Model Theory of Fields and Absolute Galois Theory

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What information about a field K does its absolute Galois group Gal(K) contain?
What can we learn about its absolute Galois group by studying the theory of a field K?

Basics

Let K and L be fields such that $K \subseteq L$ is a Galois extension, namely an algebraic field extension which is normal and separable. Let K^{sep} and K^{alg} denote the separable respectively algebraic closure of K.

Definition

(i) We define the Galois group of L over K, denoted by $\text{Gal}(L/K)$, to be

$$\text{Gal}(L/K) := \text{Aut}(L/K).$$

(ii) We define the absolute Galois group of K to be

$$Gal(K) := \text{Gal}(K^{\text{sep}}/K).$$

(iii) A profinite group is a group which is an inverse limit of finite groups, i.e. $G \cong \varprojlim_{i\in I} G_i$ for a directed family $\{G_i\}_{i\in I}$ of finite groups.

Properties[1]

- $\text{Aut}(K^{\text{sep}}/K) \cong \text{Aut}(K^{\text{alg}}/K)$,
- absolute Galois groups are the inverse limit over all Galois groups of finite intermediate extensions, i.e.

$$\text{Gal}(K) \cong \varprojlim_{i\in I} \text{Gal}(L_i/K),$$

where $\mathcal{L} = \{L_i \,|\, i \in I\}$ is the collection of all intermediate fields such that $K \subset L_i$ is a finite Galois extension.

Facts[1]

- $\text{Gal}(K) = \{e\} \Leftrightarrow K = K^{\text{sep}}$,
- (Artin-Schreyer) $\text{Gal}(K)$ is finite but non trivial iff K is a real closed field,
- if K is a pseudofinite field (i.e. a model of the theory of finite fields) then $\text{Gal}(K) \cong \hat{\mathbb{Z}}$, but not every K with $\text{Gal}(K) \cong \hat{\mathbb{Z}}$ is pseudofinite.

Results

Definition A field K is said to be elementary characterized by its absolute Galois group if for all fields L

$$\text{Gal}(L) \cong \text{Gal}(K) \Leftrightarrow L \equiv K.$$

Theorem[2] A field K is elementary characterized by $\text{Gal}(K)$ iff K is elementary equivalent to one of the following

1. $\mathbb{R}$,
2. a finite extension L of $\mathbb{Q}_p$ with $([L : L^{\text{ab}}], \frac{p}{p-1} \cdot m) = 1$, where p is some prime, L^{ab} is the maximal abelian subextension of $L/\mathbb{Q}_p$ and m denotes the number of roots of unity μ_L in L,
3. $L((\mathbb{Z}_{(q)}))$, the generalized power series field over L with exponents from $\mathbb{Z}_{(q)} = \mathbb{Q} \cap \mathbb{Z}_q$, where L is as in 2. and q is some prime,
4. $L((\mathbb{Z}_{(p)}))$, where L is a field such that
   (i) $\text{char}(L) = 0$,
   (ii) $[L : \mathbb{Q}]_{\text{trdeg}} < \infty$,
   (iii) L admits no proper abelian extension,
   (iv) L has a henselian valuation with residual characteristic p,
   (v) L is elementarily characterized by $\text{Gal}(L)$ only among all fields of characteristic $\neq p$,
   (vi) $\text{cd}_p \,\text{Gal}(L) = 1$,
5. a field L elementarily characterized by $\text{Gal}(L)$ such that
   (i) $\text{Gal}(L)$ is not pro-solvable,
   (ii) for all fields F if $\text{Gal}(F) \cong \text{Gal}(L)$ then $\text{char}(F) = 0$ and $[F : \mathbb{Q}]_{\text{trdeg}} = \infty$.

Definition A field K is large if it satisfies one of the following equivalent conditions:

(i) Each absolutely irreducible curve over K with a simple K-rational point has infinitely many K-rational points,
(ii) each function field of one variable F/K with a prime divisor of degree 1 has infinitely many such divisors,
(iii) K is existentially closed in $K((t))$.

Definition A profinite group is said to be a pro-p group if it is the inverse limit of p-groups.

Theorem[3] Let K be a field such that $\text{Gal}(K)$ is a pro-p group for some prime number p. Then K is large.

Definition A field K is pseudo algebraically closed (PAC) if every absolute irreducible variety V over K has an K-rational point.

Definition A field K is bounded if it has only finitely many Galois extensions of degree n for every integer n.

In terms of the absolute Galois group a field is bounded iff its absolute Galois group has only finitely many closed subgroups of index n for every integer n.

Theorem[4] The theory of a bounded PAC field is simple.

Theorem[5] A PAC field whose theory is simple, is bounded.

Theorem[6] If K is an infinite ω-stable field, then K is algebraically closed.

This implies that strongly minimal fields are algebraically closed. There is even a stronger result:

Theorem[7] Infinite superstable fields are algebraically closed.

Definition A structure is minimal if every (with parameters from the structure) definable subset is finite or cofinite.

Theorem[8] A minimal field of non-zero characteristic is algebraically closed.

Lemma Let K be a minimal field. Then
(i) if K is large then it is algebraically closed,
(ii) K has no proper solvable extensions.

Theorem[8] Let K be field with $\text{char}(K) > 0$. Suppose every $\exists\forall$- definable subset is either finite or cofinte. Then K is algebraically closed.

Related Questions

- Are the classes defined in conditions 4. and 5. empty?
- Does the same or a similar result hold if we study fields which are elementarily characterized by the theory of their absolute Galois group (in the language of inverse systems) instead of its isomorphism type?
- The above Theorem shows that a field with absolute Galois group isomorphic to $\mathbb{Z}_p$ is large. Does this also hold for fields with absolute Galois groups isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$, where $p \neq q$ are prime numbers?
- Does $\text{Gal}(K) \cong \mathbb{Z}_p$ imply that K is henselian or PAC?
- Are all minimal fields already algebraically closed?
- Are all fields in which every non-constant polynomial has a cofinite image already algebraically closed?

[1] eg Serge Lang, *Algebra*, Addison-Wesley Publishing Company, Inc., 1993 and Ribes/Zalesskii, *Profinite groups*, Springer-Verlag, Berlin, 2000
[2] Jochen Koenigsmann, *Elementary characterization of fields by their absolute Galois group*, Siberian Advances in Mathematics 14 (2004), no. 3, 16–42
[3] Moshe Jarden, *On Ample Fields*, Arch. Math. (Basel) 80 (2003), no. 5, 475–477
[4] Zoé Chatzidakis and Anand Pillay, *Generic structures and simple theories*, Ann. Pure Appl. Logic 95 (1998), no. 1-3, 71–92
[5] Zoé Chatzidakis, *Simplicity and independence for pseudo-algebraically closed fields*, Models and computability (Leeds, 1997), LMS Lecture Note Ser. (259), 41–61, Cambridge Univ. Press
[6] Angus Macintyre, *On ω_1-categorical theories of fields*, Fund. Math. 71 (1971), pp. 1-25
[7] Gregory Cherlin and Saharon Shelah, *Superstable fields and groups*, Ann. Math. Logic 18 (1980), no. 3, 227–270
[8] Frank O. Wagner, *Minimal fields*, J. Symbolic Logic 65 (2000), no. 4, 1833–1835