SRI 1404
Categoricity & Quasiminimality

[struck out] An uncountable structure M is uncountably cat^l ← cble langs
if $N \equiv M$ and $|N| = |M|$ then $N \cong M$

Examples: $\langle \mathbb{Q}; + \rangle$ is the unique divisible torsion-free group of Q-din^n = card^y = $2^{\aleph_0}$

$\langle \mathbb{C}; +, \cdot \rangle$ is the unique alg^y closed field of char 0 of tr.deg = card^y = $2^{\aleph_0}$

[struck out] X principal $\mathbb{C}^*$ bundle
$\downarrow \pi$ in language with π,
$\mathbb{C}$ $+, \cdot$ on $\mathbb{C}$
and $*: \mathbb{C}^* \times X \to X$
for action of $\mathbb{C}^*$ on fibres

This is u.c.:

$X_1 \quad X_2$
$\downarrow \quad \downarrow$
$\mathbb{C}_1 \cong \mathbb{C}_2$

$x_1 \searrow \swarrow x_2$ $\mathbb{C}$ — match up fibres arbitrarily preserving $\mathbb{C}^*$ action

M is s.m. if def^ble subsets are uniformly finite/cofinite i.e. if for every $\phi(x, \bar{y})$, exists $n \in \mathbb{N}$ st. for all $\bar{b} \in M^n$
$|\phi(M, \bar{b})| \leq n$
or $|\neg\phi(M, \bar{b})| \leq n$

Th^m: s.m. ⇒ u.c.
pf: $acl(A) := \bigcup_{\phi, \bar{a} \in A, |\phi(M, \bar{a})| < \omega} \phi(M, \bar{a}) \subseteq M$
$(A \subseteq M)$
Fact (Exchange) $c \in acl(Ab) \setminus acl(A) \Rightarrow b \in acl(Ac)$
X acl-independent if $\forall Y \subsetneq X$ $acl(Y) \not\ni X \neq Y$
$A = acl(A) \Rightarrow A = acl(X)$ some $X \subseteq A$ ind^t ("basis")
$dim(A) := |X|$ — well-defined
$|A| = dim(A)$ if $|A| > \aleph_0$

If $N \equiv M$, $|N| = |M|$
X_N, X_M bases
$|X_N| = |X_M|$
build iso^m $N \cong M$ over arb^y bij^n $X_N \cong X_M$ □

Fact (Zilber Ladder Th^m):
any u.c. structure is built from a s.m. structure by iteratedly taking G-bundles" for definable [struck out] groups G.

Finding s.m. sets

- If eg. $\phi(x, y, z)$ has finitely many solutions for z for a given x, y then $\phi(a, b, c)$ witnesses $c \in acl(a, b)$
e.g. (i) $\langle \mathbb{C}; x^2(y - \omega z) + y^2(z - \omega x) + z^2(x - \omega y) = 0 \rangle$
(ii) $\langle \mathbb{C}; x + y + z = 0 \rangle$
(iii) $\langle \mathbb{C}; xy^2 + yz^2 + zx^2 = 0 \rangle$

(i): $x^2(y - \omega z) + \ldots$
$= (x - \omega y)(y - \omega z)(z - \omega x)$

union of 3 such hyperplanes

$acl(A) = A \cup \omega A \cup \omega^2 A$

(ii) $x + y = z \Leftrightarrow \exists 0. \; 0+0+0 = 0 \land \exists w. (x + y + w = 0 \land z + 0 + w = 0))$
so this is a disguised form of $\langle \mathbb{C}; + \rangle$
$acl(A) = \langle A \rangle_{\mathbb{Q}}$ divisible subgp^n gen^d

(iii) Can recover + and ·
so $acl(A)$ = field-theoretic alg^c closure (A)

Trichotomy Conj^re:
"(i)-(iii) only possibilities for s.m."
roughly:
if acl non-degenerate ($acl(A \cup B) \neq acl(A) \cup acl(B)$ some A, B)
(i.e. acl not given just by binary rel^ns)
then get group,
if have more than a group (not ℤ5)
get field
[struck out] and then the structure is a finite cover of an ACF
[struck out] Zilber-Rabinovich: true for [struck out] $\langle \mathbb{C}$; polynomial [struck out] equations$\rangle$
Hr: False in general

Ab Initio
Idea: synthesise [struck out] non-degenerate ternary relation R with finite fibres of proj^n onto pairs of coords and "no self-intersection"

so for finite A,
$dim(A) \leq |A| - |R \cap A^3| =: \delta(A)$
and actually $dim(A) = \min_{B \supseteq A} \delta(B)$

Build our structure out of finite structures $(A; R)$ with $\delta(A) \geq 0$
and "strong" embeddings $(A; R) \leq (B; R) \Leftrightarrow \delta(A) \leq \delta(A')$ for $A \subseteq A' \subseteq B$

Quasiminimal structures
Idea: relax "finite" in the above to "countable"
e.g. $\langle \mathbb{C}; e^x + e^y + e^z = 0 \rangle$
[struck out], $\langle \mathbb{C}; x^\lambda + y^\lambda + z^\lambda = 0 \rangle$ $(x^\lambda = e^{\lambda(\log x + 2\pi i \mathbb{Z})})$ $\lambda \in \mathbb{C}$

Def^n: uncble M is quasimin^l if for every $\phi(x, \bar{y})$, for each $\bar{b} \in M^n$, one of $\phi(M, \bar{b})$, $\neg\phi(M, \bar{b})$ is cble.

Fact: M u.c. ⇒ for X def^ble inf., $|X(M)| = |M|$
So q.m. not s.m. ⇒ not u.c.

wrong logic!
Def^n: L language
formulae of $L_{\omega_1,\omega}(Q)$ recursively generated from L by
$\neg, \bigwedge_i \phi_i$ (cble conjunctions), $\exists x. \phi$, $Qx\, \phi$ "there exist $> \aleph_0$ many x st."

Fact: any cble structure is the unique model of an $L_{\omega_1,\omega}$ sentence

Def^n: uncble q.m. M in a cble lang is categorically q.m. (QMF, qps)
if (i) q.cl satisfies exchange
(ii) $\bar{c} \equiv \bar{c}' \Rightarrow \bar{c} \equiv_{L_{\omega_1,\omega}(Q)} \bar{c}'$ ("smallness")
(iii) If $A = qcl(A)$ cble, given $\bar{c}$, exists $\bar{b} \in A$ s.t. for $\bar{a}, \bar{a}' \in A$, if $\bar{a}\bar{c} \equiv \bar{a}'\bar{c}$ then $\bar{a}\bar{b} \equiv \bar{a}'\bar{b}$ ("ω-stability")

Th^m (A) any cat^l q.m. structure is the unique model in its cardinal of some $L_{\omega_1,\omega}(Q)$ sentence
(B) $|M| = \aleph_1$, $L_{\omega_1,\omega}(Q)$-q.m.
if for no countable fragment of $L_{\omega_1,\omega}(Q)$ does M satisfy (i)-(iii)
then it is consistent that no such sentence exists;
$V = L \Rightarrow Q_{\aleph_1}^{2^{\aleph_0} < 2^{\aleph_1}}$

Examples: $\langle \mathbb{C}; e^x + e^y + e^z = 0 \rangle$: countable cover of $\langle \mathbb{C}^*; x + y + z = 0 \rangle$ ← s.m. with no structure on fibres
- Ab initio where acl-rigid R to [struck out] (i.e. rich + ccp) [struck out]

- $\langle \mathbb{C}; e^x + e^y + e^z = 0, +, x \mapsto \lambda x \rangle$ $\lambda \in \mathbb{R}$
For all but cbly many λ:
this is "a Hr^i construction with cble closure" and ω-stable so cqm
$\delta(\bar{x}) = ld_{\mathbb{Q}(\lambda)}(\bar{x}) + td(e^{\bar{x}}) - ld_{\mathbb{Q}}(\bar{x})$