MATH 4L03 Assignment #5

Due: Friday, November 22, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

Unless otherwise noted, you may argue informally about the satisfaction (or not) of formulas by structures, rather than working through, in each case, the formal definition of satisfaction given in the textbook.

1. Show that the following logical implications are not valid by constructing suitable structures to witness this:

   (a) $\forall x \exists y F(y) = x \models \forall x \forall y (F(x) = F(y) \rightarrow x = y)$

   (b) $\{\forall x (P(x) \rightarrow Q(x)), \forall x (P(x) \rightarrow R(x))\} \models \exists x (Q(x) \wedge R(x))$

2. Let ϕ and ψ be first order formulas such that the variable x does not occur freely in ϕ. Show that the formulas $\forall x(\phi \rightarrow \psi)$ and $(\phi \rightarrow \forall x \psi)$ are logically equivalent.

3. Determine which of the following sentences are logically valid:

   (a) $\exists x (U(x) \rightarrow \forall y U(y))$.

   (b) $(\forall x (U(x) \vee V(x)) \rightarrow (\forall x U(x) \vee \exists x V(x)))$.

4. Let LO be the theory of linear orders in the language that has equality and the 2-place relation symbol $\leq$. Consider the sentences

   $$\alpha : \forall x \exists y (x \leq y \wedge \neg x = y \wedge \forall z ((x \leq z \wedge \neg x = z) \rightarrow y \leq z)),$$

   $$\beta : \forall x \exists y (y \leq x \wedge \neg x = y \wedge \forall z ((z \leq x \wedge \neg x = z) \rightarrow z \leq y)),$$

   and

   $$\gamma : \forall x (\exists y (x \leq y \wedge \neg x = y) \wedge \exists y (y \leq x \wedge \neg x = y)).$$

   (a) Find a model of $LO \cup \{\alpha, \beta\}$

   (b) Find a model of $LO \cup \{\alpha, \neg\beta, \neg\gamma\}$

   (c) Find a model of $LO \cup \{\alpha, \neg\beta, \gamma\}$.