MATH 4L03 Assignment #4

Due: Friday, November 8, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

Unless otherwise noted, you may argue informally about the satisfaction (or not) of formulas by structures, rather than working through, in each case, the formal definition of satisfaction given in the textbook.

1. Do exercise 4.11 (b) on page 150 of the textbook.
2. Do exercise 4.12 from page 150 of the textbook.
3. Do exercise 4.13 from page 151 of the textbook.
4. Let L be the first order language that has a single 2 place relation symbol E. Show that if $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ is an L-structure that satisfies the following two sentences:

$$\forall x\,(E(x,x))$$

$$\forall x \forall y \forall z\,((E(x,y) \wedge E(y,z)) \rightarrow E(z,x))$$

   then it also satisfies the sentence

$$\forall x \forall y\,(E(x,y) \rightarrow E(y,x))\,.$$

   Note that this provides a slightly shorter axiomatization of the class of equivalence relation structures.

5. (a) Using the same language L as in the previous question, show that there is no L-structure $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ which satisfies the sentences:

$$\exists x \forall y\,(E(x,y))$$

$$\exists x \forall y\,(\neg E(x,y))$$

$$\forall x \forall y((E(x,y) \rightarrow E(y,x))\,.$$

(b) Is there an L-structure $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ that satisfies the sentences:

$$\forall x \exists y \, (E(x, y))$$

$$\forall x \forall y \, (E(x, y) \to \neg E(y, x))$$

$$\forall x \forall y \forall z \, ((E(x, y) \wedge E(y, z)) \to E(x, z))?$$

6. Let $\underline{\mathbb{N}}$ be the structure $\langle \mathbb{N}, +, \cdot, 0, 1 \leq \rangle$, where the function symbols $+$ and $\cdot$, the constant symbols 0 and 1 and the predicate symbol $\leq$ have their usual interpretations on the set of natural numbers. Determine which of the following sentences are satisfied by the structure $\underline{\mathbb{N}}$:

   i) $\forall x \exists y (x = (y + y) \vee x = (y + y) + 1)$,
   ii) $\forall x \forall y \exists z ((x + z) = y)$
   iii) $\forall x \forall y (x \leq y \leftrightarrow \exists z ((x + z) = y))$.

7. Let L be a first order language with equality. For each natural number n, find sentences α_n, β_n and γ_n such that for all normal L-structures $\mathcal{A}$:

   (a) $\mathcal{A} \models \alpha_n$ iff A has exactly n elements,
   (b) $\mathcal{A} \models \beta_n$ iff A has at least n elements,
   (c) $\mathcal{A} \models \gamma_n$ iff A has at most n elements.

   Find a set Σ of sentences such that a normal L-structure $\mathcal{A} \models \Sigma$ iff A is infinite. Note that the set Σ must consist of infinitely many sentences (we will prove this later).

The following are bonus questions. The first was originally included with Assignment #3.

B1 In this problem, we assume that McMaster has a **countably infinite** number of students $S = \{s_0, s_1, \ldots, s_n, \ldots\}$ and that C is the set of courses that are on offer to them. Due to resource limitations, each student in S will be assigned to exactly one class from C. Also, each course $c \in C$ has its enrolment capped at some finite number e_c. Each student $s \in S$ provides a **finite** set $C_s \subseteq C$ of the courses that they are willing to register in.

   For $A \subseteq S$, a function $\alpha : A \to C$ is a **good** assignment for A if

- For each $s \in A$, $\alpha(s) \in C_s$ (so α assigns to s one of the courses they selected), and
- for each class $c \in C$, $|\alpha^{-1}(c)| \leq e_c$ (so no class is over-enrolled by α).

Suppose that for each **finite** subset A of S there is some good assignment $\alpha : A \to C$ for A. Prove that there is some good assignment $\alpha : S \to C$ for the entire set S. In your solution you should formulate this situation within propositional logic and then use the Compactness Theorem.

B2 Do exercise 4.24 from the textbook.