonometric polynomials are dense in $C(\mathbb{T})$, we have

$$\int_0^{2\pi} f(t) g(t) \, dt = 0$$

for all $g \in C(\mathbb{T})$. Now let E be a measurable subset of $\mathbb{T}$. By Lusin's theorem there is a sequence of continuous functions $\{g_n\}_{n=1}^\infty$ such that $g_n = \chi_E$ except on a set of measure at most 2^{-n} and where $\|g_n\|_\infty = 1$ for all $n \geq 1$. Thus $g_n \to \chi_E$ almost everywhere on $\mathbb{T}$, and the dominated convergence theorem shows that

$$\int_E f(t) \, dt = 0.$$

With E equal $\{t : f(t) > 0\}$ and $\{t : f(t) < 0\}$, we see that $f = 0$ a.e.

Now we prove that $\mathcal{F}$ is not onto by contradiction. If $\mathcal{R}_\mathcal{F} = \ell_0^\infty(\mathbb{Z})$, then the open mapping theorem shows that there is $\delta > 0$ such that

$$\left\|\widehat{f}\right\|_{\ell_0^\infty(\mathbb{Z})} \geq \delta \|f\|_{L^1(\mathbb{T})}, \qquad f \in L^1(\mathbb{T}). \tag{5.3}$$

But (5.3) fails if we take $f = \mathcal{D}_n$ for n large, since $\left\|\widehat{f}\right\|_{\ell_0^\infty(\mathbb{Z})} = \left\|\chi_{\{-n,1-n,\dots,n-1,n\}}\right\|_{\ell_0^\infty(\mathbb{Z})} = 1$ while $\|\mathcal{D}_n\|_{L^1(\mathbb{T})} \nearrow \infty$.

5.3. **The closed graph theorem.** If X is any topological space and Y is a Hausdorff space, then every continuous map $f : X \to Y$ has a closed graph (exercise: prove this). A statement that gives conditions under which the converse holds is referred to as a "closed graph theorem". Here is an elementary example. Suppose that X and Y are metric spaces and Y is compact. If the graph of f is closed in $X \times Y$ then f is continuous. Indeed, for metric spaces it is enough to show that every sequence $\{x_n\}_{n=1}^\infty$ in X converging to a point $x \in X$ has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $f(x_{n_k}) \to f(x)$ as $k \to \infty$. However, since Y is compact, $\{f(x_n)\}_{n=1}^\infty$ has a convergent subsequence, say $f(x_{n_k}) \to y \in K$ as $k \to \infty$. Thus (x, y) is a limit point of the graph $G = \{(x, f(x)) : x \in X\}$, and since G is assumed closed, we have $(x, y) \in G$, i.e. $y = f(x)$. The next theorem gives the same conclusion for a *linear* map from one Banach space to another. Note that linearity is needed here since $f:\mathbb{R}\rightarrow\mathbb{R}$ by $f(x)=\begin{cases}\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ has a closed graph, but is not continuous at the origin.

Theorem 22. *(closed graph theorem) Suppose that X and Y are Banach spaces and $\Lambda:X\rightarrow Y$ is linear. If the graph $G=\{(x,\Lambda(x)):x\in X\}$ is closed in $X\times Y$, then Λ is continuous.*

Proof: The product $X\times Y$ is a Banach space with the norm $\|(x,y)\|=\|x\|_X+\|y\|_Y$. Since Λ is linear and the graph G of Λ is closed, G is also a Banach space. Now the projection $\pi_1:X\times Y\rightarrow X$ by $(x,y)\rightarrow x$ is a continuous linear map from the Banach space G *onto* the Banach space X, and the open mapping theorem thus implies that π_1 is an open map. However, π_1 is clearly one-to-one and so the inverse map $\pi_1^{-1}:X\rightarrow G$ exists and is continuous. But then the composition $\pi_2\circ\pi_1^{-1}:X\rightarrow Y$ is also continuous where $\pi_2:X\times Y\rightarrow Y$ by $(x,y)\rightarrow y$. We are done since $\pi_2\circ\pi_1^{-1}=\Lambda$.

As a consequence of the closed graph theorem, we obtain the automatic continuity of symmetric linear operators on a Hilbert space.

Theorem 23. *(Hellinger and Toeplitz) Suppose that T is a linear operator on a Hilbert space H satisfying $\langle Tx,y\rangle=\langle x,Ty\rangle$ for all $x,y\in H$. Then T is continuous.*

Proof: It is enough to show that T has a closed graph G. So let (x,z) be a limit point of G. Then there is a sequence $\{x_n\}_{n=1}^{\infty}\subset X$ such that $x_n\rightarrow x$ and $Tx_n\rightarrow z$. For every $y\in H$ the symmetry hypothesis now shows that

$$\langle T(x_n-x),y\rangle=\langle x_n-x,Ty\rangle\rightarrow 0$$

as $n\rightarrow\infty$. But we also have

$$\langle T(x_n-x),y\rangle=\langle Tx_n,y\rangle-\langle Tx,y\rangle\rightarrow\langle z,y\rangle-\langle Tx,y\rangle$$

as $n\rightarrow\infty$. Thus $\langle z-Tx,y\rangle=0$ for all $y\in H$ and so $z=Tx$, which shows that $(x,z)\in G$.

5.3.1. *Interpolating sequences.* The closed graph theorem finds many applications within the theory of interpolating sequences, typical of its use more widely in analysis. We describe one such example here. Given a finite subset $Z=\{z_j\}_{j=1}^{J}$ of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, and a sequence of data $\xi=\left\{\xi_j\right\}_{j=1}^{J}$ in $\mathbb{C}$, it is easy to see that there is a bounded holomorphic function f in the disk that interpolates the data, i.e. $f\in H^{\infty}(\mathbb{D})$ and $f(z_j)=\xi_j$ for $1\leq j\leq J$. Indeed, the polynomial $f(z)=\sum_{j=1}^{J}\xi_j\prod_{i\neq j}\frac{z-z_i}{z_j-z_i}$ is a solution (although typically with a much larger supremum norm than necessary). In the 1950's Buck raised the question of whether or not there exists an *infinite* subset $Z=\{z_j\}_{j=1}^{\infty}$ of $\mathbb{D}$ that is interpolating for $H^{\infty}(\mathbb{D})$, i.e. for every bounded sequence $\xi=\left\{\xi_j\right\}_{j=1}^{\infty}$ of complex numbers there is $f\in H^{\infty}(\mathbb{D})$ such that $f(z_j)=\xi_j$ for $1\leq j<\infty$. In 1958 Carleson gave an affirmative answer and moreover characterized all such interpolating sequences in the disk.

Implicit in Carleson's solution, and explicitly realized by Shapiro and Shields in 1961, is the equivalence of this problem with certain Hilbert space analogues which we now describe. Let $H^2(\mathbb{D})$ be the Hardy space consisting of all holomorphic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ in the unit disk with $\|f\|_{H^2(\mathbb{D})}=\sqrt{\sum_{n=0}^{\infty}|a_n|^2}<\infty$. The Hardy space $H^2(\mathbb{D})$ can be identified with the closed subspace $\mathcal{H}^2(\mathbb{T})$ of $L^2(\mathbb{T})$ given by

$$\mathcal{H}^2(\mathbb{T})=\left\{f\in L^2(\mathbb{T}):\widehat{f}(n)=0\text{ for }n<0\right\}.$$

Indeed, simply associate each $f(z)=\sum_{n=0}^{\infty}a_nz^n$ in the unit disk with the square integrable function f^* on the circle satisfying $\widehat{f^*}(n)=\begin{cases}a_n & \text{for} \quad n\geq 0\\ 0 & \text{for} \quad n<0\end{cases}$. The inner product on $H^2(\mathbb{D})$ is then defined by that on $\mathcal{H}^2(\mathbb{T})$ inherited from $L^2(\mathbb{T})$:

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{\mathbb{T}}f^*\left(e^{i\theta}\right)\overline{g^*\left(e^{i\theta}\right)}d\theta=\sum_{n=0}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)}=\sum_{n=0}^{\infty}a_n\overline{b_n}$$

if $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$. Cauchy's theorem applied to polynomials, and then followed by a limiting argument, shows that

$$f(z)=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f^*\left(e^{i\theta}\right)}{e^{i\theta}-z}d\left(e^{i\theta}\right)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{f^*\left(e^{i\theta}\right)}{1-e^{-i\theta}z}d\theta=\langle f,k_z\rangle$$

where the "reproducing kernel" $k_z\in H^2(\mathbb{D})$ satisfies $k_z(\zeta)=\frac{1}{1-\overline{z}\zeta}$ and $k_z^*\left(e^{i\theta}\right)=\frac{1}{1-\overline{z}e^{i\theta}}$. We note in passing that if $f(r_kz)$ is a rapidly converging sequence in $H^2(\mathbb{D})$ (where the radii r_k tend to 1 from below), then $f^*\left(e^{i\theta}\right)=\lim_{k\to\infty}f\left(r_ke^{i\theta}\right)$ for a.e. θ. Moreover, Fatou's theorem shows that we actually have $f^*\left(e^{i\theta}\right)=\lim_{r\to 1^-}f\left(re^{i\theta}\right)$ for a.e. θ, but we will not need either of these facts here.

The above computations show that for fixed $z\in\mathbb{D}$, the linear functional $f\to f(z)$ is continuous on $H^2(\mathbb{D})$ with norm $\|k_z\|=\sqrt{\langle k_z,k_z\rangle}=\sqrt{k_z(z)}=\left(1-|z|^2\right)^{-\frac{1}{2}}$. Thus the map $f\to\left\{f(z_j)\sqrt{1-|z_j|^2}\right\}_{j=1}^{\infty}$ is bounded with norm 1 from $H^2(\mathbb{D})$ to $\ell^\infty(Z)$. The relevant questions that Shapiro and Shields then asked were these. When is this map onto (respectively into) the smaller Hilbert space $\ell^2(Z)$? In other words, when is the restiction map $Rf=\{f(z_j)\}_{j=1}^{\infty}$ onto (respectively into) the weighted space

$$\ell^2(\mu)=\left\{\xi=\left\{\xi_j\right\}_{j=1}^{\infty}:\|\xi\|_{\ell^2(\mu)}=\sqrt{\sum_{j=1}^{\infty}\left|\xi_j\right|^2\mu_j}<\infty\right\}$$

with weight $\mu_j=1-|z_j|^2=\frac{1}{k_{z_j}(z_j)}$? We will see in a moment that these questions are equivalent to Buck's question above. But first we show how the closed graph theorem yields an unexpected control in the way we may interpolate the data $\xi\in\ell^2(\mu)$ when R maps $H^2(\mathbb{D})$ onto (but not necessarily into) $\ell^2(\mu)$.

Lemma 8. *Suppose that $\ell^2(\mu)\subset R\left(H^2(\mathbb{D})\right)$. Then there is a constant C such that for every $\xi\in\ell^2(\mu)$, there is $f\in H^2(\mathbb{D})$ satisfying*

(1) $Rf=\xi$,
(2) $\|f\|_{H^2(\mathbb{D})}\leq C\|\xi\|_{\ell^2(\mu)}$.

Proof: Let $M=\left\{f\in H^2(\mathbb{D}):f(z_j)=0,1\leq j<\infty\right\}$. Then M is closed since point evaluations are continuous, and we have $H^2(\mathbb{D})=M\oplus M^\perp$. Let $P^\perp$ denote projection of $H^2(\mathbb{D})$ onto $M^\perp$. Then if $\xi\in\ell^2(\mu)$ and $f_\xi\in H^2(\mathbb{D})$ satisfies $Rf_\xi=$

ξ, we have that $RP^{\perp}f_{\xi} = \xi$ with $P^{\perp}f_{\xi}$ the *unique* such element in $M^{\perp}$. Thus $\Lambda\xi = P^{\perp}f_{\xi}$ defines a linear map from $\ell^2(\mu)$ to $M^{\perp}$ with $R\Lambda\xi = \xi$. Clearly Λ has closed graph since if $\left(\xi^k, \Lambda\xi^k\right) \to (\xi, h)$ in $\ell^2(\mu) \times M^{\perp}$, i.e. $\xi^k \to \xi$ in $\ell^2(\mu)$ and $\Lambda\xi^k \to h$ in $M^{\perp}$, then both

$$\xi_j^k \to \xi_j \text{ and } \Lambda\xi^k(z_j) \to h(z_j)$$

as $k \to \infty$ for each fixed j. However $\Lambda\xi^k(z_j) = \left(RP^{\perp}f_{\xi^k}\right)_j = \xi_j^k$ then shows that $h(z_j) = \xi_j = \Lambda\xi(z_j)$ for all j and hence by uniqueness in $M^{\perp}$ that $\Lambda\xi = h$, i.e. (ξ, h) is in the graph of Λ. The closed graph theorem now implies that Λ is continuous and we may take $f = \Lambda\xi$ in the conclusion of the lemma.

An even easier argument using the closed graph theorem shows that if R maps $H^2(\mathbb{D})$ into $\ell^2(\mu)$, then it does so boundedly.

Before stating Carleson's theorem, we derive an interesting consequence of the interpolation control given by (2) in the lemma above.

Corollary 3. *If $\ell^2(\mu) \subset R\left(H^2(\mathbb{D})\right)$ then $Z = \{z_j\}_{j=1}^{\infty}$ is separated in the Poincaré metric: there is a positive constant c such that*

$$|z_i - z_j| \geq c \min\{1 - |z_i|, 1 - |z_j|\}, \qquad i \neq j.$$

Proof: Fix j and let ξ satisfy $\xi_i = \delta_j^i$. By the lemma there is $f \in H^2(\mathbb{D})$ such that $f(z_i) = \delta_j^i$ and $\|f\|_{H^2(\mathbb{D})} \leq C\|\xi\|_{\ell^2(\mu)} = C\sqrt{1 - |z_j|^2}$ with $C \geq 1$. Thus for $i \neq j$ and any complex scalar λ,

$$\begin{aligned}
1 &= \left(f(z_j) - \lambda f(z_i)\right)^2 = \left\langle f, k_{z_j} - \lambda k_{z_i}\right\rangle^2 \\
&\leq \|f\|_{H^2(\mathbb{D})}^2 \left\|k_{z_j} - \lambda k_{z_i}\right\|_{H^2(\mathbb{D})}^2 \\
&\leq C\left(1 - |z_j|^2\right)\left\{\left\|k_{z_j}\right\|_{H^2(\mathbb{D})}^2 + |\lambda|^2 \|k_{z_i}\|_{H^2(\mathbb{D})}^2 - 2\operatorname{Re}\left\langle k_{z_j}, \lambda k_{z_i}\right\rangle\right\} \\
&= C\left(\frac{1}{k_{z_j}(z_j)}\right)\left\{k_{z_j}(z_j) + |\lambda|^2 k_{z_i}(z_i) - 2\operatorname{Re}\lambda k_{z_i}(z_j)\right\} \\
&= C\left\{2 - 2\frac{|k_{z_i}(z_j)|}{\sqrt{k_{z_i}(z_i)}\sqrt{k_{z_j}(z_j)}}\right\}
\end{aligned}$$

if we choose $\lambda = \frac{|k_{z_i}(z_j)|\sqrt{k_{z_j}(z_j)}}{k_{z_i}(z_j)\sqrt{k_{z_i}(z_i)}}$. Thus we have

$$1 - \left|\frac{z_i - z_j}{1 - \overline{z_j}z_i}\right|^2 = \frac{\left(1 - |z_i|^2\right)\left(1 - |z_j|^2\right)}{|1 - \overline{z_j}z_i|^2} = \frac{|k_{z_i}(z_j)|^2}{k_{z_i}(z_i)\,k_{z_j}(z_j)} \leq \left(1 - \frac{1}{2C}\right)^2$$

and hence $\left|\frac{z_i - z_j}{1 - \overline{z_j}z_i}\right| \geq c > 0$ for all $i \neq j$, which easily yields the corollary.

We note that if R maps $H^{\infty}(\mathbb{D})$ onto $\ell^{\infty}(Z)$, then the open mapping theorem easily shows that Z is separated. Indeed, there is $c > 0$ such that $R(\text{unit ball in } H^{\infty}(\mathbb{D})) \supset$ $c\times$unit ball in $\ell^{\infty}(Z)$. Thus for fixed $i \neq j$ there is $f \in H^{\infty}(\mathbb{D})$ of norm at most one with $f(z_i) = c$ and $f(z_j) = 0$. If $\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}$ is the idempotent automorphism of the disk $\mathbb{D}$ that interchanges 0 and w, then $g = f \circ \varphi_{z_j}$ maps $\mathbb{D}$ to $\mathbb{D}$ and

$g(0) = f(z_j) = 0$. Thus the Schwarz lemma implies that (recall $\varphi_{z_j} \circ \varphi_{z_j}$ is the identity)

$$c = |f(z_i)| = \left| g\left(\varphi_{z_j}(z_i)\right)\right| \leq \left|\varphi_{z_j}(z_i)\right| = \left|\frac{z_j - z_i}{1 - \overline{z_j} z_i}\right|,$$

which shows as above that Z is separated. Thus separation is a necessary condition for either $\ell^2(\mu) \subset R\left(H^2(\mathbb{D})\right)$ or $\ell^\infty(Z) \subset R(H^\infty(\mathbb{D}))$, and we may as well assume it from the outset. Now we can state Carleson's theorem with contributions from Shapiro and Shields.

Theorem 24. *Suppose that $Z = \{z_j\}_{j=1}^\infty \subset \mathbb{D}$ is separated and let R be the restriction map $Rf = \{f(z_j)\}_{j=1}^\infty$. Then the following conditions are equivalent:*

1. *R maps $H^\infty(\mathbb{D})$ onto $\ell^\infty(Z)$,*
2. *R maps $H^2(\mathbb{D})$ onto $\ell^2(\mu)$,*
3. *R maps $H^2(\mathbb{D})$ into $\ell^2(\mu)$,*
4. *$\mu(T(I)) \leq C|I|$ for all arcs $I \subset \mathbb{T}$.*

Here the tent $T(I)$ over the arc I is the convex hull of I and $z_I = re^{i\theta}$ where $e^{i\theta}$ is the midpoint of I and $r = 1 - \frac{|I|}{2\pi}$.

Marshall and Sundberg identified the crucial interplay between the spaces $H^\infty(\mathbb{D})$ and $H^2(\mathbb{D})$ here, namely that $H^\infty(\mathbb{D})$ is the multiplier algebra $M_{H^2(\mathbb{D})}$ of the Hilbert space $H^2(\mathbb{D})$. They then replaced the Hardy space with the classical Dirichlet space

$$\mathcal{D}(\mathbb{D}) = \left\{ f \text{ holomorphic in } \mathbb{D} : \|f\|_{\mathcal{D}(\mathbb{D})} = \sqrt{|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dz} < \infty \right\}$$

and obtained a theorem analogous to that above but with a stronger separation condition, $\mathcal{D}(\mathbb{D})$ in place of $H^2(\mathbb{D})$, $M_{\mathcal{D}(\mathbb{D})}$ in place of $H^\infty(\mathbb{D})$, a larger measure $\widetilde{\mu}$ in place of μ where $\widetilde{\mu}_j = \frac{1}{\widetilde{k_{z_j}}(z_j)}$ and $\widetilde{k_w}(z) = \frac{1}{\overline{w}z} \log \frac{1}{1 - \overline{w}z}$ is the reproducing kernel for $\mathcal{D}(\mathbb{D})$, a more complicated capacity condition in place of (4), and finally *without* condition (2). In fact, as observed by Bishop, even when the measure $\widetilde{\mu}$ is finite, R maps $\mathcal{D}(\mathbb{D})$ onto $\ell^2(\widetilde{\mu})$ for sequences Z more general than those for which R maps $\mathcal{D}(\mathbb{D})$ into $\ell^2(\widetilde{\mu})$. Recently Arcozzi, Rochberg and Sawyer have characterized (at least when the measure $\widetilde{\mu}$ is finite) the sequences Z for which R maps $\mathcal{D}(\mathbb{D})$ onto $\ell^2(\widetilde{\mu})$ in terms of a capacity condition on the Bergman tree $\mathcal{T}$, the standard grid of points in the disk uniformly separated in the Poincaré metric and endowed with the obvious tree structure. Many open problems remain in connection with interpolating sequences.

6. Convexity theorems ([2] and [4])

If M is a closed subspace of a Hilbert space H and Λ is a bounded linear functional on M, then we can always extend Λ to a bounded linear functional Λ_{ext} on all of H with the same norm. Indeed, Theorem 4 shows that $H = M \oplus M^\perp$, and so we can simply extend Λ by the formula $\Lambda_{ext}\left(m, m^\perp\right) = \Lambda m$. The Hahn-Banach theorem shows that we can do the same for *any* subspace M (not necessarily closed) in any normed linear space. However, this requires the *axiom of choice* and opens the door to paradoxical constructions such as the Banach-Tarski paradox: the open unit ball in $\mathbb{R}^3$ can be decomposed into 5 pieces, which can then be rearranged by rigid motions (although probably not by continuous disjoint motions - but this is an open problem) into two disjoint copies of the unit ball. Of course these pieces cannot be Lebesgue measurable, and the construction depends crucially on the existence of a free subgroup of rotations in $\mathbb{R}^3$ of rank 2. Before constructing the Banach-Tarski paradox, we will use the Hahn-Banach and Banach-Alaoglu theorems to construct nontrivial finitely additive positive measures on the full power set of *abelian* groups (this also uses the axiom of choice!), thus proving the impossibility of such paradoxical constructions using only translations.

6.1. **The Hahn-Banach theorem.**

Theorem 25. *(Hahn-Banach) Let M be a proper subspace of a normed linear space X. If $f \in M^*$, then there is $F \in X^*$ such that $F\mid_M = f$ and $\|F\|_{X^*} = \|f\|_{M^*}$.*

Proof: Suppose first that the scalar field is $\mathbb{R}$. Without loss of generality $\|f\|_{M^*} = 1$. Choose $x_0 \in X \diagdown M$ and let

$$M_0 = Span\{M, x_0\} = \{x + \lambda x_0 : x \in M, \lambda \in \mathbb{R}\}.$$

Define f_0 on M_0 by $f_0(x + \lambda x_0) = f(x) + \lambda\alpha$ where $\alpha \in \mathbb{R}$ will be chosen below so that f_0 is a norm-preserving extension of f to M_0. Since f_0 is a linear extension of f for any $\alpha \in \mathbb{R}$, we need only require that $|f(x) + \lambda\alpha| \leq \|x + \lambda x_0\|$ for $x \in M, \lambda \in \mathbb{R}$, and if we write $y = \frac{x}{\lambda}$, this is equivalent to

$$-\|y + x_0\| \leq f(y) + \alpha \leq \|y + x_0\|, \qquad y \in M,$$

and with $z = -y$, to

$$f(z) - \|z - x_0\| \leq \alpha \leq f(z) + \|z - x_0\|, \qquad z \in M. \tag{6.1}$$

However, the supremum of the left side and the infimum of the right side satisfy

$$\sup_{z\in M} f(z) - \|z - x_0\| \leq \inf_{w\in M} f(w) + \|w - x_0\|$$

since

$$f(z) - f(w) = f(z - w) \leq \|z - w\| \leq \|z - x_0\| + \|w - x_0\|.$$

This shows that there exists $\alpha \in \mathbb{R}$ such that (6.1) holds, and completes the proof that f_0 is a norm-preserving extension of f to M_0.

Now let

$$\mathcal{P} = \left\{(M', f') : M \subset M' \subset X, f' \in (M')^*, f'\mid_M = f, \|f'\|_{(M')^*} = 1\right\},$$

where M' denotes a subspace of X. Define a partial order on $\mathcal{P}$ by declaring $(M', f') \preceq (M'', f'')$ if $M' \subset M''$ and $f''\mid_{M'} = f'$. The Hausdorff maximality theorem implies the existence of a maximal totally ordered subset Ω of $\mathcal{P}$. We now define

$$\widetilde{M} = \cup\left\{M' : (M', f') \in \Omega \text{ for some } f' \in (M')^*\right\}.$$

Clearly $\widetilde{M}$ is a subspace of X and if we define $\widetilde{f}$ on $\widetilde{M}$ by

$$\widetilde{f}(x) = f'(x) \text{ if } x \in M' \text{ and } (M', f') \in \Omega,$$

then $\widetilde{f}$ is a well-defined linear functional on $\widetilde{M}$ with norm 1, and in fact $\left(\widetilde{M}, \widetilde{f}\right) \in \Omega$. Now we must have $\widetilde{M} = X$ since otherwise we could adjoin a point in $X \diagdown \widetilde{M}$ and extend $\widetilde{f}$ by the argument above, contradicting the maximality of Ω. This completes the proof when the scalar field is $\mathbb{R}$.

Now suppose the scalar field is $\mathbb{C}$. Set $u = \operatorname{Re} f$. We have just proved that there is a real-linear $U \in X^*$ such that $U\mid_M = u$ and $\|U\|^* = 1$. However, there is a unique complex-linear functional Λ whose real part is U:

$$\Lambda x = Ux - iU(ix), \qquad x \in X.$$

Then $\Lambda\mid_M = f$ and

$$|\Lambda x| = \Lambda\left(\frac{|\Lambda x|}{\Lambda x}x\right) = U\left(\frac{|\Lambda x|}{\Lambda x}x\right) \leq \left\|\frac{|\Lambda x|}{\Lambda x}x\right\| = \|x\|, \qquad x \in X.$$

6.1.1. *Weak topologies.* Suppose X is a normed linear space. Then X^* separates points on X. Indeed, given $x_0 \neq 0$, let $M = Span\{x_0\}$ and define f on M by $f(\lambda x_0) = \lambda\|x_0\|$. The Hahn-Banach theorem supplies a bounded linear functional $\Lambda \in X^*$ of norm 1 such that $\Lambda\mid_M = f$. If $x_1 \neq x_2$, then with $x_0 = x_1 - x_2$ we have

$$\Lambda x_1 - \Lambda x_2 = \Lambda(x_1 - x_2) = \Lambda x_0 = f(x_0) = \|x_0\| = \|x_1 - x_2\| \neq 0.$$

The Hahn-Banach theorem actually yields an even stronger separation theorem. Namely, if M is a closed subspace of X and $x_0 \notin M$, then there is $\Lambda \in X^*$ with $\Lambda x_0 = 1$ and $\Lambda\mid_M = 0$. For this take $f(x + \lambda x_0) = \lambda$ for $x \in M$ and λ a scalar, and use $dist(x_0, M) = \delta > 0$ to obtain

$$|f(x + \lambda x_0)| = |\lambda| \leq |\lambda|\frac{\left\|\lambda^{-1}x + x_0\right\|}{\delta} = \frac{1}{\delta}\|x + \lambda x_0\|.$$

Now let Λ be a norm preserving extension of f to X.

Given a normed linear space X we define X_w to be the vector space X topologized by the *weak topology* τ_w, which is the weakest topology on X such that all the maps in X^* are continuous: more precisely the weak topology consists of all unions of finite intersections $\Lambda_1^{-1}(G_1) \cap ... \cap \Lambda_n^{-1}(G_n)$ where G_k is an open subset of the scalar field and $\Lambda_k \in X^*$. A local base is given by all unions of the sets $\Lambda_1^{-1}(B) \cap ... \cap \Lambda_n^{-1}(B)$ where B is the unit ball in the scalar field and $\Lambda_k \in X^*$.

Theorem 26. *Let X' be a separating vector space of linear functionals on the vector space X. Let τ' be the weak topology on X induced by X'. Then $X_{\tau'}$ is a locally convex topological vector space such that $X^*_{\tau'} = X'$.*

This theorem applies in particular to the vector space X^*, with X assuming the role of the separating vector space of linear functionals on the vector space X^*. Here X acts linearly on X^* by the formula

$$x(\Lambda) = \Lambda x, \qquad x \in X, \Lambda \in X^*.$$

The X topology of X^* is called the weak* topology of X^* and is denoted τ_{w^*}.

Theorem 27. *(Banach-Alaoglu) Let V be a neighbourhood of 0 in a normed linear space X and let K be the polar set of V:*

$$K = \{\Lambda \in X^* : |\Lambda x| \leq 1 \text{ for all } x \in V\}.$$

Then K is weak* compact.

Remark 9. *The above theorem actually holds in a topological vector space X with the same proof. Of course for a normed linear space X it suffices to consider only the case V is the unit ball in X.*

Proof: For every $x \in X$ choose $\gamma(x) > 0$ such that $x \in \gamma(x) V$ so that $|\Lambda x| \leq \gamma(x)$ for all $\Lambda \in K$. Let $D_x = \overline{B(0, \gamma(x))}$, the closed ball about the origin of radius $\gamma(x)$ in the scalar field. Let τ_P be the product topology on the product space $P = \prod_{x \in X} D_x$. Tychonoff's theorem implies that P is compact. Note that the elements of P are the (arbitrary) functions f on X such that

$$|f(x)| \leq \gamma(x), \qquad x \in X.$$

Now $K \subset X^* \cap P$ and we claim that

(1) The restrictions of the topologies τ_{w^*} and τ_P to K coincide.
(2) K is a closed subset of P.

With (1) and (2) proved, we immediately obtain that K is compact in the topology τ_{w^*} as required. To see (1) simply consider the following two sets for a given $\Lambda_0 \in K$, $\{x_i\}_{i=1}^n \subset X$ and $\delta > 0$:

$$\begin{aligned} W_1 &= \{\Lambda \in X^* : |\Lambda x_i - \Lambda_0 x_i| < \delta, 1 \leq i \leq n\}, \\ W_2 &= \{f \in P : |f(x_i) - \Lambda_0 x_i| < \delta, 1 \leq i \leq n\}. \end{aligned}$$

As $\{x_i\}_{i=1}^n$ ranges over all finite subsets of X and δ ranges over all positive real numbers,

- the sets W_1 form a local base for the topology τ_{w^*} at Λ_0,
- the sets W_2 form a local base for the topology τ_P at Λ_0.

Since $K \subset X^* \cap P$, we have $W_1 \cap K = W_2 \cap K$ and (1) is proved.

To see (2), suppose that f_0 is in the τ_P closure of K in P. Then we have that f_0 is linear. Indeed, simply approximate f_0 by $f \in K$ at the points $x, y, \alpha x + \beta y$ and note that the linearity of f yields $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. We also have that $|f_0(x)| \leq 1$ for $x \in V$ by again approximating f_0 by $f \in K$ at x and then using $|f(x)| \leq 1$. Thus $f_0 \in K$ and K is τ_P closed.

6.2. **Paradoxical decompositions and finitely additive measures.**

Definition 7. *Let G be a group acting on a set X. A subset E of X is* finitely G-paradoxical *if there are subsets A_i, B_j of X and group elements g_i, h_j such that*

$$\begin{aligned} E &\supset \left(\dot{\cup}_{i=1}^m A_i\right) \dot{\cup} \left(\dot{\cup}_{j=1}^n B_j\right), \\ E &= \cup_{i=1}^m g_i A_i = \cup_{j=1}^n h_j B_j. \end{aligned} \tag{6.2}$$

The notation $\dot{\cup}$ asserts that the indicated union is pairwise disjoint. Note that one can easily arrange to have each collection of sets $\{g_i A_i\}_{i=1}^m$ and $\{h_j B_j\}_{j=1}^n$ in the second line of (6.2) pairwise disjoint simply by paring the sets A_i and B_j. One can also achieve equality in the first line of (6.2), but this is harder, and is not proved until Corollary 6 below. We say that E is *countably G-paradoxical* if m, n in (6.2) are permitted to be ∞, the first infinite ordinal. By G-paradoxical we mean finitely G-paradoxical. Finally, we say that G is paradoxical if G acts on itself by left multiplication and G is G-paradoxical. The next result uses the axiom of choice.

Theorem 28. *Let G be the circle group $\mathbb{T}$ and let it act on itself $X = \mathbb{T}$ by group multiplication:*

$$e^{it} \in G \text{ sends the point } e^{ix} \in X \text{ to the point } e^{i(t+x)} \in X.$$

Then X is countably G-paradoxical.

Proof: Let M be a choice set for the equivalence classes of the relation on $\mathbb{T}$ given by declaring two points equivalent if one is obtained from the other by rotation through a rational multiple of 2π radians. Let $\{\rho_i\}_{i=1}^{\infty}$ enumerate the rotations through a rational multiple of 2π radians, and set $M_i = \rho_i M$. Then the countable paradoxical decomposition is provided by

$$\begin{aligned} X &= \left(\dot{\cup}_{i \text{ odd}} M_i\right) \dot{\cup} \left(\dot{\cup}_{i \text{ even}} M_i\right), \\ X &= \dot{\cup}_{i \text{ odd}} g_i M_i = \dot{\cup}_{i \text{ even}} h_i M_i, \end{aligned}$$

where $g_i = \rho_{\frac{i+1}{2}} \rho_i^{-1}$ for i odd, and $h_i = \rho_{\frac{i}{2}} \rho_i^{-1}$ for i even.

Corollary 4. *There is a non-Lebesgue measurable subset of* $\mathbb{T}$*.*

Proof: If A_i, B_j, g_i, h_j witness a countable paradoxical decomposition (6.2) of $\mathbb{T} = E$ with $m, n \leq \infty$, and if we assume every subset of $\mathbb{T}$ is Lebesgue measurable, then

$$\begin{aligned} 2\pi &= |G| \geq \sum_{i=1}^{m} |A_i| + \sum_{j=1}^{n} |B_j| = \sum_{i=1}^{m} |g_i A_i| + \sum_{j=1}^{n} |h_j B_j| \\ &\geq \left|\cup_{i=1}^{m} g_i A_i\right| + \left|\cup_{j=1}^{n} h_j B_j\right| = 4\pi, \end{aligned}$$

a contradiction.

Remark 10. *There exists a G_2-paradoxical subset E of the plane $\mathbb{R}^2 = \mathbb{C}$ that does* not *require the axiom of choice for its construction, namely the Sierpiński-Mazurkiewicz Paradox: let $e^{i\theta}$ be a transcendental complex number and define*

$$\begin{aligned} E &= \left\{ x = \sum_{n=0}^{\infty} x_n e^{in\theta} \in \mathbb{C} : x_n \in \mathbb{Z}_+ \text{ and } x_n = 0 \text{ for all but finitely many } n \right\}, \\ E_1 &= \{x \in E : x_0 = 0\}, \\ E_2 &= \{x \in E : x_0 > 0\}. \end{aligned}$$

Then $E = E_1 \dot{\cup} E_2 = e^{-i\theta} E_1 = E_2 - 1$.

6.2.1. *Finitely additive invariant measures.* Let G be a group acting on a set X. If there exists a finitely (countably) additive G-invariant positive nontrivial measure μ on the power set $\mathcal{P}(X)$, then there are no finitely (countably) G-paradoxical subsets E of X having positive μ-measure. In particular G itself is not finitely (countably) G-paradoxical. This is proved as in the proof of Corollary 4 above. Thus paradoxical constructions can be viewed as nonexistence theorems for invariant measures, and by the contrapositive, the construction of invariant measures precludes paradoxical decompositions. We now state the main two theorems proved in this subsubsection. The first shows that paradoxical decompositions *never* occur for abelian groups (such as the group of translations on Euclidean space $\mathbb{R}^n$), and the second shows that paradoxical decompositions *do* exist for the rotation groups on Euclidean space $\mathbb{R}^n$ when $n \geq 3$ (resulting in the Banach-Tarski paradox).

Theorem 29. *Suppose G is an abelian group and let $\mathcal{M}$ be the power set of G. There is $\mu : \mathcal{M} \to [0,1]$ satisfying*

1. $\mu(E_1 \dot{\cup} E_2) = \mu(E_1) + \mu(E_2), \qquad E_i \in \mathcal{M}$,
2. $\mu(E + a) = \mu(E), \qquad E \in \mathcal{M}, a \in G$,
3. $\mu(G) = 1$.

Definition 8. *Let G act on a set X. Subsets A and B of X are said to be G-equidecomposable, written $A \sim_G B$ or simply $A \sim B$ when G is understood, if $A = \dot{\cup}_{i=1}^n A_i$ and $B = \dot{\cup}_{i=1}^n B_i$ where $A_i = g_i B_i$ for some $g_i \in G$, $1 \leq i \leq n$.*

We will see later that E is G-paradoxical if and only if $E = A \dot{\cup} B$ where $A \sim_G E \sim_G B$.

Remark 11. *If X is Euclidean space $\mathbb{R}^n$, then G_3-equidecomposability preserves the following properties: boundedness, Lebesgue measure zero, first category, and second category.*

Theorem 30. *(Banach-Tarski paradox) The sphere $\mathbb{S}^2$ is SO_3-paradoxical and the ball $\mathbb{B}_3$ is G_3-paradoxical. Moreover, if A and B are any two bounded subsets of $\mathbb{R}^3$, each having nonempty interior, then A and B are G_3-equidecomposable.*

To prove the first theorem, we need an "invariant" Hahn-Banach theorem. This will be proved using the following invariant fixed point theorem in a topological vector space. Let K be convex and Y a vector space. We say that a map $T : K \to Y$ is *affine* if

$$T\left((1-\theta)x + \theta y\right) = (1-\theta)Tx + \theta Ty, \qquad x, y \in K,\ 0 < \theta < 1.$$

We say that a vector space X is a *topological vector space* if there is a topology τ on X satisfying

- every point of X is a closed set,
- the vector space operations are continuous with respect to τ.

Theorem 31. *(Markov and Kakutani) Let K be a nonempty compact convex subset of a topological vector space X. Suppose that $\mathcal{F}$ is a commuting family of continuous affine maps from K to K. Then there is a point $p \in K$ such that $Tp = p$ for all $T \in \mathcal{F}$.*

Proof: For $T \in \mathcal{F}$, let $T^0 = I$, $T^n = T \circ T^{n-1}$ and $T_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k$. Then T_m and T'_n commute for all $T, T' \in \mathcal{F}$ and $m, n \geq 0$. Indeed, an affine map S satisfies $S\left(\sum_{k=1}^{\ell} \theta_\ell x_\ell\right) = \sum_{k=1}^{\ell} \theta_\ell S(x_\ell)$ if $\sum_{k=1}^{\ell} \theta_\ell = 1$, and so

$$\begin{aligned} T_m \circ T'_n &= \frac{1}{m}\sum_{j=0}^{m-1} T^j \left(\frac{1}{n}\sum_{k=0}^{n-1} (T')^k\right) \\ &= \frac{1}{m}\sum_{j=0}^{m-1} \left(\frac{1}{n}\sum_{k=0}^{n-1} T^j (T')^k\right) \\ &= \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} T^j (T')^k . \end{aligned}$$

Since $T^j (T')^k = (T')^k T^j$ we immediately have $T_m \circ T'_n = T'_n \circ T_m$.

Let $\mathcal{F}^*$ be the semigroup generated by the T_n, $n \geq 1$. Since

$$f(K) \cap g(K) \supset f \circ g(K), \qquad f, g \in \mathcal{F}^*,$$

we see that $\{f(K)\}_{f \in \mathcal{F}^*}$ has the finite intersection property and hence there is $p \in \cap_{f \in \mathcal{F}^*} f(K)$. Next, $p \in T_n(K)$ implies that $p = \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n$ for some $x_n \in K$, so that

$$p - Tp = \frac{1}{n}(x_n - T^n x_n) \in \frac{1}{n}(K - K)$$

for all $n \geq 1$. Now K compact implies $K - K$ compact, hence bounded. It follows that $p = Tp$.

Theorem 32. *(invariant Hahn-Banach) Let Y be a subspace of a normed linear space X, and let $f \in Y^*$. Suppose that Γ is a collection of bounded linear operators $T : X \to X$ satisfying $\|T\| \leq 1$ for all $T \in \Gamma$ and*

$$\begin{array}{ll} a) & ST = TS \\ b) & TY \subset Y \\ c) & f \circ T\mid_Y = f \end{array}$$

for all $S, T \in \Gamma$. Then there is $F \in X^*$ such that $F\mid_Y = f$ and $F \circ T = F$ for all $T \in \Gamma$.

Proof: Assume that $\|f\|^* = 1$. Let

$$K = \left\{ \Lambda \in X^* : \|\Lambda\|^* \leq 1, \Lambda\mid_Y = f \right\}.$$

Now K is convex, and by the Hahn-Banach theorem, K is nonempty. Moreover, K is weak* closed since $\Lambda\mid_Y = f$ if and only if $\Lambda y = f(y)$ for all $y \in Y$. By the Banach-Alaoglu theorem, K is weak* compact. Below we will show that for $T \in \Gamma$ the map $\widetilde{T} : X^* \to X^*$ defined by $\widetilde{T}\Lambda = \Lambda \circ T$ is weak* continuous. We also have $\widetilde{T}K \subset K$ since $\|T\| \leq 1$. Thus the invariant fixed point theorem applied to X^* equipped with the weak* topology and $\mathcal{F} = \left\{ \widetilde{T} : T \in \Gamma \right\}$ produces the desired $F \in K$.

To see that $\widetilde{T}$ is weak* continuous, fix $\Lambda_1 \in X^*$ and consider the weak* neighbourhoods of $\Lambda_1 T$ and Λ_1 given by

$$\begin{aligned} V &= \{ L \in X^* : |Lx_i - (\Lambda_1 T) x_i| < \varepsilon, 1 \leq i \leq n \}, \\ W &= \{ \Lambda \in X^* : |\Lambda (T x_i) - \Lambda_1 (T x_i)| < \varepsilon, 1 \leq i \leq n \}, \end{aligned}$$

for $\{x_1, .., x_n\} \subset X$. Now if $\Lambda \in W$, we have $\Lambda T \in V$ which shows that $\widetilde{T}$ is weak* continuous.

Now we prove the existence theorem for finitely additive measures on abelian groups.

Proof (of Theorem 29): Suppose G is infinite. Let $X = \ell^\infty(G)$, the Banach space of bounded complex-valued functions on G with the supremum norm. Let

$$Y = \left\{ f \in X : \lim_{x \to \infty} f(x) \text{ exists} \right\},$$

where $\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ there is a finite set F such that $|f(x) - L| < \varepsilon$ for all $x \in G \diagdown F$. For $f \in Y$ let $\Lambda f = \lim_{x \to \infty} f(x)$. Then $\Lambda \in Y^*$ and $\|\Lambda\| = 1$. Let $\Gamma = \{\tau_a : a \in G\}$ where $\tau_a f(x) = f(x - a)$. Now Γ is commutative since G is abelian. Also $\|\tau_a\| = 1$, $\tau_a Y \subset Y$ and $\Lambda \circ \tau_a\mid_Y = \Lambda$ for all $a \in G$. Thus the invariant Hahn-Banach theorem implies that there is $L \in X^*$ with $\|L\| = 1$, $L\mid_Y = \Lambda$ and $L\tau_a = L$ for all $a \in G$.

Now we define $\mu(E) = L\chi_E$ for $E \in \mathcal{M}$. Clearly

$$\begin{aligned} \mu(E_1 \dot{\cup} E_2) &= L\left(\chi_{E_1 \dot{\cup} E_2}\right) = L\left(\chi_{E_1} + \chi_{E_2}\right) = L\chi_{E_1} + L\chi_{E_2} = \mu(E_1) + \mu(E_2), \\ \mu(E + a) &= L\left(\chi_{E+a}\right) = L(\tau_a \chi_E) = (L\tau_a)\chi_E = L\chi_E = \mu(E), \\ \mu(G) &= L(\chi_G) = L1 = 1. \end{aligned}$$

Finally, from $\|L\| = 1$ and $L1 = 1$ we have that $0 \le Lf \le 1$ whenever $0 \le f \le 1$, which yields $\mu : \mathcal{M} \to [0,1]$. Indeed, if $Lf = \alpha + i\beta$ then for $t \in \mathbb{R}$,

$$L\left(f - \frac{1}{2} + it\right) = \left(\alpha - \frac{1}{2}\right) + i(\beta + t).$$

Since $\left\|f - \frac{1}{2}\right\| \le \frac{1}{2}$ we have

$$\left(\alpha - \frac{1}{2}\right)^2 + (\beta + t)^2 = \left|L\left(f - \frac{1}{2} + it\right)\right|^2 \le \left\|f - \frac{1}{2} + it\right\|^2 \le \frac{1}{4} + t^2,$$

for all $t \in \mathbb{R}$, and this implies $\beta = 0$ and then $0 \le \alpha \le 1$.

6.2.2. *Paradoxical decompositions.* We obtain the strong form of the Banach-Tarski paradox in four steps.

- First, we prove that the free nonabelian group F_2 of rank 2 is paradoxical.
- Second, we show that the special orthogonal group SO_3 in three dimensions contains a copy of F_2.
- Third, we lift the paradoxical decomposition from SO_3 to the sphere $\mathbb{S}^2$ on which it acts "almost" without nontrivial fixed points.
- Fourth, we extend the paradox to bounded sets with nonempty interior with the help of the proof of the Schröder-Bernstein theorem.

First step: We prove that F_2 is paradoxical. Let F_2 consist of all finite "words" in $\sigma, \sigma^{-1}, \tau, \tau^{-1}$ with concatenation as the group operation, and the empty word as identity 1. For $\rho \in \left\{\sigma, \sigma^{-1}, \tau, \tau^{-1}\right\}$, let $W(\rho)$ consist of all reduced words that begin with ρ (a word is *reduced* if no pair of adjacent symbols is $\sigma\sigma^{-1}$, $\sigma^{-1}\sigma$, $\tau\tau^{-1}$, or $\tau^{-1}\tau$). The following decompositions witness the paradoxical nature of F_2:

$$\begin{aligned} F_2 &= \{1\} \dot\cup W(\sigma) \dot\cup W\left(\sigma^{-1}\right) \dot\cup W(\tau) \dot\cup W\left(\tau^{-1}\right), \\ F_2 &= W(\sigma) \dot\cup \sigma W\left(\sigma^{-1}\right), \\ F_2 &= W(\tau) \dot\cup \tau W\left(\tau^{-1}\right). \end{aligned}$$

Note that we do not use the identity in these reconstructions of F_2. We can however witness the paradox with four disjoint pieces whose union is F_2 as follows. Let $S = \{\sigma^{-n}\}_{n=1}^{\infty}$ and define

$$\begin{aligned} A_1 &= \{1\} \dot\cup W(\sigma) \dot\cup S, \\ A_2 &= W\left(\sigma^{-1}\right) \setminus S, \\ A_3 &= W(\tau), \\ A_4 &= W\left(\tau^{-1}\right). \end{aligned}$$

Then $F_2 = \dot\bigcup_{i=1}^4 A_i$ and $F_2 = A_1 \dot\cup \sigma A_2$ and $F_2 = A_3 \dot\cup \tau A_4$.

Second step: To embed a copy of F_2 in SO_3 we define the 3×3 matrices:

$$\begin{aligned} \phi^{\pm} &= \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & \mp 2\sqrt{2} & 0 \\ \pm 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \\ \rho^{\pm} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & \mp 2\sqrt{2} \\ 0 & \pm 2\sqrt{2} & 1 \end{bmatrix}. \end{aligned}$$

It suffices to show that no nonempty reduced word in $\phi^{\pm}, \rho^{\pm}$ equals the identity in SO_3. Since conjugation by $\phi^{\pm}$ doesn't affect the vanishing of a word, we may assume that w is a nonempty reduced word ending in $\phi^{\pm}$.

Claim 1. *Every nonempty reduced word w in $\phi^{\pm}, \rho^{\pm}$ that ends in $\phi^{\pm}$ satisfies $w(1,0,0) = 3^{-k}\left(a, b\sqrt{2}, c\right)$ for some $a, b, c \in \mathbb{Z}$ with $3 \nmid b$, and where k is the length of w.*

We prove the claim by induction on the length k of w. The case $k = 1$ is evident upon examining the first columns of the two matrices $\phi^{\pm}$. If w of length $k \geq 2$ equals $\phi^{\pm} w'$ or $\rho^{\pm} w'$, where

$$w'(1,0,0) = 3^{1-k}\left(a', b'\sqrt{2}, c'\right), \qquad a', b', c' \in \mathbb{Z},\ 3 \nmid b',$$

then

$$\begin{aligned} \phi^{\pm} w'(1,0,0) &= 3^{-k}\left(a' \mp 4b', (b' \pm 2a')\sqrt{2}, 3c'\right), \\ \rho^{\pm} w'(1,0,0) &= 3^{-k}\left(3a', (b' \mp 2c')\sqrt{2}, c' \pm 4b'\right). \end{aligned}$$

We now see that $w(1,0,0)$ has the form $3^{-k}\left(a, b\sqrt{2}, c\right)$ for some $a, b, c \in \mathbb{Z}$, and it remains only to prove $3 \nmid b$ given that $3 \nmid b'$. There are four cases: $w = \phi^{\pm}\rho^{\pm}v$, $\rho^{\pm}\phi^{\pm}v$, $\phi^{\pm}\phi^{\pm}v$ and $\rho^{\pm}\rho^{\pm}v$ where v is possibly empty. We may suppose that $v(1,0,0) = 3^{2-k}\left(a'', b''\sqrt{2}, c''\right)$ where $a'', b'', c'' \in \mathbb{Z}$. In the first case, with $w' = \rho^{\pm} v$, we have $3 \mid a'$ (since $a' = 3a''$ by the second line displayed above) and $3 \nmid b'$ and so $3 \nmid b' \pm 2a' = b$ as required. The second case is similar. For the third case we have

$$b = b' \pm 2a' = b' \pm 2\left(a'' \mp 4b''\right) = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b'',$$

and again $3 \nmid b$ follows from $3 \nmid b'$. The fourth case is similar and this completes the proof of the claim.

Third step: To lift a paradoxical decomposition from a group to a set on which it acts is easy using the axiom of choice provided the action is with trivial fixed points. We say that a group G acts on a set X with trivial fixed points if $gx \neq x$ for all $x \in X$ and all $g \in G \setminus \{e\}$ where e denotes the identity element of G.

Proposition 1. *If G is a paradoxical group and acts on a set X with trivial fixed points, then X is G-paradoxical.*

Proof: Let A_i, B_j, g_i, h_j witness the paradoxical nature of G as in (6.2). Let M be a choice set for the G-orbits in X. Then $\{gM\}_{g \in G}$ is a partition of X because there are no nontrivial fixed points. Then $A_i^* = \dot{\cup}_{g \in A_i} gM$ and $B_j^* = \dot{\cup}_{h \in B_j} hM$ easily yield a paradoxical decomposition of X:

$$\begin{aligned} X &\supset \left(\dot{\cup}_{i=1}^m A_i^*\right) \dot{\cup} \left(\dot{\cup}_{j=1}^n B_j^*\right), \\ X &= \dot{\cup}_{i=1}^m g_i A_i^* = \dot{\cup}_{j=1}^n h_j B_j^*. \end{aligned}$$

Corollary 5. *(Hausdorff's paradox) There is a countable set $D \subset \mathbb{S}^2$ such that $\mathbb{S}^2 \setminus D$ is SO_3-paradoxical.*

Proof: Let F be a free nonabelian group of rank 2 in SO_3. Then F is countable and since each $\alpha \in F$ fixes exactly 2 points, $D = \left\{x \in \mathbb{S}^2 : \alpha x = x \text{ for some } \alpha \in F\right\}$

is countable. Then F acts on $\mathbb{S}^2 \diagdown D$ with trivial fixed points, and Proposition 1 implies that $\mathbb{S}^2 \diagdown D$ is F-paradoxical, hence also SO_3-paradoxical.

Hausdorff's paradox is already sufficient to disprove the existence of *finitely* additive rotation invariant positive measures of total mass 1 on the power set of $\mathbb{S}^2$, and hence also disproves the existence of *finitely* additive isometry invariant positive measures on the power set of $\mathbb{R}^3$ that normalize the unit cube (this was Hausdorff's motivation). *Exercise*: prove this! We can eliminate the countable set D in Hausdorff's paradox by an absorption process once we have the following lemma.

Lemma 9. *Let G act on a set X and let $E, E' \in \mathcal{P}(X)$. If $E \sim_G E'$, then E is G-paradoxical if and only E' is G-paradoxical.*

First we note that the relation $\sim_G$ is transitive. Suppose that $E \sim_G A$ and $E \sim_G B$. Then $E = \dot{\cup}_{i=1}^n A_i = \dot{\cup}_{j=1}^m B_j$ where $A = \dot{\cup}_{i=1}^n g_i A_i$ and $B = \dot{\cup}_{j=1}^n h_j B_j$ for some group elements g_i, h_j. Then $A = \dot{\cup}_{i,j=1}^{n,m} g_i (A_i \cap B_j)$ and $B = \dot{\cup}_{i,j=1}^{n,m} h_j (A_i \cap B_j)$ shows that $A \sim_G B$. From this we easily obtain the lemma. Indeed, E is G-paradoxical if and only there are disjoint subsets B_1, B_2 of E such that both $B_1 \sim_G E$ and $B_2 \sim_G E$. From $E \sim_G E'$, we have $E = \dot{\cup}_{i=1}^n A_i$ and $E' = \dot{\cup}_{i=1}^n g_i A_i$. Thus if we define $B_1' = \dot{\cup}_{i=1}^n g_i (A_i \cap B_1)$ and $B_2' = \dot{\cup}_{i=1}^n g_i (A_i \cap B_2)$, we have that B_1', B_2' are disjoint subsets of E' such that $B_1' \sim_G \dot{\cup}_{i=1}^n (A_i \cap B_1) = B_1 \sim_G E \sim_G E'$ and similarly $B_2' \sim_G E'$. This shows that E' is G-paradoxical.

Theorem 33. *(Banach-Tarski paradox) $\mathbb{S}^2$ is SO_3-paradoxical and $\mathbb{B}_3$ is G_3-paradoxical.*

Proof: Let $D = \{d_i\}_{i=1}^\infty$ be as in Hausdorff's paradox. Pick a line ℓ through the origin that misses D and fix a plane P containing ℓ. Let

$$A = \left\{ \frac{1}{n} (\theta_i - \theta_j) : n, i, j \in \mathbb{N} \right\}, \qquad \theta_i = \measuredangle (d_i, \ell),$$

where $\measuredangle (d_i, \ell)$ denotes the angle mod π through which the plane P must be rotated (in a fixed sense) about ℓ so as to contain d_i. Pick $\theta \notin A \pmod{\pi}$. Then if ρ is rotation about ℓ through angle θ, we have

$$\begin{aligned} \rho^n D \cap D &= \phi, \qquad n \geq 1, \\ \rho^m D \cap \rho^n D &= \phi, \qquad m \neq n \text{ in } \mathbb{Z}. \end{aligned}$$

Then with $\overline{D} = \dot{\cup}_{n=0}^\infty \rho^n D = D \dot{\cup} \rho \overline{D}$ we have

$$\mathbb{S}^2 = \left(\mathbb{S}^2 \diagdown \overline{D}\right) \dot{\cup} \overline{D} \sim_{SO_3} \left(\mathbb{S}^2 \diagdown \overline{D}\right) \dot{\cup} \rho \overline{D} = \mathbb{S}^2 \diagdown D,$$

and the lemma shows that $\mathbb{S}^2$ is SO_3-paradoxical.

Finally, the equality

$$\mathbb{B}_3 \diagdown \{0\} = \cup_{\omega \in S^2} \{\lambda \omega : 0 < \lambda < 1\}$$

shows that $\mathbb{B}_3 \diagdown \{0\}$ is SO_3-paradoxical, and an absorption argument as above then shows that $\mathbb{B}_3$ is G_3-paradoxical. Indeed, use a rotation ρ about a line ℓ passing through $\left(0, 0, \frac{1}{3}\right)$ but *not* passing through the origin, so that $\rho^m 0 \neq \rho^n 0$ for $m \neq n$, and set $\overline{D} = \dot{\cup}_{n=0}^\infty \rho^n 0 = \{0\} \dot{\cup} \rho \overline{D}$. Then since $\rho \in G_3$,

$$\mathbb{B}_3 = \left(\mathbb{B}_3 \diagdown \overline{D}\right) \dot{\cup} \overline{D} \sim_{G_3} \left(\mathbb{B}_3 \diagdown \overline{D}\right) \dot{\cup} \rho \overline{D} = \mathbb{B}_3 \diagdown \{0\}.$$

Remark 12. *The arguments above show that $\mathbb{S}^2$ can be duplicated using 8 pieces, and that $\mathbb{B}_3$ can be duplicated using 16 pieces. More refined arguments show that 4 pieces suffice for $\mathbb{S}^2$, and that 5 pieces suffice for $\mathbb{B}_3$. These latter results are optimal.*

Fourth step: The next result shows that if we declare $A \preceq_G B$ when A is G-equidecomposable to a *subset* of B, then the relation $\preceq_G$ is a partial ordering of the $\sim_G$ equivalence classes in $\mathcal{P}(X)$.

Theorem 34. *(Banach-Schröder-Bernstein) Suppose that a group G acts on a set X. If $A, B \in \mathcal{P}(X)$ satisfy both $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$.*

Proof: We have the following two properties of the relation $\sim_G$:

- If $A \sim_G B$, then there is a bijection $g : A \to B$ such that

$$C \sim_G g(C) \text{ whenever } C \subset A. \tag{6.3}$$

- If $A_1 \cap A_2 = \phi = B_1 \cap B_2$ and $A_i \sim_G B_i$ for $i = 1, 2$ then $A_1 \cup A_2 \sim_G B_1 \cup B_2$.

By hypothesis, $A \sim_G B_1$ and $A_1 \sim_G B$ for some $B_1 \subset B$ and $A_1 \subset A$. By the first property, there are bijections $f : A \to B_1$ and $g : A_1 \to B$ satisfying $C \sim_G f(C)$ and $D \sim_G g(D)$ whenever $C \subset A$ and $D \subset A_1$. Let $C_0 = A \setminus A_1$ and inductively $C_{n+1} = g^{-1} f(C_n)$ for $n \geq 0$. With $C = \dot{\cup}_{n=0}^{\infty} C_n$ we have

$$g(A \setminus C) = B \setminus f(C)$$

and then $A \setminus C \sim_G B \setminus f(C)$ by (6.3). But we also have $C \sim_G f(C)$ by (6.3) and the second property now yields

$$A = (A \setminus C) \dot{\cup} C \sim_G (B \setminus f(C)) \dot{\cup} f(C) = B.$$

Corollary 6. *A subset E of X is G-paradoxical if and only if there are disjoint sets $A, B \subset E$ with $A \dot{\cup} B = E$ and $A \sim_G E \sim_G B$.*

Theorem 35. *(strong form of the Banach-Tarski paradox) If A and B are any two bounded subsets of $\mathbb{R}^3$, each with nonempty interior, then A and B are G_3-equidecomposable.*

Proof: It suffices to show that $A \preceq_{G_3} B$, since interchanging A and B yields $B \preceq_{G_3} A$, and the Banach-Schröder-Bernstein theorem then shows that $A \sim_{G_3} B$. So choose solid balls K and L such that $A \subset K$ and $L \subset B$, and let n be large enough that K can be covered by n copies of L. Use the Banach-Tarski paradox to create a union S of n pairwise disjoint copies of L, and then cover K by a union of *translates* of these copies so that $K \preceq_{G_3} S$. It follows that

$$A \subset K \preceq_{G_3} S \preceq_{G_3} L \subset B,$$

and so $A \preceq_{G_3} B$.

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