MATH 4L03 Assignment #5 Solutions

Due: Friday, November 22, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

Unless otherwise noted, you may argue informally about the satisfaction (or not) of formulas by structures, rather than working through, in each case, the formal definition of satisfaction given in the textbook.

1. Show that the following logical implications are not valid by constructing suitable structures to witness this:

   (a) $\forall x \exists y F(y) = x \models \forall x \forall y (F(x) = F(y) \rightarrow x = y)$

   Solution: The sentence $\forall x \exists y F(y) = x$ asserts that the function F is onto, while the sentence $\forall x \forall y (F(x) = F(y) \rightarrow x = y)$ asserts that the function F is one to one. So to show that this logical implication is not valid, we just need to find a structure $\mathcal{A} = (A, F^{\mathcal{A}})$ such that $F^{\mathcal{A}} : A \rightarrow A$ is onto but not one to one. For this sort of function, A must be infinite. As an example, let $A = \mathbb{N}$ and for $n \in \mathbb{N}$, let $F^{\mathcal{A}}(n)$ be $n/2$ if n is even, and $(n-1)/2$ is n is odd. Then this function is onto, but not one to one.

   (b) $\{\forall x (P(x) \rightarrow Q(x)), \forall x (P(x) \rightarrow R(x))\} \models \exists x (Q(x) \wedge R(x))$

   Solution: The sentences $\{\forall x (P(x) \rightarrow Q(x)), \forall x (P(x) \rightarrow R(x))\}$ assert that the subset defined by P is contained in the subsets defined by Q and by R. The sentence $\exists x (Q(x) \wedge R(x))$ asserts that the intersection of the subsets defined by Q and by R is non-empty. To invalidate this, and also satisfy the first two sentences, it follows that the subset defined by P must be empty. Here is a structure that works: $\mathcal{A} = (\{0\}, P^{\mathcal{A}}, Q^{\mathcal{A}}, R^{\mathcal{A}})$, where $P^{\mathcal{A}}$, $Q^{\mathcal{A}}$, $R^{\mathcal{A}}$ are all the empty set.

2. Let ϕ and ψ be first order formulas such that the variable x does not occur freely in ϕ. Show that the formulas $\forall x (\phi \rightarrow \psi)$ and $(\phi \rightarrow \forall x \psi)$ are logically equivalent.

Solution:

Let $\mathcal{A}$ be a structure and $\vec{a}$ a list of elements from A such that

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} \forall x(\phi \to \psi).$$

We want to show that

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} (\phi \to \forall x\psi).$$

If $\mathcal{A} \not\models_{[\vec{x}/\vec{a}]} \phi$ then by the interpretation of the connective $\to$ it follows that

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} (\phi \to \forall x\psi)$$

as required.

On the other hand, if $\mathcal{A} \models_{[\vec{x}/\vec{a}]} \phi$, then we need to show that $\mathcal{A} \models_{[\vec{x}/\vec{a}]} \forall x\psi$ as well. To do this, let $b \in A$ be any element. By the interpretation of the $\forall x$ quantifier, we need to show that $\mathcal{A} \models_{[\vec{x}/\vec{a}][x/b]} \psi$.

Since $\mathcal{A} \models_{[\vec{x}/\vec{a}]} \forall x(\phi \to \psi)$ then $\mathcal{A} \models_{[\vec{x}/\vec{a}][x/b]} (\phi \to \psi)$. Using Theorem 4.1 we can conclude from $\mathcal{A} \models_{[\vec{x}/\vec{a}]} \phi$ that $\mathcal{A} \models_{[\vec{x}/\vec{a}][x/b]} \phi$ too. Finally, from $\mathcal{A} \models_{[\vec{x}/\vec{a}][x/b]} (\phi \to \psi)$ it follows that $\mathcal{A} \models_{[\vec{x}/\vec{a}][x/b]} \psi$ as required.

This establishes that $\forall x(\phi \to \psi)$ logically implies $(\phi \to \forall x\psi)$. A similar argument (in reverse) can be used to show that $(\phi \to \forall x\psi)$ logically implies $\forall x(\phi \to \psi)$. Note that since $(\forall x(\phi \to \psi) \to (\phi \to \forall x\psi))$ is an axiom of our system (when x isn't free in ϕ) it is important to verify that this formula is universally valid. This is what this question has established.

3. Determine which of the following sentences are logically valid:

   (a) $\exists x(U(x) \to \forall y U(y))$.

   Solution: This sentence is logically valid. Let $\mathcal{A}$ be any structure for this language. If $U^{\mathcal{A}}$ is a proper subset of A, then there is some $a \in A$ with $\mathcal{A} \models_{[x/a]} \neg U(x)$. But then $\mathcal{A} \models_{[x/a]} (U(x) \to \forall y U(y))$ and so $\mathcal{A} \models \exists x(U(x) \to \forall y U(y))$. If $U^{\mathcal{A}} = A$, then $\mathcal{A} \models \forall y U(y)$ and so $\mathcal{A} \models \exists x(U(x) \to \forall y U(y))$.

(b) $(\forall x(U(x) \vee V(x)) \rightarrow (\forall x U(x) \vee \exists x V(x)))$.

Solution: This sentence is also logically valid. Let $\mathcal{A}$ be a structure for this language and consider the subset $U^{\mathcal{A}} \cup V^{\mathcal{A}}$ of A. If this is a proper subset of A, then $\mathcal{A} \not\models \forall x(U(x) \vee V(x))$ and so

$$\mathcal{A} \models (\forall x(U(x) \vee V(x)) \rightarrow (\forall x U(x) \vee \exists x V(x))).$$

If this subset is equal to A then either $U^{\mathcal{A}} = A$ or for some $a \in A$, $a \notin U^{\mathcal{A}} = A$. In the former case, we conclude that $\mathcal{A} \models \forall x U(x)$ and so $\mathcal{A} \models (\forall x U(x) \vee \exists x V(x)))$ as required. In the latter case, since $U^{\mathcal{A}} \cup V^{\mathcal{A}} = A$ then the element a must belong to $V^{\mathcal{A}}$ and so $\mathcal{A} \models \exists x V(x)$ and hence $\mathcal{A} \models (\forall x U(x) \vee \exists x V(x)))$.

4. Let LO be the theory of linear orders in the language that has equality and the 2-place relation symbol $\leq$. Consider the sentences

$$\alpha : \forall x \exists y (x \leq y \wedge \neg x = y \wedge \forall z((x \leq z \wedge \neg x = z) \rightarrow y \leq z)),$$

$$\beta : \forall x \exists y (y \leq x \wedge \neg x = y \wedge \forall z((z \leq x \wedge \neg x = z) \rightarrow z \leq y)),$$

and

$$\gamma : \forall x (\exists y (x \leq y \wedge \neg x = y) \wedge \exists y (y \leq x \wedge \neg x = y)).$$

(a) Find a model of $LO \cup \{\alpha, \beta\}$

(b) Find a model of $LO \cup \{\alpha, \neg\beta, \neg\gamma\}$

(c) Find a model of $LO \cup \{\alpha, \neg\beta, \gamma\}$.

Solution: The sentence α asserts that for every element a in the order, there is a least element b that lies strictly above a. β asserts something similar, namely that for every element a in the order, there is a greatest element b that lies strictly below a. γ asserts that there is no least or largest element in the linear order.

So, the linear order on $\mathbb{Z}$ is a model of α and β and the linear order on $\mathbb{N}$ is a model of $\{\alpha, \neg\beta, \neg\gamma\}$. A model of $LO \cup \{\alpha, \neg\beta, \gamma\}$ is the structure with universe $\mathbb{Z} \cup \{-1/n : n \in \mathbb{N}, n > 0\}$ with the usual ordering $\leq$.