MATH 4L03 Assignment #3 Solutions

Due: Friday, October 11, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

1. Use the proof system S from the text and the lectures to show the following. You may not use the Completeness Theorem to solve these, i.e., you can't work with $\models$ in place of $\vdash$. You may use any meta-theorem that was proved in the lectures, in particular the Deduction Theorem and the Proof by Contradiction Theorem.

   (a) $\neg p \vdash (p \to q)$. Show this without using any meta-theorem, i.e., provide a complete derivation for this.

   Solution:

   i. $\neg p \to (\neg q \to \neg p)$ Ax. 1
   ii. $\neg p$ Ass.
   iii. $(\neg q \to \neg p)$ MP. 1, 2
   iv. $(\neg q \to \neg p) \to (p \to q)$ Ax. 3
   v. $(p \to q)$ MP. 3, 4

   (b) $\vdash ((\phi \to (\psi \to \theta)) \to (\psi \to (\phi \to \theta)))$,

   Solution: We can use the Deduction Theorem three times to prove this. It suffices to show that $(\phi \to (\psi \to \theta)), \psi, \phi \vdash \theta$

   i. $(\phi \to (\psi \to \theta))$ Ass.
   ii. ϕ Ass.
   iii. $\psi \to \theta$ MP. 1, 2
   iv. ψ Ass.
   v. θ MP., 3, 4

   So, by the Deduction Theorem, $(\phi \to (\psi \to \theta)), \psi \vdash \phi \to \theta$, and then $(\phi \to (\psi \to \theta)) \vdash \psi \to (\phi \to \theta)$, and finally $\vdash ((\phi \to (\psi \to \theta)) \to (\psi \to (\phi \to \theta)))$.

(c) $\vdash ((\phi \to \psi) \to (\neg\psi \to \neg\phi))$,

Solution: We will use the Deduction Theorem (twice), Proof by Contradiction, and the fact that from an inconsistent set of formulas, any formulas can be deduced (this was shown in class). By the Deduction Theorem, applied twice, it suffices to show that $(\phi \to \psi), \neg\psi \vdash \neg\phi$.

To show this, it will suffice to show, using Proof by Contradiction (the version at the bottom of page 95 of the textbook), that $(\phi \to \psi), \neg\psi, \phi \vdash \neg\psi$ and that $(\phi \to \psi), \neg\psi, \phi \vdash \psi$, since from this we can conclude that $(\phi \to \psi), \neg\psi \vdash \neg\phi$, as required.

Clearly $(\phi \to \psi), \neg\psi, \phi \vdash \phi$ and that $(\phi \to \psi), \neg\psi, \phi \vdash \neg\psi$. Then, using Modus Ponens, we can see that $(\phi \to \psi), \neg\psi, \phi \vdash \psi$. The result follows by applying the Proof by Contradiction meta-theorem (bottom of page 95 version).

Note: a good exercise is to verify that this alternate proof by contradiction meta-theorem can be established using the original version and some other results from the lectures.

(d) $\vdash (\phi \to (\neg\theta \to \neg(\phi \to \theta)))$,

Solution: We can use the Deduction Theorem and Proof by Contradiction (alternate version) to show this. By the Deduction Theorem it suffices to show that $\phi, \neg\theta \vdash \neg(\phi \to \theta)$ and using Proof by Contradiction that the set $\{\phi, \neg\theta, (\phi \to \theta)\}$ is inconsistent (since then we can conclude that $\phi, \neg\theta \vdash \neg(\phi \to \theta)$, as required).

Clearly, $\{\phi, \neg\theta, (\phi \to \theta)\}$ is inconsistent since we can deduce both $\neg\theta$ (it is an assumption) and θ (using Modus Ponens applied to the other two members of the set).

(e) If $\Gamma, \phi \vdash \neg\psi$ then $\Gamma, \psi \vdash \neg\phi$.

Solution: We use Proof by Contradiction (alternate version) for this: it suffices to show that $\Gamma \cup \{\psi, \phi\}$ is inconsistent, from which we can conclude that $\Gamma, \psi \vdash \neg\phi$, as required. But from $\Gamma \cup \{\psi, \phi\}$ we can deduce ψ and also $\neg\psi$ (since $\Gamma, \phi \vdash \neg\psi$), so this set is inconsistent.

2. Is the following true for all formulas ϕ, ψ, and θ?

$$\vdash ((\phi \to (\phi \to \neg\theta)) \to (\psi \to \theta)).$$

Solution: By the Soundness Theorem, we just need to show that this formula is not a tautology for some ϕ, θ, and ψ. If we set $\phi = p$, $\theta = q$, and $\psi = r$ and let ν be the truth assignment with $\nu(p) = F$, $\nu(q) = F$, and $\nu(r) = T$ we see that ν does not satisfy the given formula, so it is not a tautology.

3. Let Γ be a set of formulas. Prove that the following statement are equivalent:

   (a) Γ is inconsistent,

   (b) $\Gamma \vdash \neg(\phi \to \phi)$ for **all** formulas ϕ,

   (c) $\Gamma \vdash \neg(\phi \to \phi)$ for **some** formula ϕ.

Solution: To see that (a) implies (b), note that from an inconsistent set of formulas, every formula can be deduced (see Theorem 3.5), so (b) holds. Condition (b) implies (c), trivially. Now, suppose that (c) holds. To show that Γ is inconsistent, we just need to show that $\Gamma \vdash (\phi \to \phi)$, since we would have that from Γ we can deduce some formula $((\phi \to \phi))$ and its negation (this is the definition of being inconsistent). But using the Deduction Theorem, we see that from $\phi \vdash \phi$ we get that $\vdash (\phi \to \phi)$ and so that $\Gamma \vdash (\phi \to \phi)$, as claimed. Note that in class we proved directly that $\vdash (\phi \to \phi)$, without using the Deduction Theorem (since our proof of the Deduction Theorem made use of this fact).

4. In this question, all of the usual connectives, $S = \{\neg, \wedge, \vee, \to, \leftrightarrow\}$, may appear in the formulas in question, and the propositional variables that appear come from the infinite set $P = \{p_1, p_2, \dots\}$, i.e., we are considering formulas from $Form(P, S)$. Let V be the set of all truth assignments for the set of propositional variables from P, i.e., $V = \{\nu : P \to \{T, F\}\}$.

   For ϕ a formula, let $X_\phi = \{\nu \in V \mid \nu(\phi) = T\}$.

(a) Show that there exist formulas ϕ and θ such that $X_\phi = V$ and X_θ is the empty set.

Solution: If we set ϕ to be any tautology, then $X_\phi = V$ and $X_{\neg\phi} = \emptyset$.

(b) Given formulas ϕ and θ and $\nu \in X_\phi \cap X_\theta$ show that there is a formula γ such that $\nu \in X_\gamma$ and $X_\gamma \subseteq X_\phi \cap X_\theta$.

Solution: Let $\gamma = (\phi \wedge \theta)$. Since $\nu \in X_\phi \cap X_\theta$ then $\nu(\gamma) = T$ and so $\nu \in X_\gamma$. If $\nu \in X_\gamma$ then $\nu(\gamma) = T$ so $\nu(\phi) = \nu(\theta) = T$. Then $\nu \in X_\phi \cap X_\theta$, which shows that $X_\gamma \subseteq X_\phi \cap X_\theta$. In fact, $X_\gamma = X_\phi \cap X_\theta$.

(c) Let ϕ be a formula. Show that there is some formula θ such that $X_\theta = V \setminus X_\phi$, i.e., X_θ is the complement of X_ϕ in V.

Solution: The formula $\theta = \neg\phi$ will work.

(d) Let Σ be a set of formulas such that

$$\bigcap_{\phi \in \Sigma} X_\phi = \emptyset,$$

i.e., the intersection of the X_ϕ for $\phi \in \Sigma$ is the empty set. Prove that for some natural number n, there are $\phi_0, \phi_1, \ldots, \phi_{n-1} \in \Sigma$ such that

$$X_{\phi_0} \cap X_{\phi_1} \cap \cdots \cap X_{\phi_{n-1}} = \emptyset.$$

HINT: Use the Compactness Theorem.

Solution: Suppose that the intersection $X_{\phi_0} \cap X_{\phi_1} \cap \cdots \cap X_{\phi_{n-1}}$ is non-empty for every finite subset $\{\phi_0, \phi_1 \ldots, \phi_{n-1}\}$ of Γ. Then there is some truth assignment ν such that ν belongs to this intersection, which means that $\nu(\phi_j) = T$ for all $j < n$. This means that the set $\{\phi_0, \phi_1 \ldots, \phi_{n-1}\}$ is satisfiable and so the set Γ is finitely satisfiable. But then by the Compactness Theorem, Γ is satisfiable, which means that there is some ν with $\nu \in X_\phi$ for all $\phi \in \Gamma$. But then $\bigcap_{\phi \in \Sigma} X_\phi$ is non-empty, contradicting our assumption.

5. The following is a simple derivation of the formula q from the set $\{p, (p \to q)\}$:

(1)	p	Ass.
(2)	$p \to q$	Ass.
(3)	q	MP, 1, 2.

Use the proof of the Deduction Theorem to convert the above derivation into a derivation of

$$p \vdash (p \to q) \to q.$$

[The proof of the Deduction Theorem can be regarded as a description of a procedure that takes as input a derivation of $\Gamma, A \vdash B$ and produces as output a derivation of $\Gamma \vdash (A \to B)$.]

Solution: We can use the proof to construct deductions for $p \vdash ((p \to q) \to p)$, $p \vdash ((p \to q) \to (p \to q))$, and $p \vdash ((p \to q) \to q)$. For the first step, the proof of the Deduction Theorem provides the following deduction:

(1)	p	Ass.
(2)	$(p \to ((p \to q) \to p))$	Ax. 1
(3)	$((p \to q) \to p))$	MP, 1, 2.

Since the second line of the original deduction uses the Assumption Rule, applied to $(p \to q)$ then the proof of the Deduction Theorem produces the following deduction of $p \vdash ((p \to q) \to (p \to q))$ it copies the deduction in Example 3.7 (c)). To make things easier, let $\phi = (p \to q)$.

(1)	$(\phi \to ((\phi \to \phi) \to \phi))$	Ax. 1
(2)	$((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$	Ax. 2
(3)	$((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$	MP, 1, 2.
(4)	$(\phi \to (\phi \to \phi))$	Ax. 1
(5)	$(\phi \to \phi)$	MP, 3, 4

The final step, according to the proof, is to combine the first two deductions and add three new lines:

(1)	p	Ass.
(2)	$(p \to ((p \to q) \to p))$	Ax. 1
(3)	$((p \to q) \to p))$	MP, 1, 2.
(4)	$(\phi \to ((\phi \to \phi) \to \phi))$	Ax. 1
(5)	$((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$	Ax. 2
(6)	$((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$	MP, 4, 5
(7)	$(\phi \to (\phi \to \phi))$	Ax. 1
(8)	$((p \to q) \to (p \to q))$	MP, 6, 7
(9)	$(((p \to q) \to (p \to q)) \to (((p \to q) \to p) \to ((p \to q) \to q)))$	Ax. 2
(10)	$(((p \to q) \to p) \to ((p \to q) \to q))$	MP. 8, 9
(11)	$((p \to q) \to q)$	MP, 3, 10