Covers

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1. Uncountably categorical theories

Work in a saturated model of an uncountably categorical theory (countable language, infinite models).

Zilber's Ladder Theorem (1977)

Let M_0, M be $\emptyset$-definable sets and M_0 strongly minimal. Then there exists a finite sequence of $\emptyset$-definable sets $M_0, M_1, \ldots, M_k$ ending in M such that for each i (with $0 \leq i < k$) and each M_i-atom $A \subseteq M_{i+1}$, the group $\mathrm{Gal}(A/M_i)$ of M_i-elementary automorphisms of A is definable, together with its action on A, and $\mathrm{Gal}(A/M_i) \subseteq \mathrm{dcl}(M_i)$.

REMARKS:

- Taken from Boris' 1993 AMS Monograph.
- M_i-atom: M_i-definable set not properly containing another non-empty M_i-definable set.
- M_{i+1} is covered by these.
- The M_i are in *eq*.

NOTES:

(0) Any uncountably categorical structure has a strongly minimal parameter-definable subset.

(1) Definable groups emerge from a purely model-theoretic hypothesis: the binding group construction.

(2) Write $A = A(\bar{a})$ and $\mathrm{Gal}(A/M_i) = G(\bar{a})$ where $\bar{a}$ varies in some $\emptyset$-definable subset D_i of M_i.

(3) $G(\bar{a})$ is transitive on $A(\bar{a})$ and the stabilizer in $G(\bar{a})$ of some finite set of points in $A(\bar{a})$ is the identity.

(4) $\mathrm{Aut}(M_{i+1}/M_i)$ embeds into $\prod_{\bar{a} \in D_i} G(\bar{a})$.

(5) M_{i+1} is an *affine cover* of M_i; in case $A(\bar{a})$ and $G(\bar{a})$ are finite, it is a *finite cover*.

The totally categorical case.

Zilber, early 1980's

In the case where the theory is totally categorical, there is a $\emptyset$-definable strongly minimal set, which can be taken to be a pure set, or a projective space arising from a vector space of infinite dimension over a finite field.

REMARKS:

- Only definable groups here are abelian (-by-finite).
- Alternative proof using C of FSG's: Cherlin-Harrington-Lachlan, Mills, ...
- Gives a strong structure theory for totally categorical structures. For example, they are not finitely axiomatizable (Zilber), but are quasi-finite axiomatizability (Ahlbrandt-Ziegler; Cherlin; Hrushovski).

2. Finite Covers

Suppose L is a first-order language and L' extends L by, amongst other things, a single extra sort C. Suppose T' is a complete L'-theory and T is the restriction of this to the L-sorts.

Say that T is *fully embedded* in T' if, whenever M' is a model of T' with L-part M:

- the $\emptyset$-definable subsets of M^n are the same in the L and L'-senses
- the L'-definable subsets of M^n (parameters from M') are the L-definable subsets of M^n (parameters from M).

Suppose D is a $\emptyset$-definable subset in T^{eq} we say that T' is a finite cover of T (over D) if T is fully embedded in T' and there is a $\emptyset$-definable function $\pi : C \to D$ with finite fibres.

NOTE:

- By full embeddedness, if M' is saturated then the restriction map $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$ is surjective.
- The kernel of this is $\mathrm{Aut}(M'/M) \leq \prod_{d \in D} \mathrm{Aut}(\pi^{-1}(d)/M)$

Basic Problem

Given T and D, say something meaningful about the finite covers T' over D.

Even with T strongly minimal, this is a hard problem.

Sub-problem

Describe the maximal covering expansions T'.

Meaning: T is not fully embedded in any proper expansion of T'.

REMARKS:

- If T'' is an expansion of T' in which T is fully embedded, then $\mathrm{Aut}(M') = \mathrm{Aut}(M'/M)\mathrm{Aut}(M'')$.
- If $\mathrm{Aut}(M''/M) = 1$ here then say that T' splits over T.

3. An Example

Take:

- M a pure set; $n \geq 2$
- $D \subseteq M^{eq}$: n-tuples of distinct elements of M modulo the equivalence relation $\bar{x} \sim \bar{y}$ iff $\bar{x}$ is an even permutation of $\bar{y}$.
- For $w \in [M]^n$ denote the two elements of D corresponding to enumerations of w as w^+ and w^-.

Define a finite cover M_n of M over D by adding:

- An extra sort $C = \{w_0, w_1, w_2, w_3 : w \in [M]^n\}$
- A projection map $\pi : C \to D$ with $\pi(w_0) = \pi(w_2) = w^+$ and $\pi(w_1) = \pi(w_3) = w^-$.
- A 2-ary relation R on C with $R(w_i, w_{i+1})$ (mod 4) (for $w \in [M]^n$)

So M_n is obtained by freely adjoining a copy of a finite structure $\{w, w^+, w^-, w_0, w_1, w_2, w_3\}$ over each $\{w, w^+, w^-\}$ in M.

OBSERVATIONS:

- $\mathrm{Aut}(M_n/M) = \prod_{w \in [M]^n} Z_2 = Z_2^{[M]^n}$
- M_n is non-split over M: Suppose $\mathrm{Aut}(M_n) = \mathrm{Aut}(M_n/M)\mathrm{Aut}(M'')$. Then for $w \in [M]^n$ we have:
  - $\mathrm{Aut}(M_n/w)$ is transitive on $\{w_0, w_1, w_2, w_3\}$ and
  - $\mathrm{Aut}(M_n/M)$ stabilizes $\{w_0, w_2\}$ and $\{w_1, w_3\}$, so
  - $\mathrm{Aut}(M''/w)$ induces Z_4 on $\{w_0, w_1, w_2, w_3\}$. In particular,
  - $\mathrm{Aut}(M'') \to \mathrm{Aut}(M)$ is not an isomorphism, because
  - $\mathrm{Aut}(M/w) = \mathrm{Sym}(w) \times \mathrm{Sym}(M \setminus w)$.

Theorem (DE + Elisabetta Pastori, 2009)

- *M_2 has no proper covering expansion.*
- *The covering expansions of M_n over M (for $n \geq 3$) can be described. In particular, there is a unique maximal covering expansion (up to interdefinability over M).*

The proof is heavily group-theoretic.

4. Group-theoretic methods

REMINDER: Any permutation group can be considered as a topological group in which pointwise stabilizers of finite sets form a base of open neighbourhoods of the identity. A subgroup of the group of all permutations of a set is closed iff it is the automorphism group of some structure on the set.

THE AHLBRANDT-ZIEGLER APPROACH (EARLY 1990'S):

- Treat the problem of determining the (finite) covers M' of M as a question about topological group extensions:
  $1 \to \mathrm{Aut}(M'/M) \to \mathrm{Aut}(M') \xrightarrow{\rho} \mathrm{Aut}(M) \to 1.$
- If $K_0 = \mathrm{Aut}(M'/M)$ is abelian, conjugation in $\mathrm{Aut}(M')$ makes it a continuous (profinite) G-module, where $G = \mathrm{Aut}(M)$.
- Use tools from representation theory and group cohomology to study this.
- If M' splits over M, then $H^1_c(G, K_0/K)$ classifies covering expansions M'' of M' with $\mathrm{Aut}(M''/M) = K$.

Higher cohomology groups (DE + Paul Hewitt, 2006)

- For profinite G-modules K, there is a reasonably nice technology of higher cohomology groups $H^n_c(G, K)$ (obey a long exact sequence, Shapiro's Lemma ...).
- $H^2_c(G, K)$ parametrizes extensions $1 \to K \to \Gamma_1 \to G \to 1$ arising from permutation groups (and is trivial iff these all split).

Theorem (DE + P Hewitt, 2006)

Suppose M is saturated and there is a (global) type p definable over $\emptyset$ with the property that for every finite p-Morley sequence $\bar{a}$ and finite tuple c in M, there is a finite p-Morley sequence extending $\bar{a}$ with c in its definable closure. Let $G = \mathrm{Aut}(M)$ and let A be a finite abelian group, considered as a trivial G-module. Then for $n \geq 1$

$$H^n_c(G, A) = \{0\}.$$

Back to the example

NOTATION: Let $G = \mathrm{Aut}(M) = \mathrm{Sym}(M)$ and
$\Gamma_n = \mathrm{Aut}(M_n)$ and $K_n = \mathrm{Aut}(M_n/M) = Z_2^{[M]^n}$ (write additively).
So K_n is a continuous G-module and we have a non-split ses:

$$0 \to K_n \to \Gamma_n \xrightarrow{\rho} G \to 1. \tag{1}$$

We are interested in closed subgroups $\Gamma \leq \Gamma_n$ with $\rho(\Gamma) = G$. These are automorphism groups of covering expansions of M_n: call them full subgroups of Γ_n.

BASIC METHOD:

A. Work out the closed G-submodules K of K_n

B. For $K \leq K_n$, decide whether there is a full $\Gamma \leq \Gamma_n$ with $\Gamma \cap K_n = K$.

For (B), if there is such a Γ then the (non-zero) cohomology class corresponding to the extension (1) is in the image of $H^2_c(G, K) \to H^2_c(G, K_n)$. So if $H^2_c(G, K) = 0$, this cannot happen.

Details

For $0 \leq \ell \leq n$ there exists a continuous G-homomorphism $\alpha_{\ell,k} : K_\ell \to K_n$ given by, for $w \in [M]^n$ and $f \in Z_2^{[M]^\ell}$:

$$\alpha_{\ell,n}(f)(w) = \sum_{v \in [w]^\ell} f(v).$$

Moreover, any closed G-submodule K of K_n is a sum of submodules $\mathrm{im}(\alpha_{\ell,n})$. (D Gray, 1997)

Theorem

1. *If $K < K_2$ then $H_c^2(G, K) = 0$.*
2. *For $2 \leq n$ there exists a continuous homomorphism $\gamma_{2,n} : \Gamma_2 \to \Gamma_n$ which extends $\alpha_{2,n}$ (and commutes with the ρ-maps).*
3. *Any full subgroup of Γ_n contains a Γ_n-conjugate of the image of $\gamma_{2,n}$.*

The proof of (3) also uses computation of $H_c^1(G, K_n/K)$ for various $K \leq K_n$ (E. + Gray, 1998).

5. Coda

Summary:

- Zilber's Ladder Theorem...
- ... as motivation for studying finite covers
- Group-theoretic methods for analysing these...
- ... most useful in ω-categorical case.

The group-theoretic methods give a reasonably nice general theory for finite covers (M', M) with $\mathrm{Aut}(M'/M)$ finite. In this case, M' is internal to M.

Hrushovski (2006): General description of internality in terms of definable groupoid actions.

DEFINITION: Suppose L is a first-order language and L' extends L by, amongst other things, a single extra sort C. Suppose T' is a complete L'-theory and T is the restriction of this to the L-sorts and T is fully embedded in T'. Suppose that for every $M' \models T'$ with T-part M, there is a finite tuple c in M' with $M' \subseteq \mathrm{dcl}(M, c)$. Then we say that T' is an internal cover of T.

EXAMPLE: M' is a finite cover of M and $\mathrm{Aut}(M'/M)$ is finite.

Theorem (Ehud Hrushovski, 2006)

There is a correspondence between:

- *internal covers of T, and*
- *connected $\emptyset$-definable concrete groupoids in T.*

The groupoid $\mathcal{G} = (Ob\mathcal{G}, Mor\mathcal{G})$ corresponding to an internal cover (M', M) has $\mathrm{Aut}(M'/M)$ isomorphic to $Mor\mathcal{G}(a, a)$ (for all $a \in Ob\mathcal{G}$).

Note that the correspondence applies to finite covers (M', M) with $\mathrm{Aut}(M'/M)$ finite. The group-theoretic machinery works best in this case.

QUESTIONS:

- Do other finite covers of M correspond to some sort of definable object in M?
- Can what is being done by use of group cohomology of automorphism groups be replaced by arguments involving definable sets in the structure?