MATH 4L03 Midterm Test Solutions

Midterm Test
Duration of test: 50 minutes
McMaster University
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Last Name:____________________________ First Name:________________

Student No.:______________________________

Please answer all five questions. To receive full credit, provide justifications for your answers. For all questions, write your answers in the answer booklet that has been provided. Please be sure to include your name and student number on all sheets of paper that you hand in.

No aids are allowed.

Each question is worth 5 marks; the maximum number of marks is 25.

Score

Question	1	2	3	4	5	Total
Score						

The formal system S:

Formulas: The set L consists of all propositional formulas built up from propositional variables in the set $P = \{p_0, p_1, \ldots\}$ using the connectives from $\{\neg, \rightarrow\}$.

Axioms:

Axiom 1: $(\phi \rightarrow (\psi \rightarrow \phi))$,

Axiom 2: $((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)))$,

Axiom 3: $((\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi))$,

for all formulas ϕ, ψ, and θ from L.

Rules of deduction: The Rule of Assumptions, and the Rule of Modus Ponens.

1. (a) Give the definition of a tautology in propositional logic.
   (b) Which of the following propositional formulas are tautologies? Justify your answers to receive full credit.
      i. $((p \to (q \to r)) \to (\neg p \to q))$.
      ii. $(\neg p \wedge \neg q) \leftrightarrow \neg(p \vee q)$

   Solution: For a), see text. Formula (ii) is not a tautology; the assignment that sets p and q to F does not satisfy this formula. Formula (ii) is a tautology (it is an instance of De Morgan's Law).

2. (a) Give the definition of logical implication, $\Gamma \models \phi$, in propositional logic.
   (b) Let ϕ, ψ and θ be propositional formulas. Determine which of the following statements, if any, are correct. Justify your answers.
      i. If $\phi \models \theta$ or $\psi \models \theta$ then $(\phi \vee \psi) \models \theta$.
      ii. If $\phi \models \theta$ and $\psi \models \theta$ then $(\phi \vee \psi) \models \theta$.

   Solution: For a), see text.

   (i) is not correct, since, for example, if $\phi = p$, $\psi = q$ and $\theta = p$, we have that $\phi \models \theta$ but that $(\phi \vee \psi) \not\models \theta$.

   (ii) is correct, since if ν is a truth assignment with $\nu((\phi \vee \psi)) = T$ then either $\nu(\phi) = T$ or $\nu(\psi) = T$. From the assumptions, we can then conclude that $\nu(\theta) = T$ as well.

3. Let p and q be propositional variables. Produce a deduction of $p, \neg p \vdash q$ within the system S. You may not use any meta-theorems, or the Completeness Theorem, but must provide a formal deduction within S. At each step of your deduction, indicate the rule or axiom that is being used.

   Solution:

   (1) $(\neg p \to (\neg q \to \neg p))$ (Ax. 1)

   (2) $\neg p$ (Ass.)

   (3) $(\neg q \to \neg p)$ (MP, 1, 2)

   (4) $((\neg q \to \neg p) \to (p \to q))$ (Ax. 3)

(5) $(p \to q)$ (MP, 3, 4)

(6) p (Ass.)

(7) q (MP, 5, 6)

4. Let Γ be a set of propositional formulas from the set L of the formal system S.

   (a) Give the definition of Γ being inconsistent.

   (b) Prove that if Γ is an infinite inconsistent set, then there is some finite subset Δ of Γ that is also inconsistent. In your solution, you may not use the Completeness or Compactness Theorems.

   Solution: For a), see text. For (b), since Γ is inconsistent, then there is some formula θ with $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be a deduction for $\Gamma \vdash \theta$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a deduction for $\Gamma \vdash \neg\theta$. Let Δ be the set of all formulas in these two deductions that belong to the set Γ. So Δ is finite and contains all of the formulas from Γ that are used as assumptions in the two deductions. But then $\Delta \vdash \theta$ and $\Delta \vdash \neg\theta$, using the same deductions. Thus Δ is a finite subset of Γ that is inconsistent.

5. (a) State the Compactness Theorem for propositional logic.

   (b) Suppose that $\Sigma = \{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ is an infinite set of propositional formulas that is **not** satisfiable. Show that there is some natural number N such that the formula

   $$\bigwedge_{i=1}^{N} \phi_i$$

   is a contradiction (i.e., is not satisfiable). Hint: Consider using the contra-positive statement of the Compactness Theorem from part a).

   Solution: For a), see text.

   For (b), the contra-positive version of the Compactness Theorem states that if Σ is an infinite set of formulas that is not satisfiable, then there is some finite subset Δ of Σ that is not satisfiable. So for the given set Σ, we know that there is some finite subset Δ of Σ that isn't satisfiable. Each member of Δ is some formula $\phi_k \in \Sigma$ for some $k \geq 1$. Let N be the largest natural number such that $\phi_N \in \Delta$. Since Δ is finite, such a number N exists.

So $\Delta \subseteq \{\phi_1, \phi_2, \ldots, \phi_N\}$ and since Δ is not satisfiable, the set $\{\phi_1, \phi_2, \ldots, \phi_N\}$ also is not satisfiable. That is, there is no truth assignment ν with $\nu(\phi_i) = T$ for all $1 \leq i \leq N$. But this is equivalent to having that the formula $\bigwedge_{i=1}^{N} \phi_i$ is not satisfiable. In other words, that $\bigwedge_{i=1}^{N} \phi_i$ is a contradiction.