MATH 4L03 Assignment #2 Solutions

Due: Friday, September 27, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

1. For each of the following formulas find formulas that are in disjunctive normal form and conjunctive normal form that are logically equivalent to it.

   (a) $(p \wedge q) \rightarrow r$.

   (b) $(p \vee q) \wedge (\neg p \vee r)$.

   (c) $(p \vee q) \leftrightarrow c$.

   Solution:

   (a) $(p \wedge q) \rightarrow r$ is logically equivalent to $\neg p \vee \neg q \vee r$ (CNF). The DNF formula equivalent to this is long. Since the formula is logically equivalent to $\neg p \vee \neg q \vee r$, then it is the disjunction of all possible conjuncts, except for the unique one, $(p \wedge q \wedge \neg r)$ that corresponds to the only truth assignment that falsifies the formula. It looks like

   $$(\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee$$
   $$(\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \vee q \vee r).$$

   (b) $(p \vee q) \wedge (\neg p \vee r)$ is logically equivalent to

   $$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r),$$

   a formula in DNF. It is logically equivalent to the following CNF formula:

   $$(p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r).$$

   (c) $(p \vee q) \leftrightarrow c$ is logically equivalent to

   $$(p \wedge q \wedge c) \vee (p \wedge \neg q \wedge c) \vee (\neg p \wedge q \wedge c) \vee (\neg p \wedge \neg q \wedge \neg c),$$

   a formula in DNF. It is logically equivalent to the following CNF formula:

   $$(p \vee q \vee \neg c) \wedge (p \vee \neg q \vee c) \wedge (\neg p \vee q \vee c) \wedge (\neg p \vee \neg q \vee c).$$

2. Let α be a formula whose only connective symbols are $\neg$, $\vee$ and $\wedge$. Let α' be the formula obtained by replacing each occurrence of $\vee$ in α by $\wedge$, each occurrence of $\wedge$ in α by $\vee$ and each occurrence of a propositional variable by its negation.

   For example, if α is the formula:

   $$(p_1 \vee (\neg p_2 \wedge p_1))$$

   then α' is:

   $$(\neg p_1 \wedge (\neg\neg p_2 \vee \neg p_1)).$$

   Show, for any formula α, that α' is logically equivalent to $\neg\alpha$.

   Solution: The fact that α' is logically equivalent to $\neg\alpha$ will be proved by induction on the length of the formula α.

   For α of length 0, say $\alpha = p_i$ we see that α' is just $\neg p_i$. So α' is actually equal to $\neg\alpha$ and the logical equivalence follows.

   Assume the fact is true for all suitable formulas of length k or less and let α be a formula of length $k+1$ whose only logical connectives are $\vee$, $\wedge$ and $\neg$. Then there are three cases to consider:

   - $\alpha = (\beta \vee \gamma)$ for some formulas β and γ. Since these formulas are shorter than α, then our inductive hypothesis applies to them, yielding that β' is logically equivalent to $\neg\beta$ and γ' is logically equivalent to $\neg\gamma$. We see that α' is just $(\beta' \wedge \gamma')$ (this actually requires another inductive argument, but should be clear), and so by our hypothesis, we get that α' is logically equivalent to $(\neg\beta \wedge \neg\gamma)$. This formula is logically equivalent (using De Morgan's Law) to $\neg(\beta \vee \gamma)$, which is just $\neg\alpha$ as required.
   - $\alpha = (\beta \wedge \gamma)$ for some formulas β and γ. This case is similar to the previous one and can be proved by repeating the previous argument with $\wedge$ in place of $\vee$ and vice versa.
   - $\alpha = \neg\beta$. Then as β has length less than α, we know by our inductive hypothesis that β' is logically equivalent to $(\neg\beta)$. Then, $\alpha' = \neg\beta'$ and so is logically equivalent to $\neg\neg\beta$. Of course, this formula is just $\neg\alpha$ and so we are done.

3. Let τ and ρ be formulas and Γ a set of formulas with τ a tautology.

(a) Prove that $\Gamma \models \tau$.

(b) Prove that $\tau \models \rho$ if and only if ρ is a tautology.

Solution: To show that $\Gamma \models \tau$ we need to show that for all assignments ν, if ν satisfies Γ then ν satisfies τ. Since τ is a tautology, then ν satisfies τ for all assignments ν, in particular, for those which also satisfy Γ. Thus $\Gamma \models \tau$.

If ρ is any formula, then from what has been established in a), $\tau \models \rho$ if ρ is a tautology. Conversely, if $\tau \models \rho$, then consider any assignment ν. We need to show that ν satisfies ρ in order to prove that ρ is a tautology. But, since τ is a tautology, we have that ν satisfies τ. From $\tau \models \rho$ we conclude that ν satisfies ρ as required.

4. Let ϕ, ψ, and θ be formulas. Show that

$$(\phi \to (\psi \to \theta)) \models ((\phi \to \psi) \to (\phi \to \theta)).$$

Does

$$((\phi \to \psi) \to (\phi \to \theta)) \models (\phi \to (\psi \to \theta))?$$

Solution: Truth tables can be used to show that both logical implications hold, i.e., that the two formulas are logically equivalent (the table will have 8 rows) or by an informal argument: If ν is a truth assignment that falsifies $((\phi \to \psi) \to (\phi \to \theta))$ we need to show that it also falsifies $(\phi \to (\psi \to \theta))$. But if ν falsifies the first statement, then it must falsify $\phi \to \theta$ and satisfy $\phi \to \psi$, so $\nu(\phi) = \nu(\psi) = T$ and $\nu(\theta) = F$. But this assignment also falsifies $(\phi \to (\psi \to \theta))$, as required.

Conversely, if ν is a truth assignment that falsifies $(\phi \to (\psi \to \theta))$ then it follows that $\nu(\phi) = \nu(\psi) = T$ and $\nu(\theta) = F$ as well. But then ν falsifies $((\phi \to \psi) \to (\phi \to \theta))$.

5. A set of formulas Σ is called **semantically closed** if:

for every formula α, if $\Sigma \models \alpha$, then $\alpha \in \Sigma$.

(a) Prove that the set of tautologies is semantically closed. (Hint: use problem 3 a))

(b) Prove that if Γ is semantically closed, then it contains every tautology.

(c) Prove that the intersection of any collection of semantically closed sets is a semantically closed set.

(d) Prove that for every set Γ, there is a smallest set of formulas (with respect to inclusion) which contains Γ and which is semantically closed (call this set the semantic closure of Γ).

(e) What is the semantic closure of the set $\{p, \neg p\}$?

Solution:

(a) Let Ω be the set of tautologies. To show that Ω is semantically closed, we must show that for any formula α, if $\Omega \models \alpha$ then α is a tautology. So, suppose that $\Omega \models \alpha$, and let ν be a truth assignment. Since every truth assignment satisfies every tautology, then ν satisfies Ω. But then $\Omega \models \alpha$ implies that ν satisfies α. Thus we have shown that every truth assignment satisfies α and so we conclude that α is a tautology.

(b) Let Γ be semantically closed and let τ be a tautology. Then from problem #3 we see that $\Gamma \models \tau$. Since Γ is assumed to be semantically closed, it follows that $\tau \in \Gamma$.

(c) Let I be some set and $\{\Gamma_i : i \in I\}$ be a collection (indexed by I) of semantically closed sets. Let Γ be the intersection of the Γ_i, $i \in I$. To show that Γ is semantically closed, suppose that $\Gamma \models \alpha$. Since $\Gamma \subseteq \Gamma_i$ for each $i \in I$ then it follows that $\Gamma_i \models \alpha$ for each $i \in I$. Since the Γ_i are semantically closed, it follows that $\alpha \in \Gamma_i$ for each $i \in I$. This implies that α is in the intersection of the Γ_i's and so $\alpha \in \Gamma$. Thus Γ is semantically closed.

(d) The set of all formulas is semantically closed, so there is at least one semantically closed set which contains Γ. Let $\overline{\Gamma}$ be the intersection of **all** semantically closed sets which contain Γ. So, $\overline{\Gamma}$ is a subset of every semantically closed set which contains Γ. From part (c) of this problem we see that $\overline{\Gamma}$ is semantically closed (and of course, contains Γ). It is the smallest such set, since as noted earlier, $\overline{\Gamma}$ is a subset of every semantically closed set which contains Γ.

Here is another solution: Let $\overline{\Gamma}$ be the set of all formulas which are logically implied by Γ. Then Γ is contained in this set, since $\Gamma \models \gamma$ for all $\gamma \in \Gamma$. This set is semantically closed, for if $\overline{\Gamma} \models \alpha$, then $\Gamma \models \alpha$ and thus, by definition, $\alpha \in \overline{\Gamma}$.

$\overline{\Gamma}$ is the smallest semantically closed set which contains Γ since if Δ is any other semantically closed set which contains Γ, then for $\alpha \in \overline{\Gamma}$ we have that $\Gamma \models \alpha$, implying that $\Delta \models \alpha$, which in turn implies that $\alpha \in \Delta$ since Δ is semantically closed. Thus $\overline{\Gamma} \subseteq \Delta$.

(e) Since every formula is logically implied by $\{p_1, \neg p_1\}$ it follows that the set of all formulas is the semantic closure of this set.

6. Let Σ be a set of formulas and α and β be formulas.

(a) Show that if either $\Sigma \models \alpha$ or $\Sigma \models \beta$ then $\Sigma \models (\alpha \vee \beta)$.

(b) Show, by example, that the statement: "if $\Sigma \models (\alpha \vee \beta)$ then either $\Sigma \models \alpha$ or $\Sigma \models \beta$" is false in general.

Solution: Suppose that $\Sigma \models \alpha$. Then for all truth assignments ν, if ν satisfies Σ then ν satisfies α. But then by the definition of the connective $\vee$, we have that ν satisfies $(\alpha \vee \beta)$, and so we have shown that $\Sigma \models (\alpha \vee \beta)$. Similarly, if $\Sigma \models \beta$ we can conclude that $\Sigma \models (\alpha \vee \beta)$.

Let Σ be the set $\{(p_1 \vee p_2)\}$, $\alpha = p_1$ and $\beta = p_2$. Then, $\Sigma \models (\alpha \vee \beta)$, since $\Sigma = \{(\alpha \vee \beta)\}$ but $\Sigma \not\models \alpha$ and $\Sigma \not\models \beta$.

BONUS: Suppose that $\theta \in Form(P, \{\neg, \leftrightarrow\})$. Prove that θ is a tautology if and only if every propositional variable occurs an even number of times in θ and the connective $\neg$ occurs an even number of times in θ.

Solution: We first deal with the negation connective by showing that the following property holds for all formulas $\theta \in Form(P, \{\neg, \leftrightarrow\})$.

> If θ has an even number of occurrences of $\neg$ then it is logically equivalent to a formula θ' that only contains $\leftrightarrow$ and such that the number of occurrences of each variable in θ is the same as in θ'. If θ has an odd number of occurrences of $\neg$ then it is logically equivalent to a formula $\neg\theta'$ where θ' only contains $\leftrightarrow$ and such that the number of occurrences of each variable in θ is the same as in θ'.

We prove this by induction on the length of θ. If θ has length 0 then it is a variable, and there is nothing to show. If the length of θ is $n+1$ for some $n \geq 0$ and the condition holds for shorter formulas, then there are two similar cases to consider: θ has an even number of occurrences of $\neg$ or it has an odd number. Suppose that the former holds. There are two subcases to consider: $\theta = \neg\alpha$ or $\theta = (\alpha \leftrightarrow \beta)$. In the first subcase, since θ has an even number of occurrences of $\neg$, then α has an odd number and so by induction, $\alpha \equiv \neg\alpha'$, for some α' as in the stated condition. But then $\theta \equiv \neg\neg\alpha' \equiv \alpha'$ and we are done, since θ and α' have the same number of occurrences of each variable.

In the other subcase, $\theta = (\alpha \leftrightarrow \beta)$ and we have, by induction, that the condition holds for α and for β. There are two possibilities: either both α and β contain an even number of $\neg$'s or both contain an odd number. In the former, we can argue as in the previous subcase to conclude that $\theta \equiv (\alpha' \leftrightarrow \beta')$, and in the latter we conclude that

$$\theta \equiv (\neg\alpha' \leftrightarrow \neg\beta') \equiv \neg\neg(\alpha' \leftrightarrow \beta') \equiv (\alpha' \leftrightarrow \beta'),$$

showing that the condition holds for θ in this case. The other case, where the number of occurrences of $\neg$ in θ is odd can be handled similarly.

This shows that the condition holds for all formulas $\theta \in Form(P, \{\neg, \leftrightarrow\})$. From this we can see that if θ has an odd number of $\neg$'s then it isn't a tautology. If $\theta \equiv \neg\theta'$ as in the condition, then the truth assignment that sets all variables to T will satisfy θ' (this should be proved by induction on the length of θ') and so falsifies $\neg\theta'$ and hence falsifies θ.

So, now we need only consider formulas in which $\neg$ occurs an even number of times. Using the above condition, we need only consider those formulas in which $\neg$ doesn't occur at all, i.e., only those formulas $\theta \in Form(P, \{\leftrightarrow\})$. We will show that such a formula is a tautology if and only if each variable occurs an even number of times in θ.

We will prove, by induction on the length of $\theta \in Form(P, \{\leftrightarrow\})$ that:

> If ν is a truth assignment, then $\nu(\theta) = T$ if and only if the number of occurrences, counting repetitions, of variables p in θ for which $\nu(p) = F$ is even.

When θ has length 1, this condition can be seen to hold. Suppose that $\theta = (\alpha \leftrightarrow \beta)$ and the condition holds for α and β. Let ν be a truth assignment such that the number of occurrences of variables p in θ for which $\nu(p) = F$ is even. There are two cases to consider. In the first, for both α and β, the number of such variables is even. In this case, by induction, $\nu(\alpha) = \nu(\beta) = T$ and so $\nu(\theta) = T$. In the other case, the number of occurrences of variables for which $\nu(p) = F$ is odd for both α and β. Then by induction, $\nu(\alpha) = \nu(\beta) = F$, and so $\nu(\theta) = T$.

Conversely, suppose that ν is a truth assignment such that $\nu(\theta) = T$. Then either $\nu(\alpha) = \nu(\beta) = T$ or $\nu(\alpha) = \nu(\beta) = F$. In the first case, we have by induction that the number of occurrences of variables p with $\nu(p) = F$ in α is even and is also even for β. But then the number of occurrences of such variables in θ is also even, as required. The final case, where $\nu(\alpha) = \nu(\beta) = F$ can be handled similarly.

So we see that if $\theta \in Form(P, \{\leftrightarrow\})$ is such that each variable occurs an even number of times, then every truth assignment must satisfy θ and so θ is a tautology. On the other hand, if some variable p occurs an odd number of times in θ then the truth assignment that sets p to F and all other variables to T will falsify θ, demonstrating that θ is not a tautology.