MATH 4L03 Assignment #6 Solutions

Due: Wednesday, December 4, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

You may not use the Completeness Theorem in this assignment to show that deductions of various formulas exist. Other theorems, such as the Soundness Theorem and any meta-theorem can be used in your solutions, unless otherwise indicated.

To shorten some of the deductions you may apply the Gen rule to more than one quantifier at a time. For example if the formula $\phi(x, y)$ has been deduced, and the variables x and y don't appear freely in any of the assumptions used in the deduction, then you can deduce $\forall x \forall y \phi(x, y)$ in one step.

Similarly, you may use a variant of axiom 4 that handles more than one quantifier at a time. For example if $\phi(x, y, z)$ is a formula and τ_1, τ_2 and τ_3 are terms that are freely substitutable for x, y, and z respectively, then the following formula can be treated as an instance of Axiom 4:

$$(\forall x \forall y \forall z \phi(x, y, z) \to \phi(\tau_1, \tau_2, \tau_3)).$$

1. Let L be the first order language that has a single 2 place relation symbol E. Let

$$\Gamma = \{\forall x E(x, x), \forall x \forall y \forall z\, (E(x, y) \to (E(y, z) \to E(z, x)))\}.$$

Produce a deduction for $\Gamma \vdash \forall x \forall y\, (E(x, y) \to E(y, x))$. You may use any of the meta-theorems from Section 5.2 of the text.

In assignment #4, you were asked to show that a minor variant of Γ logically implies this sentence. Now you are asked to show that this sentence can be deduced from Γ.

Solution:

1 $\forall x E(x, x)$ (Ass.)

2 $(\forall x E(x,x) \to E(x,x))$ (Ax 4)

3 $E(x,x)$ (MP 1, 2)

4 $\forall x \forall y \forall z\, (E(x,y) \to (E(y,z) \to E(z,x)))$ (Ass.)

5 $\forall x \forall y \forall z\, (E(x,y) \to (E(y,z) \to E(z,x))) \to (E(x,x) \to (E(x,y) \to E(y,x)))$ (Ax 4, 3 times)

6 $(E(x,x) \to (E(x,y) \to E(y,x)))$ (MP 4, 5)

7 $E(x,y) \to E(y,x)$ (MP 3, 6)

8 $\forall y(E(x,y) \to E(y,x))$ (Gen. 7)

9 $\forall x \forall y(E(x,y) \to E(y,x))$ (Gen. 8)

2. Let L be the first order language with two 1-place relation symbols A and B.

   (a) Show that there is a deduction for

   $$\vdash \forall x(A(x) \to B(x)) \to (\forall x A(x) \to \forall x B(x)).$$

   Hint: Use the deduction theorem twice.

   Solution: Using the Deduction Theorem twice, it suffices to show that

   $$\forall x(A(x) \to B(x)), \forall x A(x) \vdash \forall x B(x).$$

   1 $\forall x(A(x) \to B(x))$ (Ass.)

   2 $\forall x A(x)$ (Ass.)

   3 $\forall x(A(x) \to B(x)) \to (A(x) \to B(x))$ (Ax 4)

   4 $(A(x) \to B(x))$ (MP 1, 3)

   5 $\forall x A(x) \to A(x)$ (Ax 4)

   6 $A(x)$ (MP 2, 5)

   7 $B(x)$ (MP 4, 6)

   8 $\forall x B(x)$ (Gen. 7)

   (b) Show that there is a deduction for

   $$\vdash \forall x(A(x) \to B(x)) \to (\exists x A(x) \to \exists x B(x)).$$

   Solution: We first show that there is a deduction for

   $$\forall x(A(x) \to B(x)), \forall x \neg B(x) \vdash \forall x \neg A(x).$$

1 $\forall x(A(x) \to B(x))$ (Ass.)
2 $\forall x(A(x) \to B(x)) \to (A(x) \to B(x))$ (Ax 4)
3 $A(x) \to B(x)$ (MP 1, 2)
4 $\forall x \neg B(x)$ (Ass.)
5 $\forall x \neg B(x) \to \neg B(x)$ (Ax 4)
6 $\neg B(x)$ (MP 4, 5)
7 $(A(x) \to B(x)) \to (\neg B(x) \to \neg A(x))$ (Tautology)
8 $(\neg B(x) \to \neg A(x))$ (MP 3, 7)
9 $\neg A(x)$ (MP 6, 8)
10 $\forall x \neg A(x)$ (Gen. 9)

From this deduction, we get, from the Deduction Theorem, that

$$\forall x(A(x) \to B(x)) \vdash \forall x \neg B(x) \to \forall x \neg A(x).$$

Using that

$$\vdash (\forall x \neg B(x) \to \forall x \neg A(x)) \to (\neg \forall x \neg A(x) \to \neg \forall x \neg B(x)),$$

since this formula is an instance of a tautology, we can obtain the desired deduction by one more application of Modus Ponens, since $(\exists x A(x) \to \exists x B(x))$ is an abbreviation for $\neg \forall x \neg A(x) \to \neg \forall x \neg B(x)$.

(c) Is there a deduction for

$$\vdash [\forall x A(x) \to \forall x B(x)] \to [\forall x(A(x) \to B(x))]?$$

Solution: No, since this formula is not universally valid. To see this, consider the structure $\mathcal{C} = (\{0,1\}, A^{\mathcal{C}}, B^{\mathcal{C}})$ with $A^{\mathcal{C}} = \{0\}$ and $B^{\mathcal{C}} = \{1\}$. Then $\mathcal{C} \models \forall x A(x) \to \forall x B(x)$ but $\mathcal{C} \not\models \forall x(A(x) \to B(x))$.

3. Let L be the first order language that has one 2-place relation symbol E and one 1-place function symbol g. Show that there is a deduction of

$$\vdash (\forall x E(g(x), x) \to \forall x \exists y E(y, x)).$$

Hint: Use the proof by contradiction meta-theorem and the deduction theorem.

Solution: We show that $\forall x E(g(x), x) \vdash \forall x \exists y E(y, x)$ and then use the Deduction Theorem. To show this new deduction, we first show that

$$\forall x E(g(x), x), \forall y \neg E(y, x) \vdash E(g(x), x)$$

and that

$$\forall x E(g(x), x), \forall y \neg E(y, x) \vdash \neg E(g(x), x)$$

then apply the proof by contradiction meta-theorem to conclude that $\forall x E(g(x), x) \vdash \neg \forall y \neg E(y, x)$ and then apply the Generalization Rule to get that $\forall x E(g(x), x) \vdash \forall x \exists y E(y, x)$, as required.

The following is a deduction for the first one:

1 $\forall x E(g(x), x)$ (Ass.)

2 $(\forall x E(g(x), x) \rightarrow E(g(x), x))$ (Ax 4)

3 $E(g(x), x)$ (MP 1, 2)

and the following is a deduction for the second one:

1 $\forall y \neg E(y, x)$ (Ass.)

2 $(\forall y \neg E(y, x) \rightarrow \neg E(g(x), x))$ (Ax 4)

3 $\neg E(g(x), x)$ (MP 1, 2)

4. Let G be the set of group axioms found on page 217 of the text (at the start of section 5.1). Produce a deduction of

$$G \vdash \forall x \forall y (((x + (-x)) + y) = y).$$

Solution:

1 $\forall x (x + (-x) = 0)$ (G Axiom)

2 $\forall x (x + (-x) = 0) \rightarrow (x + (-x) = 0)$ (Ax 4)

3 $(x + (-x) = 0)$ (MP 1, 2)

4 $(x + (-x) = 0) \rightarrow (x + y = x + y \rightarrow (x + (-x)) + y = 0 + y)$ (Thm 5.10)

5 $(x + y = x + y \rightarrow (x + (-x)) + y = 0 + y)$ (MP 3, 4)

6 $\forall x(x = x)$ (Ax 6)

7 $(\forall x(x = x) \to x + y = x + y)$ (Ax 4)

8 $x + y = x + y$ (MP 6, 7)

9 $(x + (-x)) + y = 0 + y$ (MP 5, 8)

10 $\forall x(0 + x = x)$ (G Axiom)

11 $(\forall x(0 + x = x) \to 0 + y = y)$ (Ax 4)

12 $0 + y = y$ (MP 10, 11)

13 $((x + (-x)) + y = 0 + y \to (0 + y = y \to (x + (-x)) + y = y))$ (Thm. 5.9)

14 $(0 + y = y \to (x + (-x)) + y = y)$ (MP 9, 13)

15 $(x + (-x)) + y = y$ (MP 12, 14)

16 $\forall y(x + (-x)) + y = y$ (Gen. 15)

17 $\forall x \forall y(x + (-x)) + y = y$ (Gen. 16)