MATH 4L03 Assignment #3

Due: Friday, October 11, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

1. Use the proof system S from the text and the lectures to show the following. You may not use the Completeness Theorem to solve these, i.e., you can't work with $\models$ in place of $\vdash$. You may use any meta-theorem that was proved in the lectures, in particular the Deduction Theorem and the Proof by Contradiction Theorem.

   (a) $\neg p \vdash (p \to q)$. Show this without using any meta-theorem, i.e., provide a complete derivation for this.

   (b) $\vdash ((\phi \to (\psi \to \theta)) \to (\psi \to (\phi \to \theta)))$,

   (c) $\vdash ((\phi \to \psi) \to (\neg\psi \to \neg\phi))$,

   (d) $\vdash (\phi \to (\neg\theta \to \neg(\phi \to \theta)))$,

   (e) If $\Gamma, \phi \vdash \neg\psi$ then $\Gamma, \psi \vdash \neg\phi$.

2. Is the following true for all formulas ϕ, ψ, and θ?

$$\vdash ((\phi \to (\phi \to \neg\theta)) \to (\psi \to \theta)).$$

3. Let Γ be a set of formulas. Prove that the following statement are equivalent:

   (a) Γ is inconsistent,

   (b) $\Gamma \vdash \neg(\phi \to \phi)$ for **all** formulas ϕ,

   (c) $\Gamma \vdash \neg(\phi \to \phi)$ for **some** formula ϕ.

4. In this question, all of the usual connectives, $S = \{\neg, \wedge, \vee, \to, \leftrightarrow\}$, may appear in the formulas in question, and the propositional variables that appear come from the infinite set $P = \{p_1, p_2, \ldots\}$, i.e., we are considering formulas from $Form(P, S)$. Let V be the set of all truth assignments for the set of propositional variables from P, i.e., $V = \{\nu : P \to \{T, F\}\}$.

   For ϕ a formula, let $X_\phi = \{\nu \in V \mid \nu(\phi) = T\}$.

(a) Show that there exist formulas ϕ and θ such that $X_\phi = V$ and X_θ is the empty set.

(b) Given formulas ϕ and θ and $\nu \in X_\phi \cap X_\theta$ show that there is a formula γ such that $\nu \in X_\gamma$ and $X_\gamma \subseteq X_\phi \cap X_\theta$.

(c) Let ϕ be a formula. Show that there is some formula θ such that $X_\theta = V \setminus X_\phi$, i.e., X_θ is the complement of X_ϕ in V.

(d) Let Σ be a set of formulas such that

$$\bigcap_{\phi \in \Sigma} X_\phi = \emptyset,$$

i.e., the intersection of the X_ϕ for $\phi \in \Sigma$ is the empty set. Prove that for some natural number n, there are $\phi_0, \phi_1, \ldots, \phi_{n-1} \in \Sigma$ such that

$$X_{\phi_0} \cap X_{\phi_1} \cap \ldots \cap X_{\phi_{n-1}} = \emptyset.$$

HINT: Use the Compactness Theorem.

5. The following is a simple derivation of the formula q from the set $\{p, (p \to q)\}$:

(1)	p	Ass.
(2)	$p \to q$	Ass.
(3)	q	MP, 1, 2.

Use the proof of the Deduction Theorem to convert the above derivation into a derivation of

$$p \vdash (p \to q) \to q.$$

[The proof of the Deduction Theorem can be regarded as a description of a procedure that takes as input a derivation of $\Gamma, A \vdash B$ and produces as output a derivation of $\Gamma \vdash (A \to B)$.]

This will be added as a bonus question for assignment #4, since we won't cover the related course material until Thursday, 10/10.

BONUS In this problem, we assume that McMaster has a **countably infinite** number of students $S = \{s_0, s_1, \ldots, s_n, \ldots\}$ and that C is the set of courses that are on offer to them. Due to resource limitations, each student in S will be assigned to exactly one class from C. Also, each course $c \in C$ has its enrolment capped at some finite number e_c. Each student $s \in S$ provides a **finite** set $C_s \subseteq C$ of the courses that they are willing to register in.

For $A \subseteq S$, a function $\alpha : A \to C$ is a **good** assignment for A if

- For each $s \in A$, $\alpha(s) \in C_s$ (so α assigns to s one of the courses they selected), and
- for each class $c \in C$, $|\alpha^{-1}(c)| \leq e_c$ (so no class is over-enrolled by α).

Suppose that for each **finite** subset A of S there is some good assignment $\alpha : A \to C$ for A. Prove that there is some good assignment $\alpha : S \to C$ for the entire set S. In your solution you should formulate this situation within propositional logic and then use the Compactness Theorem.