MATH 4L03 Assignment #2

Due: Friday, September 27, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

1. For each of the following formulas find formulas that are in disjunctive normal form and conjunctive normal form that are logically equivalent to it.

   (a) $(p \wedge q) \to r$.

   (b) $(p \vee q) \wedge (\neg p \vee r)$.

   (c) $(p \vee q) \leftrightarrow c$.

2. Let α be a formula whose only connective symbols are $\neg$, $\vee$ and $\wedge$. Let α' be the formula obtained by replacing each occurrence of $\vee$ in α by $\wedge$, each occurrence of $\wedge$ in α by $\vee$ and each occurrence of a propositional variable by its negation.

   For example, if α is the formula:

   $$(p_1 \vee (\neg p_2 \wedge p_1))$$

   then α' is:

   $$(\neg p_1 \wedge (\neg\neg p_2 \vee \neg p_1)).$$

   Show, for any formula α, that α' is logically equivalent to $\neg\alpha$.

3. Let τ and ρ be formulas and Γ a set of formulas with τ a tautology.

   (a) Prove that $\Gamma \models \tau$.

   (b) Prove that $\tau \models \rho$ if and only if ρ is a tautology.

4. Let ϕ, ψ, and θ be formulas. Show that

   $$(\phi \to (\psi \to \theta)) \models ((\phi \to \psi) \to (\phi \to \theta)).$$

   Does

   $$((\phi \to \psi) \to (\phi \to \theta)) \models (\phi \to (\psi \to \theta))?$$

5. A set of formulas Σ is called **semantically closed** if:

   for every formula α, if $\Sigma \models \alpha$, then $\alpha \in \Sigma$.

   (a) Prove that the set of tautologies is semantically closed. (Hint: use problem 3 a))

   (b) Prove that if Γ is semantically closed, then it contains every tautology.

   (c) Prove that the intersection of any collection of semantically closed sets is a semantically closed set.

   (d) Prove that for every set Γ, there is a smallest set of formulas (with respect to inclusion) which contains Γ and which is semantically closed (call this set the semantic closure of Γ).

   (e) What is the semantic closure of the set $\{p, \neg p\}$?

6. Let Σ be a set of formulas and α and β be formulas.

   (a) Show that if either $\Sigma \models \alpha$ or $\Sigma \models \beta$ then $\Sigma \models (\alpha \vee \beta)$.

   (b) Show, by example, that the statement: "if $\Sigma \models (\alpha \vee \beta)$ then either $\Sigma \models \alpha$ or $\Sigma \models \beta$" is false in general.

BONUS: Suppose that $\theta \in Form(P, \{\neg, \leftrightarrow\})$. Prove that θ is a tautology if and only if every propositional variable occurs an even number of times in θ and the connective $\neg$ occurs an even number of times in θ.