Priority option: the value of being a leader

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We consider the strategic interaction between two firms competing for the opportunity to invest in a project with uncertain future values. Starting in complete markets, we provide a rigorous characterization of the strategies followed by each firm in continuous time in the context of a timing/coordination game. In particular, the roles of leader and follower emerge from the resulting symmetric, Markov, sub-game perfect equilibrium. Comparing the expected value obtained by each firm in this case with that obtained when the roles of leader and follower are predetermined, we are able to calculate the amount of money that a firm would be willing to spend in advance (either by paying a license or acquiring market power) to have the right to be the leader in a subsequent game – what we call the priority option. We extend these results to incomplete markets by using utility-indifference arguments.

Keywords: Real options; strategic competition; duopoly; timing/coordination games; incomplete markets; utility-indifference;

1. Introduction

The need to incorporate strategic interactions within the real options paradigm for decision making under uncertainty has long been recognized in the literature. In a recent review article, Azevedo and Paxson [1] present an exhaustive survey ranging from the pioneering contributions of Smets [13] and Grenadier [8] to a catalogue of more than fifty articles dealing with different aspects of the intersection between game theory and real options. The majority of the papers surveyed deal with what they call the "standard" real option game: two symmetric risk-neutral firms competing to invest in a single underlying project in a continuous-time framework. A large portion of the review is dedicated to the intricacies involved in a rigorous formulation of games in continuous time. The authors also make passing references to "non-standard" formulations, such as risk-averse firms, asymmetric competition, multiple players, and sequential games.

Against this landscape, we can situate our paper as a contribution to a rigorous treatment of the duopoly game for both complete and incomplete markets. In the context of the standard real option game described above, Grenadier [9] characterizes a solution in terms of the stochastic demand Y_t associated with the underlying project: for an initial demand Y_0 below a threshold

Y_L both firms defer investment, whereas if Y_0 is higher than a threshold Y_F both firms invest immediately. When the demand is in the interval $[Y_L, Y_F)$, each firm has an incentive to invest provided the other firm defers investment. Grenadier [9] derives an equilibrium consisting of mixed strategies resulting in the possibility of investment occurring sequentially, with one of the firms emerging as the leader and the other as the follower, or simultaneously. However insightful, the derivation is heuristic and does not satisfactorily address some delicate questions regarding the timing of the decision, such as what happens to the continuous-time process (Y_t) while the two firms play the many rounds of a potentially infinite game. In this paper, we recast the result of Grenadier [9] into the timing/coordination game framework of Dutta and Rustichini [5] and Thijssen [14], while extending it to incomplete markets.

In a more recent contribution, Bensoussan, Diltz and Hoe [2] consider real option games in both complete and incomplete markets under the assumption of preassigned leader and follower roles. They justify this type of Stackelberg game by saying that the roles are predetermined by regulatory or competitive advantages. By contrast, in our analysis these roles are endogenously determined in equilibrium. It is intuitive that a preassigned leader should extract a higher value from the game than one that emerges from a competitive equilibrium, since the former does not face the threat of early investment from a competitor. By comparing the expected value for the leader in our formulation with the leader value obtained in Bensoussan *et al.* [2], we are able to quantify the value of such competitive advantage that gives one firm priority to invest over the other. We call this the *priority option* and compute its value for both complete and incomplete markets.

The paper is organized as follows. In Section 2 we analyze the standard real option game described above in the context of a timing/coordination game. As usual we begin by obtaining the optimal strategy for a follower given that one of the firms has already invested in the project, which reduces to a classical optimal stopping problem. We then consider the value realized by a leader who knows that the other firm will act according to the optimal follower strategy just obtained. Because of market completeness, this can be computed as an expected value of discounted future cash flows taking into account the fact that the follower invests at a future optimal stopping time. Based on these values, we prove the main result of this section in Theorem 4: below a threshold Y_L neither firm has any incentive to invest in the project, above a higher threshold Y_F both firms invest immediately, and in the intermediate region $[Y_L, Y_F]$ the two firms play a timing/coordination game. This is a rigorous version of the result of Grenadier mentioned above, with the delicate issue of what happens at the end points Y_L and Y_F fully explored.

We compute the priority option value in a complete market in Section 2.4. A firm with the preassigned role of a leader faces an optimal stopping problem with a payoff function equal to the leader value computed in the previous sections. As in Bensoussan *et al.* [2], this gives rise to an obstacle problem where the obstacle is not differentiable at Y_F, leading to a solution characterized by three different thresholds, rather than the more familiar single-threshold solution for classical optimal stopping problems. Below the first threshold Y_1 the leader has no incentive to invest, but invests in the interval $[Y_1, Y_2]$. When the demand is in the interval $[Y_2, Y_3]$ the leader does not have an incentive to invest either, and instead waits for it either to rise to Y_3 or to *drop* to Y_2. This is because, after investing in the project, the leader faces a decrease in value once the follower also invests at $Y_F \in (Y_2, Y_3)$, but a *preassigned* leader does not need to worry about preemption from the other firm and therefore has the luxury to wait until demand is sufficiently away from Y_F. The surprising feature is that this occurs in both directions: when demand is higher than

Y_3 then the preassigned leader invests simply because demand is high enough to justify both firms being in the market, whereas when demand is lower than Y_2 the entry threshold for the follower Y_F is so far away that the leader reaps the benefits from being alone in the market for a sufficiently long time to justify investment. The difference between the value obtained by a preassigned leader and the value expected by each firm when the roles are not predetermined is the priority option value computed in Proposition 7.

In the complete market case, we can equivalently choose either the demand process (Y_t) or the corresponding project value (V_t) process as the underlying variable of interest, since they are related through expectations of discounted future cash flows under the unique risk-neutral measure as, for example, in expression (2.18). Because this is no longer true in incomplete markets, in Section 3 we focus directly on project values, which are treated as lump-sums received by the firms at the time of investment. The optimal stopping problem for the follower then reduces to that treated in Henderson [10], from which we can directly read both the follower investment threshold V_F and follower value function through expression (3.8). Accordingly, upon entrance of the follower in the market, the leader experiences a loss modelled as a one-time reduction in project value by a fraction $(1-b)$, where $0 < b < 1$ measures the residual value left for the leader. Because the leader anticipates this loss, its utility indifference value enters the calculation of the value function for the leader in (3.14).

As it turns out, the calculation of the investment threshold V_L for the leader is significantly more complicated in incomplete markets. In Proposition 9 we derive sufficient conditions for the existence of V_L and characterize it as the unique solution of a transcendental equation. The conditions are written in terms of the residual fractional value b for the leader, showing that the leader has an incentive to invest provided b is large enough. Moreover, the conditions depend on the level of risk aversion and we verify numerically that V_L approaches V_F as risk aversion increases. In other words, when firms are sufficiently risk averse, there is no discernible advantage of being a leader in an incomplete market. We present our last two results in Section 3.4, where we calculate the value function for a preassigned leader and the corresponding priority option value. As in complete markets, the preassigned leader faces a stopping problem with a payoff given by the value function previously obtained in (3.14). Once more, lack of differentiability at V_F leads to a solution in terms of three thresholds, with interpretations analogous to the ones discussed in the complete market case. We then find the priority option value through utility indifference arguments by comparing the value functions of a leader with and without a preassigned role. Section 4 concludes the paper with a discussion of the results and suggestions for further research.

2. Option to Invest in Complete Markets

Let us consider two firms that can invest in an uncertain project by paying a fixed sunk cost of K. Notice that this does not mean that the investment cost is paid all at once, but merely corresponds to the present value of the investment cost at the time the option to invest is exercised. Each firm can alternatively invest in a riskless bank account at a fixed interest rate r. Once the investment is made, the project immediately starts to produces a cash flow at the rate $D_{Q(t)}Y_t$, where Y_t is the underlying stochastic demand, $Q(t)$ is the number of firms which have invested in the project by date t, and D reflects the corresponding inverse demand curve. We assume that $D_1 > D_2$, which reflects the fact that there is an advantage in being the first firm to invest. We also assume that $D_0 = 0$, meaning that the project is in an idle state before the option to invest is exercised.

The stochastic demand (Y_t) is a diffusion process with dynamics given by

$$\frac{dY_t}{Y_t} = \nu dt + \eta dW_t, \tag{2.1}$$

where (W_t) is a standard one-dimensional Brownian motion under the probability measure $\mathbb{P}$. We impose market completeness by assuming that (Y_t) is perfectly correlated with a traded financial asset whose price dynamics is given by

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t = r dt + \sigma dW_t^{\mathbb{Q}}, \tag{2.2}$$

where $W_t^{\mathbb{Q}} = W_t + \lambda t$ is a Brownian motion under the unique risk-neutral measure $\mathbb{Q}$ and $\lambda = (\mu - r)/\sigma$ is the Sharpe ratio for the traded financial asset. It follows that the dynamics of the stochastic demand under $\mathbb{Q}$ is

$$\frac{dY_t}{Y_t} = (\nu - \eta\lambda)dt + \eta dW_t^{\mathbb{Q}} = (r + \eta(\xi - \lambda))dt + \eta dW_t^{\mathbb{Q}}, \tag{2.3}$$

where $\xi = (\nu - r)/\eta$ plays the role of a Sharpe ratio for the project. In what follows, we denote by $Y_s^{t,y}$ the solution of (2.3) for $s \geq t$ with initial value y at time t, that is, $Y_t^{t,y} = y$.

Because the market is complete, we can use the risk-neutral dynamics above and the discount rate r to calculate the present value of a stream of future cash flows. For example, when both firms have already invested, so that the instantaneous cash flow per unit of demand is D_2, we find that the value at time t of all future cash flows obtained from the project given is

$$\begin{aligned} V^F(y) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-r(s-t)} D_2 Y_s^{t,y} ds|\right] &= \int_t^\infty e^{-r(s-t)} D_2 \mathbb{E}^{\mathbb{Q}}\left[Y_s^{t,y}\right] ds \\ &= \int_t^\infty e^{-r(s-t)} D_2 y e^{(r+\eta(\xi-\lambda))(s-t)} ds \\ &= \frac{D_2 y}{\eta(\lambda - \xi)}. \end{aligned} \tag{2.4}$$

In other words, if we define

$$\delta := \eta(\lambda - \xi), \tag{2.5}$$

we observe that the instantaneous cash flow $D_2 Y_t$ is equal to δ times the project value at t, from which we see that δ plays the role of a dividend rate, so that the dynamics (2.3) can be written as

$$\frac{dY_t}{Y_t} = (r - \delta)dt + \eta dW_t^{\mathbb{Q}}. \tag{2.6}$$

2.1. *Follower Value*

We denote a firm by L (“Leader") if it is the first to invest, by F (“Follower") if it is the second to invest and by S if both firms invest simultaneously. Notice that these roles are not predetermined, but follow from equilibrium considerations based on the values that we shall derive next.

We start with the follower value. If one of the firms has already invested, the remaining firm has an option to invest in the project at a random time τ by paying the fixed cost K and start receiving cash flows with present value given by $V^F(Y_\tau) = D_2 Y_\tau/\delta$, according to (2.4). Assuming that this option has infinite maturity, we recognize this as a perpetual early-exercise option with payoff $(D_2 Y_\tau/\delta - K)^+$. Since the market is assumed to be complete, the value of this option is given by

$$F(y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} \left(\frac{D_2 Y_\tau^{0,y}}{\delta} - K \right)^+ \mathbf{1}_{\{\tau < \infty\}} \right], \tag{2.7}$$

where $\mathcal{T} := \mathcal{T}_{[0,\infty]}$ denotes the collection of all $\mathbb{F}$-stopping times with values in $[0,\infty]$. As it is well known (see Appendix A), the dynamic programming equation corresponding to this optimal stopping problem is the variational inequality

$$\min \left(rF - \frac{\eta^2}{2} y^2 F'' - (r-\delta) y F', F - \left(\frac{D_2 y}{\delta} - K \right)^+ \right) = 0, \tag{2.8}$$

supplemented by the conditions $F(v) \geq 0$ and $F(0) = 0$. Since the obstacle function

$$g(y) = \left(\frac{D_2 y}{\delta} - K \right)^+ \tag{2.9}$$

has polynomial growth, we can use a classical verification argument to show that a candidate solution to (2.8) is indeed the value function $F(y)$ in (2.7). Following the heuristic derivation presented in Chapter 5 of [4], we find that $F(y)$ has the form presented in the next proposition and shown in Figure 1.

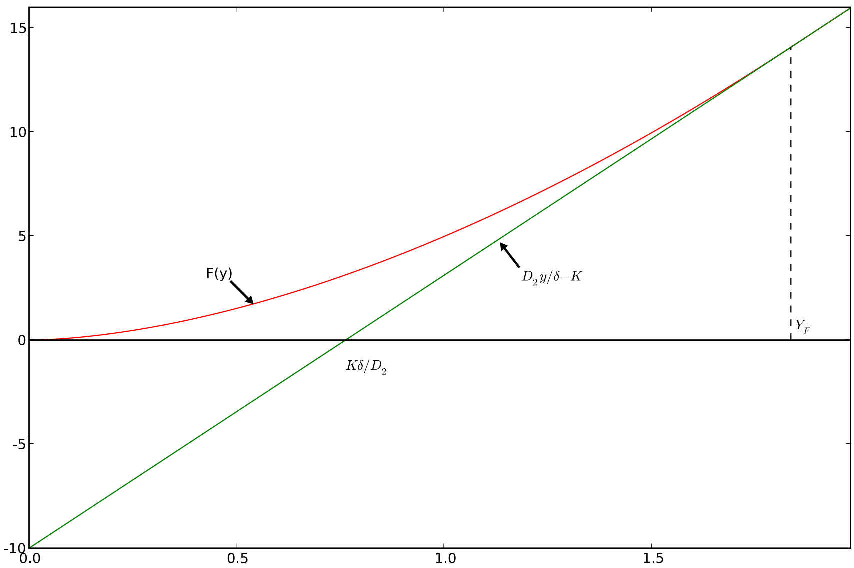

Fig. 1. Follower value as a function of demand in a complete market. The straight line $D_2 y/\delta - K$ is the payoff obtained from exercise at a level of demand equal to y, with optimal exercise corresponding to the threshold $y = Y_F$. When the parameters are set as $(K, \nu, \eta, \mu, \sigma, r, D_1, D_2) = (10, 0.01, 0.2, 0.04, 0.3, 0.03, 1, 0.35)$, we have $Y_F = 1.83$ and $\beta = 1.71$.

Proposition 1. *Provided $\delta = \eta(\lambda - \xi) > 0$, the value $F(y)$ of being the follower at a demand level y is*

$$F(y) = \begin{cases} \dfrac{K}{\beta - 1}\left(\dfrac{y}{Y_F}\right)^{\beta} & \text{if } y \leq Y_F, \\ \dfrac{D_2 y}{\delta} - K & \text{if } y > Y_F, \end{cases} \tag{2.10}$$

where Y_F is a threshold given by

$$Y_F = \frac{\delta K \beta}{D_2(\beta - 1)}, \tag{2.11}$$

and

$$\beta := \left(\frac{1}{2} - \frac{r-\delta}{\eta^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\eta^2}\right)^2 + \frac{2r}{\eta^2}} > 1. \tag{2.12}$$

Proof. It suffices to show that $F(\cdot)$ satisfies the conditions of Theorem 10.4.1 in [12], which we reproduce in Appendix A for convenience. Set

$$\mathcal{D} = \{y \in \mathbb{R} : F(y) > g(y)\} = (0, Y_F), \tag{2.13}$$

and observe that the constant β and the threshold Y_F were chosen so that $F(y)$ satisfies the matching and smooth pasting conditions

$$F(Y_F) = g(Y_F) \tag{2.14}$$

$$F'(Y_F) = g'(Y_F). \tag{2.15}$$

We then have that $F \in C^1(G) \cap C(\overline{G})$ and $F \in C^2(G \backslash \partial\mathcal{D})$ with locally bounded second order derivative near $\partial\mathcal{D}$, so conditions (i) and (v) of Theorem 13 are satisfied. Conditions (ii), (iii), (iv), (viii) and (ix) hold by construction of $\mathcal{D}$ and F together with elementary properties of the process Y. Condition (vii) holds by observing that β is the positive root of the characteristic equation

$$\frac{\eta^2}{2}\beta(\beta - 1) + (r - \delta)\beta - r = 0, \tag{2.16}$$

whereas condition (vi) follows from a direct calculation. □

Remark 1. Observe that our $F(y)$ is the same as the function $F(X)$ in equation (5) of [9] provided one identifies our $\eta(\lambda-\xi)$ with Grenadier's $(r-\alpha)$, which follows from the fact that his stochastic demand has a risk-neutral expected growth rate equal to α.

2.2. *Leader Value*

After exercising the investment option, the leader has no further decisions to take. Therefore, the value of being the leader at a time t when the demand is y can be calculated directly from the cash flows obtained from the underlying project from t onwards. Observe, however, that these cash flows depend on the exercise decision of the follower: if $y > Y_F$, then it is optimal for the follower to exercise at time t and the project will have the value $D_2 y/\delta$ given in (2.4). On the other hand, if $y \leq Y_F$ the follower will wait to invest until

$$\tau_F(y) = \inf\{s \geq t : Y_s^{t,y} = Y_F\} \tag{2.17}$$

and the project will have value

$$\begin{aligned}
V^L(y) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau_F} e^{-r(s-t)} D_1 Y_s^{t,y} ds + \int_{\tau_F}^{\infty} e^{-r(s-t)} D_2 Y_s^{t,y} ds\right] \\
&= D_1 \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\infty} e^{-r(s-t)} Y_s^{t,y} ds\right] + (D_2 - D_1)\mathbb{E}^{\mathbb{Q}}\left[\int_{\tau_F}^{\infty} e^{-r(s-t)} Y_s^{t,y} ds\right] \\
&= \frac{D_1 y}{\delta} + (D_2 - D_1)\,\mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau_F - t)}\mathbb{E}^{\mathbb{Q}}\left[\int_{\tau_F}^{\infty} e^{-r(s-\tau_F)} Y_s^{\tau_F, Y_F} ds\right]\right] \\
&= \frac{D_1 y}{\delta} + \frac{(D_2 - D_1)\,Y_F}{\delta}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau_F - t)}\right] \\
&= \frac{D_1 y}{\delta} - \frac{(D_1 - D_2)\,Y_F}{\delta}\left(\frac{y}{Y_F}\right)^{\beta}, \qquad (2.18)
\end{aligned}$$

where we used the well-known expression for the Laplace transform of $(\tau_F - t)$. We can summarize these observations as follows:

Proposition 2. *Provided $\delta = \eta(\lambda - \xi) > 0$, the value of becoming a leader at a demand level y is*

$$L(y) = \begin{cases} \dfrac{D_1 y}{\delta} - \dfrac{(D_1 - D_2)}{D_2}\dfrac{K\beta}{\beta - 1}\left(\dfrac{y}{Y_F}\right)^{\beta} - K & \text{if } y \le Y_F, \\ \dfrac{D_2 y}{\delta} - K & \text{if } y > Y_F. \end{cases} \qquad (2.19)$$

Moreover, the value obtained by both firms from simultaneous exercise at a demand level y is

$$S(y) = \frac{D_2 y}{\delta} - K. \qquad (2.20)$$

Proof. Becoming a leader at demand level y means paying the investment cost K in exchange of a project with value either equal to $D_2 y/\delta$ if $y > Y_F$ or equal to $V^L(y)$ given in (2.18) if $y \le Y_F$, which proves (2.19). Notice that this corresponds to the situation where one of the firms invests at this level of demand while the other adopts the optimal strategy of a follower. Alternatively, whenever both firms invest simultaneously they each pay the investment cost K in exchange of a project with value equal to $D_2 y/\delta$, which proves (2.20). □

Remark 2. Our functions $L(y)$ and $S(y)$ coincide with the functions $L(X) - K$ and $S(X) - K$ in equations (9) and (10) of [9] under the same identification discussed in Remark 1. Observe, however, that we derived the value of being a leader directly from the cash flows obtained from the underlying project, without having to solve the differential equation (7) in [9]. This approach has advantages when we extend the model to incomplete markets in Section 3.

Remark 3. We chose to treat the stochastic demand (Y_t) as the state variable, primarily because we wanted a direct comparison with the setting in Grenadier [9]. The analysis in Bensoussan *et al.* [2], on the other hand, uses the underlying project value (V_t) itself as the state variable. In a complete market, these formulations are equivalent, since we easily move back and forth between the two simply by calculating the discounted expected value of all future cash flows generated by the project. This is because market completeness implies that there is a unique way to compute expectations and to do the discounting, namely through the unique risk-neutral measure $\mathbb{Q}$ and using the risk-free rate r, as in the calculation leading to (2.4) for example. If we identify the project value in [2] as corresponding to the situation with just one firm present in the market (i.e, with the inverse demand function equal to D_1), then a similar calculation shows that the project value for the follower obtained in (2.4) satisfies $V^F(Y_t) = (1-a)V_t$, with $a := (D_1 - D_2)/D_1$.

2.3. *Equilibrium Strategies*

The purpose of this section is to derive an equilibrium strategy for both firms based on the values just calculated. We start with the following technical proposition, which was stated without proof in Grenadier [9]. The proof which we present in Appendix B is a fuller version of a similar proof in the Appendix of [8], adapted to our purposes.

Proposition 3 (Grenadier (2000)). *There exists a unique point $Y_L \in (0, Y_F)$ such that*

$$\begin{cases} L(y) < F(y) & \text{for } y < Y_L, \\ L(y) = F(y) & \text{for } y = Y_L, \\ L(y) > F(y) & \text{for } Y_L < y < Y_F, \end{cases} \tag{2.21}$$

and

$$\begin{cases} S(y) < \min(L(y), F(y)) & \text{for } y < Y_F, \\ S(y) = F(y) & \text{for } y \geq Y_F. \end{cases} \tag{2.22}$$

We will now focus on symmetric, Markov, sub-game perfect equilibrium exercise strategies. To do so in our continuous-time setting we use the method of Thijssen *et al.* [15] which extended the original ideas of Fudenberg and Tirole [7] to a stochastic setting. The main idea is to set up a combined timing/coordination game, where the first-stage timing game is in continuous-time, while the discrete-time coordination stage is used to resolve priority when both firms want to invest at the same instant. A complementary treatment is given in [5,14].

Formally, the time domain is extended to index pairs $(t,z) \in \mathcal{T} \equiv \mathbb{R}_+ \times \mathbb{Z}_+$ where t represents the continuous "process time," and z counts the discrete rounds of a coordination game. The information filtration $\mathbb{F} \equiv (\mathcal{F}_{t,z})$ now relies on the lexicographic ordering $\leq_L$, $\mathcal{F}_{t,z} \subseteq \mathcal{F}_{t',z'} \subseteq \mathcal{F}$ whenever $(t,z) \leq_L (t',z') \in \mathcal{T}$ and the underlying stochastic process is extended to $Y_{(t,z)} := Y_t$. A *simple strategy* is a pair of $\mathbb{F}$-adapted processes $(G^i_{(t,z)}, p^i_{(t,z)})$ with $(G^i_{(t,z)}, p^i_{(t,z)}) \in [0,1] \times [0,1]$ for all $(t,z) \in \mathcal{T}$, and $G^i_{(t,z)}$ a non-decreasing càdlàg process (see [7]). The interpretation of $G^i_{(t,z)}$ is as the cumulative probability of firm i exercising by instant (t,z), whereas $p^i_{(t,z)}$ denotes the probability of exercising in a coordination stage game described below. We focus on instantaneously-stationary Markovian strategies with $p^i_{(t,z)} \equiv p_i(Y_t)$. In the option exercise game with a Markov underlying state variable (Y_t) we may without loss of generality focus on hitting-time strategies, such that $G^i_t = G^i_{(t,z)} = 1_{\{t \leq \tau(y^i)\}}$, where $\tau(y^i) = \inf\{t \geq 0 : Y_t \geq y^i\}$ is the first hitting time by (Y_t) of the half-interval $[y^i, \infty)$.

We now consider the timing game starting at initial point $Y_0 = y$. Based on Proposition 3, there are three different regions to consider. If $y < Y_L$, neither firm wants to invest and both wait for the demand to rise. If $y > Y_F$, both firms want to invest and it does not matter who is the leader or the follower, because the second firm will exercise the option to invest immediately after the first.

The interesting region is $Y_L \leq y \leq Y_F$, where according to Proposition 3 each firm prefers to be the leader, but at the same time both firms are worse off by investing simultaneously

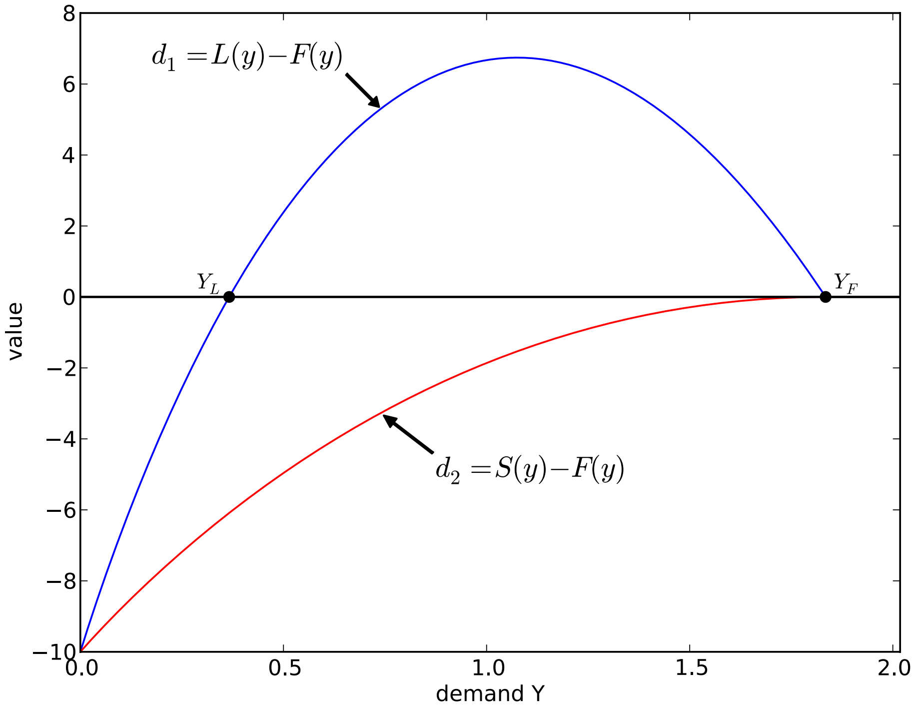

Fig. 2. Differences in values for leader, follower and simultaneous exercise. With the same parameters used in Figure 1, we have $Y_L = 0.37$ and $Y_F = 1.83$.

than being the follower, as can be seen in Figure 2. This is precisely the situation where the coordination game is played, with the z-th round at "time" $(0, z)$ having payoffs

	Invest	Wait
Invest	$(S(y), S(y))$	$(F(y), L(y))$
Wait	$(L(y), F(y))$	Repeat

In this infinite 2-by-2 game, two stationary strategy types are possible. The case where $\Delta G_0^i > 0$ with

$$\Delta G_t^i := G_t^i - \lim_{s \nearrow t} G_s^i, \tag{2.23}$$

and $p_i(y) = 0$ corresponds to a one-shot strategy, where firm i attempts to exercise the option "once" (that is, at $(0, 0)$) with probability ΔG_0^i and then waits infinitesimally. The case when $p_i(y) > 0$ means that at each stage $(0, z)$, firm i will exercise with probability $p_i(y)$.

For an initial condition $Y_L < y < Y_F$, it follows from right-continuity of G^i that only the coordination game using $p_i(y)$'s is possible. Given a mixed strategy $(p_1(y), p_2(y))$ with $\max(p_1(y), p_2(y)) > 0$, at least one firm will immediately exercise a.s., and the probabilities of the three possible outcomes are

$$
\begin{aligned}
a_1(y) &= \frac{p_1(y)(1-p_2(y))}{p_1(y)+p_2(y)-p_1(y)p_2(y)} && \text{(firm 1 exercises)},\\
a_2(y) &= \frac{(1-p_1(y))p_2(y)}{p_1(y)+p_2(y)-p_1(y)p_2(y)} && \text{(firm 2 exercises)},\\
a_S(y) &= \frac{p_1(y)p_2(y)}{p_1(y)+p_2(y)-p_1(y)p_2(y)} && \text{(simultaneous exercise)}.
\end{aligned}
$$

These probabilities are computed by adding the corresponding probabilities for each round of the infinite game, for example:

$$
a_1(y) = p_1(y)(1-p_2(y)) + p_1(y)(1-p_1(y))(1-p_2(y))^2 + \ldots = \sum_{k=1}^{\infty} p_1(y)(1-p_1(y))^{k-1}(1-p_2(y))^k. \tag{2.24}
$$

Thus, the expected payoff for firm 1 when $Y_L < y < Y_F$ is

$$
E_1(y; p_1, p_2) = a_1(y)L(y) + a_2(y)F(y) + a_S(y)S(y), \tag{2.25}
$$

and similarly for firm 2. Maximizing (2.25) with respect to p_1 subject to $p_1 = p_2$, we find that the unique symmetric Nash equilibrium of the stage game is given by

$$
\widehat{p}(y) = \frac{L(y) - F(y)}{L(y) - S(y)}, \qquad Y_L < y < Y_F. \tag{2.26}
$$

It follows that the probability of simultaneous investment for a given demand $y \in (Y_L, Y_F)$ is

$$
a_S(y) = \frac{L(y) - F(y)}{L(y) + F(y) - 2S(y)}, \tag{2.27}
$$

and the probability of sequential investment with firm i emerging as the leader is given by

$$
a_{seq}(y) = \frac{1}{2}(1 - a_S(y)) = \frac{F(y) - S(y)}{L(y) + F(y) - 2S(y)}. \tag{2.28}
$$

Figure 3 illustrates the behaviour of these probabilities.

Remark 4. We could alternatively skip the "extensive-form" description of the stage game above and simply postulate directly that, if both firms wish to exercise at the same time, then the expected payoff for firm 1 would be of the form $E_1(y) = a_1(y)L(y) + a_2(y)F(y) + a_S(y)S(y)$ for some triple $a_1(y) + a_2(y) + a_S(y) = 1$, and similarly for firm 2. This "normal-form" approach avoids the need to define the extended time index $\mathcal{T}$, but provides no explicit mechanism to determine the $a(y)$'s, relying instead on economic principles such as rent equalization (see for example [7] or [14, Section 4]).

Let us now consider what happens at the end-points of the interval (Y_L, Y_F). L'Hôpital's rule gives that $\widehat{p}(Y_F-) = 1$ so that both firm exercise simultaneously at Y_F. The situation at Y_L is more involved and corresponds to one of the subtleties of a continuous-time formulation. First observe that $\widehat{p}(Y_L+) = 0$ which implies that $a_1(Y_L) = a_2(Y_L) = \frac{1}{2}$. Next note that at the stopping time $\tau(Y_L) = \inf\{t : Y_t = Y_L\}$, we have that $G^i_{\tau(Y_L)-} = 0$ and $G^i_{\tau(Y_L)} = 1$, so that $\Delta G^i(\tau(Y_L)) = 1$ even though $p_i(Y_L) = 0$. The key is that $\inf\{t : Y_t > Y_L\} = \tau(Y_L)$ and in equilibrium $p_i(y) > 0$ for all $y > Y_L$, so at least one of the two firms will exercise immediately after $\tau(Y_L)$. Thus, at Y_L each firm becomes the leader with probability 1/2 and exercises the

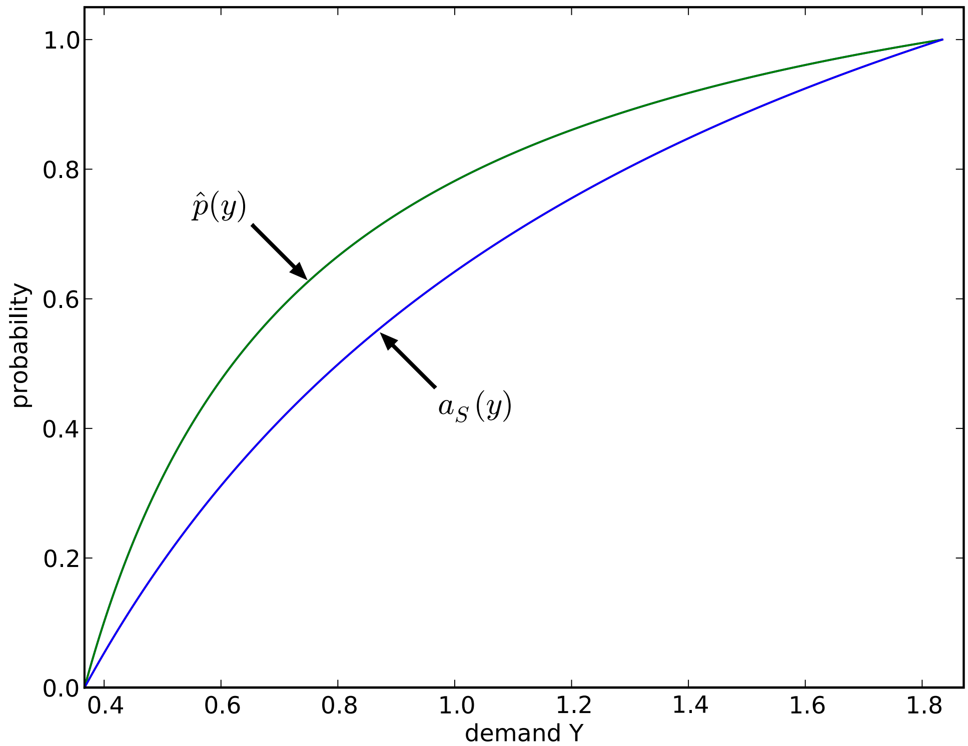

Fig. 3. Optimal probability $\widehat{p}(y)$ for each firm to exercise and probability a_S of simultaneous exercise by both firms. Parameter values are the same as in Figure 1.

option, while the other becomes the follower and postpones exercise until the demand rises to Y_F.

The equilibrium strategies just introduced can be summarized in the next theorem, which extends the result first established in [9]:

Theorem 4. *There exists a symmetric, Markov, sub-game perfect equilibrium with strategies depending on the level of demand as follows:*

(i) *If $y < Y_L$, both firms wait for the demand to rise and reach Y_L.*
(ii) *At $y = Y_L$, there is no simultaneous exercise and each firm has an equal probability of emerging as a leader while the other becomes a follower and waits until demand rises to Y_F.*
(iii) *If $Y_L < y < Y_F$, each firm chooses a mixed strategy consisting of exercising the option to invest with probability $\widehat{p}(y)$. The resulting equilibrium yields simultaneous exercise with probability $a_S(y)$ given in (2.27) and the case where one firm emerges as the leader and the other waits until demand rises to Y_F with probability $(1 - a_S(y))$.*
(iv) *If $y \geq Y_F$, both firms invest immediately.*

We conclude this section showing that the optimal probability $\widehat{p}$ corresponds to the probability that makes each firm indifferent between being the follower or playing the game described above, which is a restatement of the concept of rent equalization from Fudenberg and Tirole [7] that implies that in equilibrium the timing value of the leader option completely vanishes due to strategic preemption.

Proposition 5 (Rent equalization)**.** *The equilibrium described in Theorem 4 makes the expected payoff for each firm to be equal to* $F(y)$ *for all* $y \in [0, \infty)$*.*

Proof. Observe that, because the firms are symmetric, they have equal probabilities of emerging as the leader when sequential exercise occurs in the critical region $Y_L < y < Y_F$. Therefore, the expected payoff for each firm as a function of demand is

$$\begin{cases} F(y) & \text{if } 0 \leq y < Y_L, \\ \frac{1}{2}(L(Y_L) + F(Y_L)) & \text{if } y = Y_L, \\ \frac{1}{2}(1 - a_S(y))(F(y) + L(y)) + a_S(y)S(y) & \text{if } Y_L < y < Y_F, \\ S(y) & \text{if } y \geq Y_F. \end{cases} \tag{2.29}$$

Using the expression for the probability of simultaneous exercise in (2.27) gives the result. □

2.4. *Priority Option Value*

We have assumed so far that the roles of leader and follower are not predetermined, but rather the outcome of the equilibrium strategies described in Theorem 4. Alternatively, we could consider a Stackelberg game where the roles of the firms are predetermined exogenously. A problem of this type was treated in Bensoussan *et al.* [2], where it was argued that the advantageous role of being a leader can be determined, for example, by regulations or competitive advantages. Our purpose in this section is to obtain a value for this advantage, which we call the *priority option.*

For this, assume as in [2] that the roles of leader and follower are predetermined. In other words, the leader has the option to invest in the project knowing that the follower is forbidden to invest until the leader has done so. That is, the leader can invest in the project at a random time τ and receive the payoff $L(Y_\tau)$ according to (2.19). Therefore, the value function for the leader in this case is

$$L^\pi(y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} L(Y_\tau^{0,y})^+ \mathbf{1}_{\{\tau < \infty\}} \right], \tag{2.30}$$

where the superscript π is meant to indicate that the leader now has the priority to invest. As before, the dynamic programming equation associated with this optimal stopping problem is

$$\min \left(rL^\pi - \frac{\eta^2}{2} y^2 (L^\pi)'' - (r - \delta) y (L^\pi)', L^\pi - L^+ \right) = 0. \tag{2.31}$$

As in [2], the fact that the leader anticipates the rational exercise decision by the follower at the threshold Y_F yields a payoff function that is $C^0(0, \infty)$ but not $C^1(0, \infty)$. Indeed, we find that

$$L'(y) = \begin{cases} \dfrac{D_1}{\delta} - \dfrac{(D_1 - D_2)\beta}{\delta} \left(\dfrac{y}{Y_F} \right)^{\beta - 1} & \text{if } y < Y_F, \\ \dfrac{D_2}{\delta} & \text{if } y \geq Y_F, \end{cases} \tag{2.32}$$

so that

$$L'(Y_F-) = \frac{D_1}{\delta} - \frac{(D_1 - D_2)\beta}{\delta} < \frac{D_2}{\delta} = L'(Y_F+). \tag{2.33}$$

We also have that L is strictly concave for $0 \le y < Y_F$, since

$$L''(y) = -\frac{(D_1 - D_2)\beta(\beta - 1)}{\delta Y_F} \left(\frac{y}{Y_F}\right)^{\beta-2} < 0. \tag{2.34}$$

Finally, it is relatively straightforward to show that it satisfies the bounds

$$\frac{D_2 y}{\delta} - K \le L(y) \le \frac{D_2 y}{\delta}. \tag{2.35}$$

We therefore conclude that our obstacle $L(y)$ has the same properties as the obstacle $\Psi(v)$ in [2], which implies that our value function $L^\pi(y)$ inherits the properties of their value function. In particular, the polynomial bounds in (2.35) allow us to use a verification argument to establish that the value function is smoother than the obstacle itself, as shown in the next theorem. Before we state the result, let us define the constants Y_1 and A_1 as

$$Y_1 := \frac{\delta K \beta}{D_1(\beta - 1)}, \tag{2.36}$$

$$A_1 := \frac{1}{Y_1^\beta} \frac{K}{\beta - 1} \left(\frac{D_2}{D_1}\right)^\beta \left[\left(\frac{D_1}{D_2}\right)^\beta - \beta\left(\frac{D_1}{D_2}\right) + \beta\right], \tag{2.37}$$

and the constants Y_2, Y_3, A_2, A_3 as a solution of the nonlinear system of equations

$$\begin{cases} A_2 Y_2^\beta + A_3 Y_2^{\beta_1} = \dfrac{D_1 Y_2}{\delta} - \dfrac{(D_1 - D_2) Y_F}{\delta} \left(\dfrac{Y_2}{Y_F}\right)^\beta - K, \\ \beta A_2 Y_2^{\beta-1} + \beta_1 A_3 Y_2^{\beta_1 - 1} = \dfrac{D_1 Y_2}{\delta} - \dfrac{(D_1 - D_2)\beta Y_F}{\delta} \left(\dfrac{Y_2}{Y_F}\right)^\beta, \\ A_2 Y_3^\beta + A_3 Y_3^{\beta_1} = \dfrac{D_2 Y_3}{\delta} - K, \\ \beta A_2 Y_3^{\beta-1} + \beta_1 A_3 Y_3^{\beta_1 - 1} = \dfrac{D_2 Y_3}{\delta}. \end{cases} \tag{2.38}$$

Theorem 6. *Let Y_1 and A_1 be given by* (2.36) *and* (2.37) *and assume that the nonlinear system* (2.38) *has a unique solution given by the constants Y_2, Y_3, A_2, A_3. If*

$$0 < Y_1 < Y_2 < Y_F < Y_3, \tag{2.39}$$

then the solution to (2.30) *is given by*

$$L^\pi(y) = \begin{cases} A_1 y^\beta & \text{if } 0 \le y < Y_1, \\ L(y) & \text{if } Y_1 \le y \le Y_2, \\ A_2 y^\beta + A_3 y^{\beta_1} & \text{if } Y_2 < y < Y_3, \\ L(y) & \text{if } y \ge Y_3, \end{cases} \tag{2.40}$$

where β and β_1 are the positive and negative roots of the characteristic equation (2.16).

Proof. We proceed again by verifying the conditions in Theorem 13. Set

$$\mathcal{D} = (0, Y_1) \cup (Y_2, Y_3), \tag{2.41}$$

and consider the matching and smooth pasting conditions

$$L^{\pi}(Y_i) = L(Y_i), \tag{2.42}$$

$$(L^{\pi})'(Y_i) = L'(Y_i), \tag{2.43}$$

for $i = 1, 2, 3$. Observe that Y_1 and A_1 in (2.36) and (2.37) correspond to (2.42) and (2.43) with $i = 1$, whereas the nonlinear system (2.38) corresponds to (2.42) and (2.43) with $i = 2, 3$. We then see that $L^{\pi} \in C^1(G) \cap C(\overline{G})$ and $L^{\pi} \in C^2(G \backslash \partial\mathcal{D})$ with locally bounded second order derivative near $\partial\mathcal{D}$, so conditions (i) and (v) are satisfied. Conditions (ii), (iii), (iv), (viii) and (ix) hold by construction of $\mathcal{D}$ and L^{π} and elementary properties of the process Y. Condition (vii) holds by observing that β and β_1 are the positive and negative roots of the characteristic equation (2.16), whereas condition (vi) follows from a direct calculation using the explicit expression for L in (2.19). □

Although in general (2.38) does not have an explicit solution, we can easily solve it numerically and verify whether condition (2.39) is satisfied in practice for given values of the underlying parameters.

The obstacle $L(y)$ and the function $L^{\pi}(y)$ are shown in Figure 4. We see that the optimal investment strategies for a predetermined leader and follower are as follows:

(i) For $0 \le y < Y_1$ the leader waits to invest until the demand rises to $y \ge Y_1$ and the follower waits to invest until the demand rises to $y \ge Y_F$.
(ii) For $Y_1 \le y \le Y_2$ the leader invests immediately and the follower waits to invest until the demand rises to $y \ge Y_F$.
(iii) For $Y_2 < y < Y_3$ the leader waits until either the demand drops to $y \le Y_2$, in which case the follower waits to invest until it rises to $y \ge Y_F$, or the demand rises to $y \ge Y_3$, in which case the follower exercises immediately since $y \ge Y_F$.
(iv) For $y \ge Y_3$ both the leader and the follower invest immediately.

Notice that Y_1 given in (2.36) coincides with the exercise threshold for a monopolistic firm when the cash flow per unit of demand is D_1. In other words, when the initial demand is low, the priority option allows the leader to act as a monopolistic firm and ignore the follower's actions.

Furthermore, observe that at $y = Y_1$ we have

$$L(Y_1) - F(Y_1) = \frac{K}{\beta - 1}\left(\frac{D_2}{D_1}\right)^{\beta}\left[\left(\frac{D_1}{D_2}\right)^{\beta} - \beta\left(\frac{D_1}{D_2}\right) + \beta - 1\right] > 0, \tag{2.44}$$

since $\beta > 1$ and $D_1 > D_2$. It then follows that $Y_L < Y_1$, which shows that the priority option delays the investment decision for the leader. All this suggests that the priority option can be quite valuable. To compute its actual value, we proceed as follows.

According to Proposition 5, the expected payoff for each firm when the roles of leader and follower are not predetermined is $F(y)$ for all values of y. The premium π for the priority option, which we assume to be offered only at time 0, first to one firm (randomly chosen) and then, if declined, to the other, is given by the next proposition, whose proof follows by direct substitution of the values calculated in Theorem 6 and Proposition 5. The priority option value is illustrated in Figure 5.

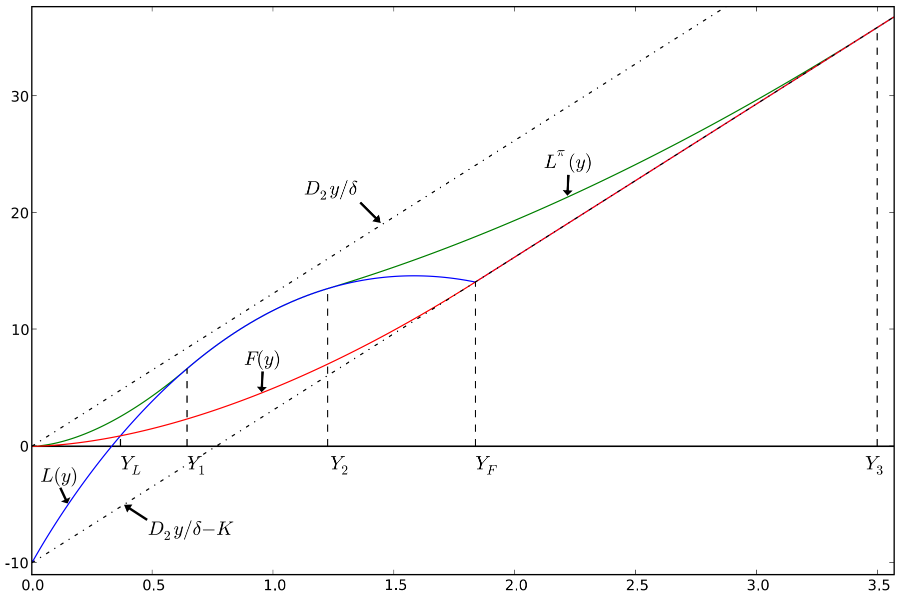

Fig. 4. Obstacle problem with predetermined roles, showing the value function L^π for the predetermined leader, the payoff L obtained upon investment, and the corresponding value function F for the follower. Parameter values are the same as in Figure 1, which results in $Y_1 = 0.64$, $Y_2 = 1.22$ and $Y_3 = 3.50$.

Proposition 7. *The value of $\pi(y) = L^\pi(y) - F(y)$ of the priority option is given by*

$$\pi(y) = \begin{cases} \left[\left(\frac{D_1}{D_2}\right)^\beta - \beta\left(\frac{D_1}{D_2}\right) + \beta - 1\right]\frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta & \text{if } 0 \le y < Y_1, \\ \frac{D_1}{\delta}y - K - \left[\beta\left(\frac{D_1}{D_2}\right) - \beta + 1\right]\frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta & \text{if } Y_1 \le y < Y_2, \\ \left(A_2 Y_F^\beta - \frac{K}{\beta-1}\right)\left(\frac{y}{Y_F}\right)^\beta + A_3 y^{\beta_1} & \text{if } Y_2 \le y < Y_F, \\ A_2 y^\beta + A_3 y^{\beta_1} - \frac{D_2 y}{\delta} + K & \text{if } Y_F \le y < Y_3, \\ 0 & \text{if } Y_3 \le y. \end{cases} \tag{2.45}$$

Fig. 5. Priority option value $\pi = L^\pi - F$. Parameter values are the same as in Figure 1.

As we have just argued, the value of being a predetermined Leader for the firm that buys the priority option is L^π. We now compute the value of the other firm, the predetermined Follower, denoted by F^π. When the roles were not pre-determined, the follower value F was found in Section 2.1 through a standard approach of maximizing expected cash flow over all possible investment strategies. However, with a priority option, the Follower is constrained to first wait for the Leader to invest. The following Lemma provides the structure of F^π:

Lemma 8. *We have*

$$F^\pi(y) = \begin{cases} F(Y_1)\left(\frac{y}{Y_1}\right)^\beta & \text{if } 0 < y \le Y_1, \\ \frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta & \text{if } Y_1 < y < Y_2, \\ F(Y_2)\left(\frac{y}{Y_2}\right)^A \frac{\left(\frac{Y_3}{y}\right)^B - \left(\frac{Y_3}{y}\right)^{-B}}{\left(\frac{Y_3}{Y_2}\right)^B - \left(\frac{Y_3}{Y_2}\right)^{-B}} + F(Y_3)\left(\frac{y}{Y_3}\right)^A \frac{\left(\frac{y}{Y_2}\right)^B - \left(\frac{y}{Y_2}\right)^{-B}}{\left(\frac{Y_3}{Y_2}\right)^B - \left(\frac{Y_3}{Y_2}\right)^{-B}} & \text{if } Y_2 \le y < Y_3, \\ \frac{D_2 y}{\delta} - K & \text{if } Y_3 \le y, \end{cases} \tag{2.46}$$

with

$$A = \frac{1}{2} - \frac{r-\delta}{\eta^2},$$
$$B = \sqrt{A^2 + \frac{2r}{\eta^2}} = \beta - A.$$

Proof. Given the initial project value $Y_0 = y$, four cases must be considered. First, if $y \le Y_1$, the Leader will invest as soon as $Y_t \ge Y_1$. Consequently,

$$F^\pi(y) = \mathbb{E}_y^{\mathbb{Q}}\left[e^{-r\tau_1}\right] F(Y_1) = F(Y_1)\left(\frac{y}{Y_1}\right)^\beta. \tag{2.47}$$

Next, if $Y_1 \le y \le Y_2$, the Leader invests immediately and

$$F^\pi(Y) = F(Y) = \frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta. \tag{2.48}$$

Third, suppose that $Y_2 < y < Y_3$. This is the most complicated case. Indeed, the Leader will delay investment until the exit time τ_H defined by

$$\tau_H = \inf\{t \ge 0 : Y_t \notin (Y_2, Y_3)\}. \tag{2.49}$$

At the investment time τ_H, the value of the Follower would be $F(Y_{\tau_H})$. Hence, his expected value at $t = 0$ is

$$F^\pi(y) = F(Y_2)\,\mathbb{E}_y^{\mathbb{Q}}\left[e^{-r\tau_H}1_{\{Y_{\tau_H}=Y_2\}}\right] + F(Y_3)\,\mathbb{E}_y^{\mathbb{Q}}\left[e^{-r\tau_H}1_{\{Y_{\tau_H}=Y_3\}}\right]. \tag{2.50}$$

Using the formula for the Laplace transform of a two-sided exit time of a geometric Brownian motion $Y_t = ye^{(\mu-\frac{1}{2}\nu^2)t+\nu W_t^Q}$, we obtain the expression on the third line of (2.46).

Finally, if $y \ge Y_3$, Leader invests immediately at $t = 0$ and since $y \ge Y_3 > Y_F$, the Follower invests immediately afterwards, leading to $F^\pi(y) = F(y) = D_2 y/\delta - K$. □

We note that $F^\pi(y) = F(y)$ on $y \in [Y_1, Y_2] \cup [Y_3, \infty)$, but is strictly smaller $F^\pi(y) < F(y)$ on $y \in (0, Y_1) \cup (Y_2, Y_3)$. In the former case, the priority option causes the predetermined follower to lose value from potentially winding up a leader, while in the latter case the priority option causes the predetermined follower to postpone investment until $Y_t \notin [Y_2, Y_3]$, which may be sub-optimal (and strictly sub-optimal when $y > Y_F$ and the follower wishes to invest immediately).

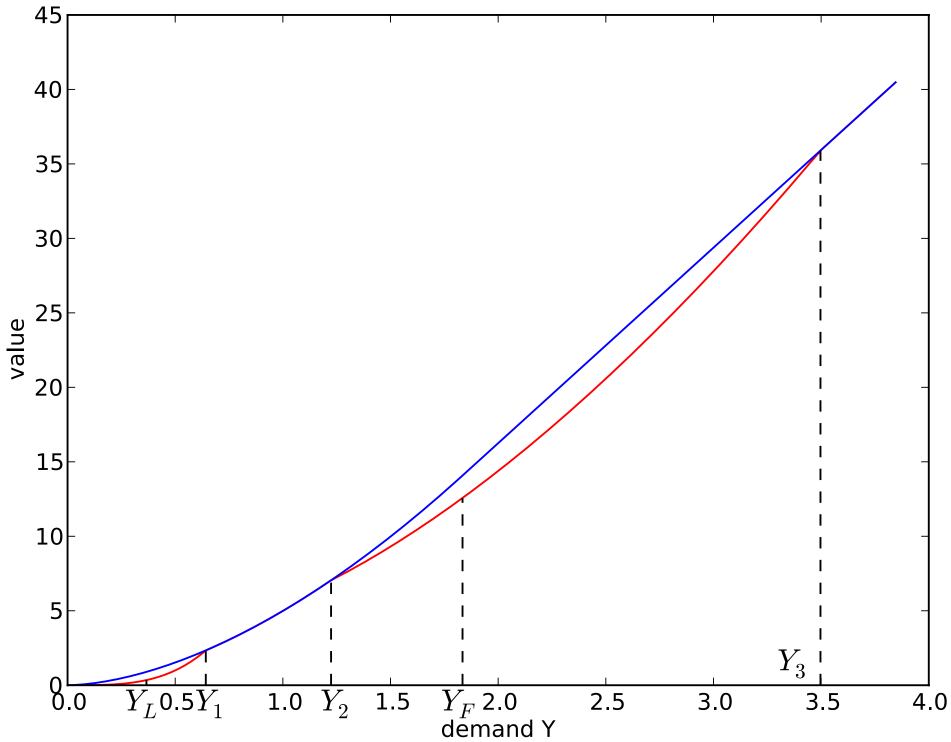

Fig. 6. Value F^{π} for a predetermined follower compared with the follower value F obtained when roles are not predetermined. Parameter values are the same as in Figure 1.

3. Option to invest in incomplete markets

We consider again two firms with the option to invest in the project for a sunk cost of K but now drop the assumption of a complete market. Indeed, many real options involve non-traded underlying assets, such as real estate prices, pharmaceutical developments, etc. As observed in Remark 3, in the complete market case we can equivalently choose either the stochastic demand (Y_t) or the project value (V_t) as the state variable and convert one into another through expected value of discounted cash flows. In incomplete markets this is more delicate, since cash flows received at different times cannot be easily compared. In this section we follow one of the approaches used in [2] and treat project values as lump-sum payoffs instead of present values of future cash flows and compare payoffs at different times using certainty equivalent arguments in the context of optimal utility of terminal wealth. The alternative, also considered in [2], of dealing with instantaneous cash flows associated with the stochastic demand in the context of optimal utility of consumption will not be pursued here.

Accordingly, the project value V_t is now assumed to be partially correlated with a traded asset P_t as follows:

$$\frac{dV_t}{V_t} = \nu dt + \eta(\rho dW_t + \sqrt{1-\rho^2} dW_t^0), \tag{3.1}$$

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t, \tag{3.2}$$

where $\rho \in (-1, 1)$ is a constant and (W_t^0) is a Brownian motion independent of (W_t). Observe that the dynamics takes place under the physical measure $\mathbb{P}$ and that (3.1) reduces to a complete market in the limit $\rho \to 1$. The case $\rho \to -1$ also corresponds to a complete market, with the obvious modifications in the corresponding hedging strategies.

For a monopolistic firm, investing in the project at time t means receiving a lump sum equal to V_t. In the duopoly case considered here, if a firm invests after another firm has already invested it receives a reduced lump sum equal to $(1-a)V_t$, for some $0 < a < 1$, whereas the other firm keeps a fraction bV_t of the original project value, with $0 < b < 1$. Setting $b = a$ corresponds to the framework of [2], where the total project value remains the same and is divided between the two firms according to the proportions a and $1-a$. Setting $b = (1-a)$ is analogous to the framework used in Section 2, with the total project value in the presence of both firms becoming $(1-a+b)V_t$ and being divided equally between the firms.

Next we assume that both firms act as utility maximizing agents with an exponential utility function

$$U(x) = -e^{-\gamma x}, \tag{3.3}$$

where $\gamma > 0$ is the risk aversion coefficient. In addition to investing in the project, the firms can allocate an amount θ_t to be invested at time t in the traded asset with price P_t. As usual in this type of problems, we take the money market account as the numeraire, or equivalently, set $r = 0$. In this way, the wealth associated with the trading strategy θ evolves according to

$$dX_t^\theta = \theta_t \frac{dP_t}{P_t} = \theta_t \sigma(\lambda dt + dW_t). \tag{3.4}$$

For $u \geq t$, we denote the solution of this equation starting at x at time t by $X_u^{t,x,\theta}$.

3.1. *Follower Value Function*

As before, we denote a firm by L if it is the first to invest, by F if it is the second to invest, and by S if both firms invest simultaneously, and consider first the case where these roles are not predetermined.

Starting with the follower, given that one of the firms has already invested in the project, the remaining firm has an option to invest in the project at a stopping time τ by paying the sunk cost[a] K and receive a lump sum $(1-a)V_\tau$, where $0 < a < 1$. For an infinite maturity, this is again a perpetual early-exercise option with payoff $((1-a)V_\tau - K)^+$. However, since the market is incomplete, we cannot value this option using risk neutral expectations as in (2.7). Instead, we follow Henderson and Hobson [10] and define the value function for the follower as

$$f(x, v) = \sup_{(\tau,\theta)} \mathbb{E}\left[e^{\frac{\lambda^2 \tau}{2}} U\left(X_\tau^{0,x,\theta} + \left((1-a)V_\tau^{0,v} - K\right)^+\right)\right]. \tag{3.5}$$

[a]Because we use the bank account as the numeraire, or equivalently set $r = 0$, this corresponds to a sunk cost that increases at a rate r in units of currency but is constant when expressed in units of the bank account.

The choice of the discount factor $e^{\frac{\lambda^2\tau}{2}}$ is explained in the Appendix of [10] and leads to a horizon unbiased optimization problem. It then follows from [10] that if we set

$$\beta \equiv \beta(\rho) := 1 + \frac{2(\rho\lambda - \xi)}{\eta} > 1, \tag{3.6}$$

and define $V_F = V^*/(1-a)$, where $V^* \equiv V^*(\rho)$ is the solution to the nonlinear equation

$$\kappa(V^* - K) = \log\left[1 + \frac{\kappa V^*}{\beta}\right], \qquad \kappa := \gamma(1-\rho^2), \tag{3.7}$$

then $(1-a)V_F > K$ and the follower value function is given by $f(x,v) = -e^{-\gamma(x+F(v))}$, where

$$F(v) = \begin{cases} -\frac{1}{\kappa}\log\left[1 - \left(1 - e^{-\kappa((1-a)V_F - K)}\right)\left(\frac{v}{V_F}\right)^{\beta}\right] & \text{if } 0 \le v \le V_F, \\ (1-a)v - K & \text{if } v > V_F. \end{cases} \tag{3.8}$$

3.2. *Leader Value Function*

As before, after investing in the project at time t at level v, the leader has no further decisions to take. However, the value received by the leader upon investment must take into account that the follower will also invest in the project at the stopping time

$$\tau_F(v) = \inf\{u \ge t : V_u^{t,v} \ge V_F\}. \tag{3.9}$$

If $v > V_F$ the follower invests immediately and receives a fraction $(1-a)v$ of the project value, so that the payoff for the leader is $(bv - K)$, where $0 < b < 1$ is a the fraction of the project value received by the leader.

Conversely, if $v \le V_F$ then the follower will invest in the project at $\tau_F \equiv \tau_F(v) > t$ and receive $(1-a)V_{\tau_F}^{t,v} - K$. In the complete market case it was straightforward to take this into account when calculating the project value for the leader in expression (2.18). For the incomplete market case, we use the following argument along the lines of [2]. We represent the reduction in project value experienced by the leader upon entrance of the follower as a lump sum loss $(1-b)V_{\tau_F}^{t,v}$ at τ_F and consider its utility indifference value for the leader at time t. For this, and taking without loss of generality $t = 0$, consider

$$h(x,v) = \sup_{\theta} \mathbb{E}\left[e^{\frac{\lambda^2\tau}{2}} U\left(X_{\tau_F}^{0,x,\theta} - (1-b)V_{\tau_F}^{0,v}\right)\right], \tag{3.10}$$

which corresponds to utility the leader obtains by optimally allocating wealth over the interval $(0, \tau_F)$ in anticipation of the loss in project value at τ_F. Following the same steps as in [2], it is straightforward to show that

$$h(x,v) = U(x)\left[1 - \left(1 - e^{\kappa(1-b)V_F}\right)\left(\frac{v}{V_F}\right)^{\beta}\right]^{\frac{1}{1-\rho^2}}, \tag{3.11}$$

with β defined in (3.6). We then define the utility indifference value $H_F(v)$ for the reduction in project value experienced by the leader through the equality

$$h(x - H_F(v), 0) = h(x,v), \tag{3.12}$$

from which it follows that

$$H_F(v) = \frac{1}{\kappa}\log\left[1 - \left(1 - e^{\kappa(1-b)V_F}\right)\left(\frac{v}{V_F}\right)^{\beta}\right]. \tag{3.13}$$

Using $H_F(v)$, we can incorporate the expected reduction in project value at τ_F into the value function $\ell(x,v) = -\exp(-\gamma(x + L(v)))$ for the leader simply by setting

$$L(v) = \begin{cases} v - H_F(v) - K & \text{if } v \le V_F, \\ bv - K & \text{if } v > V_F. \end{cases} \tag{3.14}$$

It can be checked that $\lim_{v \nearrow V_F} H_F(v) = (1-b)V_F$ so that $L(v)$ is continuous at $v = V_F$.

Finally, when the firms are symmetric, they should receive the same project value when exercising simultaneously. Imposing continuity of the value function for the leader and the follower at V_F requires that we choose $b = 1-a$ as in Section 2, leading to a value function for simultaneous exercise of the form $s(x,v) = -\exp(-\gamma(x + S(v)))$, where

$$S(v) = (1-a)v - K. \tag{3.15}$$

Remark 5. Our derivation of the value function for the leader differs from Bensoussan *et al.* [2] in two respects. First, our reduction in project value for the leader is given as $(1-b)v$ for an arbitrary factor $0 < b < 1$, whereas [2] sets $b = a$, meaning that the leader experiences a reduction in project value exactly equal to the lump sum received by the follower. Secondly, [2] considers utility indifference arguments from the point of view of the follower, again implicitly assuming that any amount received by the follower is subtracted from the total wealth of the leader. We disagree with this approach and prefer to use the utility indifference of the leader directly since the Stackelberg game involves no direct transactions between leader and follower.

3.3. *Equilibrium Strategies*

As in the complete market case, we start with a technical result comparing the value functions obtained in the previous section.

Proposition 9. *Setting $(1-a) = b$, assume that $b_1^* < b < 1$ where b_1^* satisfies*

$$b_1^* \left(1 - e^{-\kappa\left(\frac{V^*}{b_1^*} - K\right)}\right) = 1 - e^{-\kappa(V^* - K)}. \tag{3.16}$$

In addition, if $\kappa V^* < \beta$, assume that $b_2^* < b < 1$, where b_2^* satisfies

$$e^{-\kappa\left(\frac{V^*}{b_2^*} - K\right)} = 2e^{-\kappa(V^* - K)} - 1. \tag{3.17}$$

Then there exist a unique point V_L such that

$$\begin{cases} L(v) < F(v) & \text{for } v < V_L, \\ L(v) = F(v) & \text{for } v = V_L, \\ L(v) > F(v) & \text{for } V_L < v < V_F, \end{cases} \tag{3.18}$$

and

$$\begin{cases} S(v) < \min\left(L(v), F(v)\right) & \text{for } v < V_F, \\ S(v) = F(v) & \text{for } v \ge V_F. \end{cases} \tag{3.19}$$

Proof. For any $v < V_F$, let

$$d_1(v) := L(v) - F(v) = v - K + \frac{1}{\kappa}\log\left[\frac{1 - C_1\left(\frac{v}{V_F}\right)^\beta}{1 - C_2\left(\frac{v}{V_F}\right)^\beta}\right].$$

where

$$C_1 := \left(1 - e^{-\kappa((1-a)V_F - K)}\right), \qquad C_2 := \left(1 - e^{\kappa a V_F}\right). \tag{3.20}$$

Observe first that $d_1(0) = -K < 0$ and $d_1(V_F) = 0$. It is also easy to see that $d_1(v)$ is continuously differentiable with

$$d_1'(v) = 1 + \frac{1}{\kappa}\frac{\beta v^{\beta-1}}{V_F^\beta}\frac{C_2 - C_1}{(1 - C_1\left(\frac{v}{V_F}\right)^\beta)(1 - C_2\left(\frac{v}{V_F}\right)^\beta)}, \tag{3.21}$$

so that we have

$$\begin{aligned} d_1'(V_F) &= 1 + \frac{1}{\kappa}\frac{\beta}{V_F}\frac{C_2 - C_1}{(1 - C_1)(1 - C_2)} \\ &= 1 + \frac{1}{\kappa}\frac{\beta}{V_F}\frac{e^{-\kappa(V_F - K)} - 1}{e^{-\kappa((1-a)V_F - K)}} \\ &= 1 - \frac{\beta b}{\kappa V^*}\frac{1 - e^{-\kappa(\frac{V^*}{b} - K)}}{e^{-\kappa(V^* - K)}} \\ &= 1 - \frac{b\left(1 - e^{-\kappa(\frac{V^*}{b} - K)}\right)}{1 - e^{-\kappa(V^* - K)}} < 0, \end{aligned}$$

since $b > b_1^*$ defined in (3.16). Finally, we have that

$$d_1''(v) = \frac{1}{\kappa}\frac{\beta v^{\beta-1}}{V_F^\beta}\frac{C_2 - C_1}{\left(1 - C_1\left(\frac{v}{V_F}\right)^\beta\right)\left(1 - C_2\left(\frac{v}{V_F}\right)^\beta\right)} \tag{3.22}$$

$$\times\left[\frac{\beta - 1}{v} + \frac{C_1\beta\frac{v^{\beta-1}}{V_F^\beta}}{1 - C_1\left(\frac{v}{V_F}\right)^\beta} + \frac{C_2\beta\frac{v^{\beta-1}}{V_F^\beta}}{1 - C_2\left(\frac{v}{V_F}\right)^\beta}\right]. \tag{3.23}$$

To determine the sign of this second derivative, observe first that $V_F > (1-a)V_F > K$ implies that

$$C_2 - C_1 = e^{\kappa a V_F}\left(e^{-\kappa(V_F - K)} - 1\right) < 0, \tag{3.24}$$

so that the factor appearing in (3.22) is always strictly negative. For the remaining expression, observe that if $\kappa V^* > \beta$ holds, then

$$2e^{-\kappa(V^* - K)} - 1 < 0 < e^{-\kappa(\frac{V^*}{b} - K)}, \tag{3.25}$$

for any $0 < b < 1$. Conversely, if $\kappa V^* < \beta$ and $b > b_2^*$, we have that

$$2e^{-\kappa(V^* - K)} - 1 < e^{-\kappa(\frac{V^*}{b} - K)}, \tag{3.26}$$

since b_2^* is defined by (3.17) and the map $b \to e^{-\kappa(\frac{V^*}{b}-K)}$ is increasing. In either case we find that

$$\begin{aligned} C_1 + C_2 &= 2C_1C_2 + e^{\kappa a V_F}\left[1 - 2e^{-\kappa((1-a)V_F - K)} + e^{-\kappa(V_F-K)}\right] \\ &= 2C_1C_2 + e^{\kappa a V_F}\left[1 - 2e^{-\kappa(V^*-K)} + e^{-\kappa(\frac{V^*}{b}-K)}\right] \\ &> 2C_1C_2, \end{aligned}$$

which in turn implies the following chain of inequalities

$$\begin{aligned} & C_1 + C_2 > 2C_1C_2\left(\frac{v}{V_F}\right)^\beta \\ \iff \quad & C_1\left[1 - C_2\left(\frac{v}{V_F}\right)^\beta\right] > -C_2\left[1 - C_1\left(\frac{v}{V_F}\right)^\beta\right] \\ \iff \quad & \frac{C_1\beta\frac{v^{\beta-1}}{V_F^\beta}}{1 - C_1\left(\frac{v}{V_F}\right)^\beta} > -\frac{C_2\beta\frac{v^{\beta-1}}{V_F^\beta}}{1 - C_2\left(\frac{v}{V_F}\right)^\beta} \\ \iff \quad & \frac{C_1\beta\frac{v^{\beta-1}}{V_F^\beta}}{1 - C_1\left(\frac{v}{V_F}\right)^\beta} + \frac{C_2\beta\frac{v^{\beta-1}}{V_F^\beta}}{1 - C_2\left(\frac{v}{V_F}\right)^\beta} > 0. \end{aligned}$$

Since $\beta > 1$, we conclude that the expression inside the square brackets in (3.23) is always positive, which implies that d_1 is strictly concave on $v \in (0, V_F)$. We conclude that there exist a unique root $V_L < V_F$ for $d_1(v)$ in the interval $[0, V_F]$. A similar calculation using $d_2(v) := S(v) - F(v)$ concludes the proof. □

The behaviour of the incomplete market thresholds V_L and V_F with respect to risk aversion is illustrated in Figures 7 and 8. In both figures we see that higher risk aversion diminishes the follower's option value for waiting, which implies that V_F decreases in γ. However, for V_L there are two effects in play, preemption and risk aversion, and their interaction is ambiguous. Indeed, when risk aversion is close to zero, the situation is similar to the complete market analyzed in Section 2: once a leader invests, a rational follower waits until V_F; the benefits of being alone in the market make preemption the dominating effect and give rise to a low investment threshold V_L for the leader. As risk aversion increases, the leader weighs the loss of project value due to the subsequent entrance by the follower more heavily, resulting in a lower leader value L in (3.14). Since V_L is defined by $L(V_L) = F(V_L)$, its exact behavior depends on how fast the follower value F in (3.8) decreases with risk aversion. Ultimately, for high enough values of risk aversion, both Figures 7 and 8 show that V_L coincides with V_F, meaning that any remaining advantage of being a leader disappears.

As before, we will focus on symmetric, Markov, sub-game perfect equilibrium exercise strategies. In particular, we assume that both firms have the same level of initial wealth x and the same utility function. These assumptions are relaxed in Appendix C, where we offer some remarks on how to treat an example of an asymmetric case. According to the previous proposition, if $v < V_L$, neither firm wants to invest and both wait for the project value to rise, whereas if $v > V_F$, both firms invest immediately.

For the region $V_L < v < Y_F$, both firms wish to be the only investor and we consider again a mixed strategy in the instantaneous stage game consisting of exercising the option to invest with

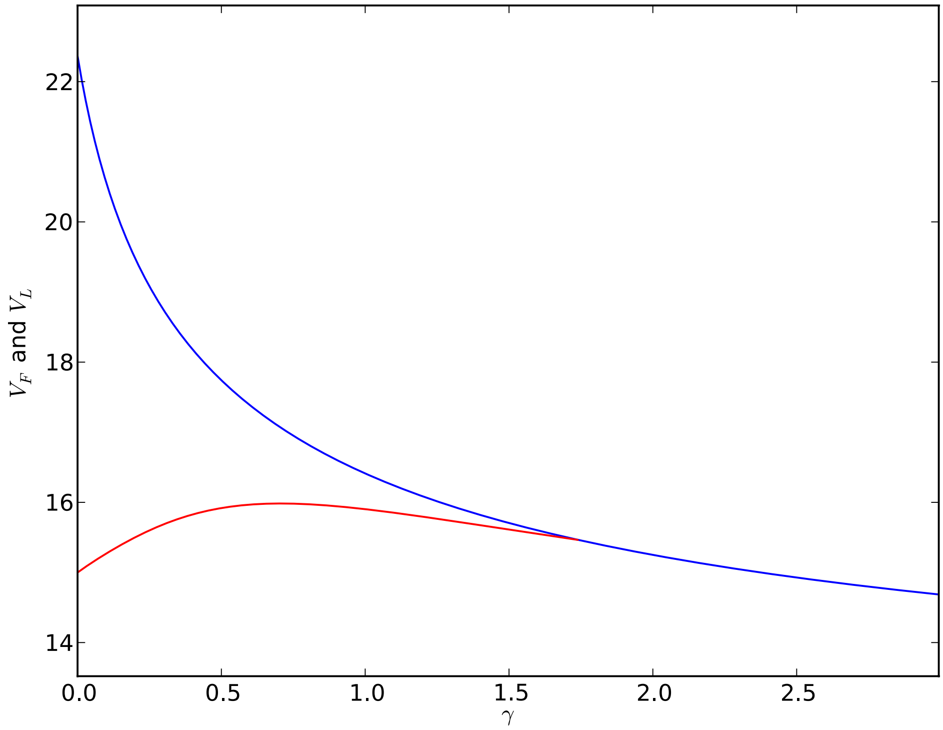

Fig. 7. Leader and Follower investment thresholds in the incomplete market case, when $a = 0.2$. Parameters are the same as Figure 1, with $\rho = 0.8$.

stationary probabilities $p_i(v)$ and $p_j(v)$ and the three possible outcomes described immediately before equation (2.23). The expected utility for firm i is then

$$U_i\,(x, v; p_i, p_j) = \left[p_i(1-p_j)\ell(x,v) + p_j(1-p_i)f(x,v) + p_i p_j s(x,v)\right] \sum_{k=0}^{\infty}(1-p_i)^k(1-p_j)^k$$
$$= \frac{p_i(1-p_j)\ell(x,v) + p_j(1-p_i)f(x,v) + p_i p_j s(x,v)}{1-(1-p_i)(1-p_j)}. \tag{3.27}$$

Again, firm i wants to maximize this expected value with respect to p_i knowing that firm j can choose the same strategy. A calculation similar to the complete market case then leads to the following optimal probability to invest (independent of x):

$$\widehat{p}(v) = \frac{\ell(x,v) - f(x,v)}{\ell(x,v) - s(x,v)} = \frac{L(v) - F(v)}{L(v) - S(v)}. \tag{3.28}$$

Equilibrium strategies for the two firms can now be characterized analogously to the complete market case:

Theorem 10. *Assume that the hypotheses in Proposition 9 hold and that both firms have the same initial wealth x and risk aversion γ. Then there exists a symmetric, Markov, sub-game*

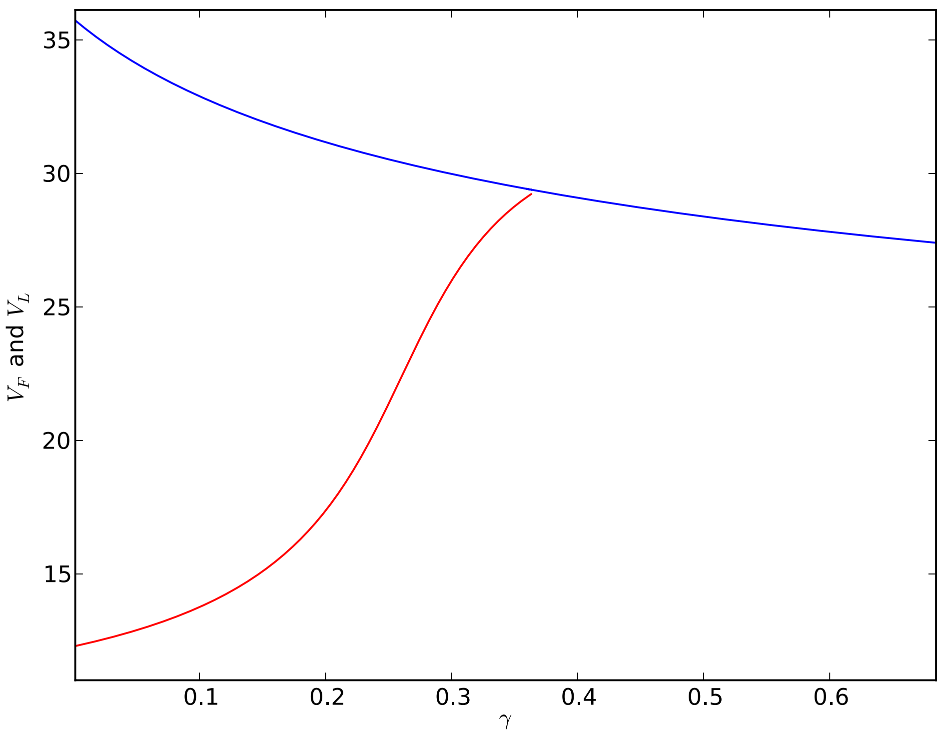

Fig. 8. Leader and Follower investment thresholds in the incomplete market case, when $a = 0.5$. Parameters are the same as Figure 1, with $\rho = 0.8$.

perfect equilibrium with strategies depending on the level of demand as follows:

(i) *If $v < V_L$, both firms wait for the project value to rise and reach V_L.*

(ii) *At $v = V_L$, there is no simultaneous exercise and each firm has an equal probability of emerging as a leader while the other becomes a follower and waits until the project value rises to V_F.*

(iii) *If $V_L < v < V_F$, each firm chooses a mixed strategy consisting of exercising the option to invest with probability $\widehat{p}(v)$. There can be an equilibrium with simultaneous exercise with probability*

$$a_S(v) = \frac{L(v) - F(v)}{L(v) + F(v) - 2S(v)}, \tag{3.29}$$

and an equilibrium where one firm emerges as the leader and the other waits until demand rises to V_F with probability $(1 - a_S(v))$.

(iv) *If $v \geq V_F$, both firms invest immediately.*

As in the complete market case, the same argument used to establish Proposition 5 shows that the mixed strategies described above lead to each firm being indifferent between being the follower or playing the game:

Proposition 11 (Rent Equalization)**.** *The equilibrium described in Theorem 10 makes the expected utility for each firm to be equal to $f(x, v)$.*

3.4. *Priority to Invest*

As in Section 2.4, we assume now that the roles of leader and follower are predetermined. In other words, the leader has the option to invest in the project knowing that the follower is forbidden to invest until the leader has done so. That is, the leader can invest in the project at a random time τ and receive the value function $\ell(X_\tau, V_\tau) = -\exp(-\gamma(X_\tau + L(V_\tau)))$ according to (3.14). Therefore, the value function for the leader in this case is

$$\ell^\pi(x, v) = \sup_{\tau,\theta} \mathbb{E}\left[e^{\frac{\lambda^2\tau}{2}} U(X_\tau^{0,x,\theta} + L(V_\tau^{0,v})^+) \mathbf{1}_{\{\tau<\infty\}}\right], \tag{3.30}$$

where the superscript π is meant to indicate that the leader now has the priority to invest. As it is well known (see for example [10] or [11]), the dynamic programming equation associated with the combined optimal stopping and optimal control problem (3.30) is the following nonlinear variational inequality:

$$\min\left(-\frac{1}{2}\lambda^2\ell^\pi - \frac{1}{2}\eta^2 v^2 \ell^\pi_{vv} - \nu v \ell^\pi_v + \frac{(\rho\sigma\eta y \ell^\pi_{xv} + \mu\ell^\pi_x)^2}{2\sigma^2\ell^\pi_{xx}}, \ell^\pi - U(x + L(v)^+)\right) = 0. \tag{3.31}$$

Following [16], we set

$$\ell^\pi(x, v) = -U(x)\Sigma^{\frac{1}{1-\rho^2}}(v) = e^{-\gamma x}\Sigma^{\frac{1}{1-\rho^2}}(v) \tag{3.32}$$

and obtain an equation that is linear in the derivatives of Σ:

$$\min\left(-\frac{1}{2}\eta^2 v^2 \Sigma'' - \left(\nu - \frac{\rho\eta\mu}{\sigma}\right) v, \Sigma + e^{-\kappa L(v)^+}\right) = 0. \tag{3.33}$$

By its turn, (3.33) is the dynamic programming equation associated with the optimal stopping problem

$$\Sigma(v) = \sup_{\tau\in\mathcal{T}} \widetilde{\mathbb{E}}\left[-e^{-\kappa L(V_\tau^{0,v})^+}\right], \tag{3.34}$$

where $\widetilde{\mathbb{E}}[\cdot]$ denotes the expectation with respect to the minimal martingale measure for this problem (see [6]), under which the project value and the traded asset follow the dynamics

$$\frac{dV_t}{V_t} = \left(\nu - \frac{\rho\eta\mu}{\sigma}\right) dt + \eta(\rho dW_t^{\mathbb{Q}} + \sqrt{1-\rho^2} dW_t^0), \tag{3.35}$$

$$\frac{dP_t}{P_t} = \sigma dW_t^{\mathbb{Q}}, \tag{3.36}$$

with $W^{\mathbb{Q}} = W_t + \frac{\mu}{\sigma} t$ as before.

Since the obstacle $\Psi(v) = -\exp(-\kappa L(v)^+)$ in (3.34) is bounded, we can use verification to obtain the solution presented in the next theorem. Before we state the result, let us define the constants V_1 and B_1 as a solution to the system

$$-1 + B_1 V_1^\beta = -e^{-\kappa L(V_1)}, \tag{3.37}$$

$$\beta B_1 Y_1^{\beta-1} = \kappa e^{-\kappa L(V_1)} L'(V_1). \tag{3.38}$$

Substituting (3.37) into (3.38) we find that V_1 satisfies the nonlinear equation

$$\kappa L(V_1) = \log\left[1 + \frac{\kappa}{\beta} V_1 L'(V_1)\right]. \tag{3.39}$$

We show in Appendix D that (3.39) has a solution V_1 in the interval $(0, V_F)$ provided $b > b_3^*$, where

$$\beta - \kappa\left(\frac{1}{b_3^*} - 1\right) V^* = \beta e^{-\kappa\left(\frac{1}{b_3^*}-1\right)V^*}. \tag{3.40}$$

Substituting V_1 back into (3.37) gives the constant B_1. Next define the constants V_2, V_3, B_2, B_3 as a solution to the nonlinear system

$$\begin{aligned} B_2 + B_3 V_2^{\beta} &= -e^{-\kappa L(V_2)}, \\ \beta B_3 V_2^{\beta_1 - 1} &= \kappa e^{-\kappa L(V_2)} L'(V_2), \\ B_2 + B_3 V_3^{\beta} &= -e^{-\kappa L(V_3)}, \\ \beta B_3 V_3^{\beta_1 - 1} &= \kappa e^{-\kappa L(V_3)} L'(V_3). \end{aligned} \tag{3.41}$$

Theorem 12. *Let* V_1 *and* B_1 *be given by* (3.37) *and* (3.38) *and assume that the nonlinear system* (3.41) *has a solution* (V_2, V_3, B_2, B_3). *If*

$$0 < V_1 < V_2 < V_F < V_3, \tag{3.42}$$

then

$$\Sigma(v) = \begin{cases} -1 + B_1 v^{\beta} & \text{if } 0 \le v < V_1, \\ -e^{-\kappa L(v)} & \text{if } V_1 \le v \le V_2, \\ B_2 + B_3 v^{\beta} & \text{if } V_2 < v < V_3, \\ -e^{-\kappa L(v)} & \text{if } v \ge V_3, \end{cases} \tag{3.43}$$

where $\beta \equiv \beta(\rho)$ *is defined in* (3.6).

Proof. We proceed once more by verifying the conditions in Theorem 13. Set

$$\mathcal{D} = (0, V_1) \cup (V_2, V_3) \tag{3.44}$$

and observe that the constants $V_1, V_2, V_3, B_1, B_2, B_3$ were defined so that the value matching and smooth pasting conditions

$$\Sigma(V_i) = -e^{-\kappa L(V_i)}, \tag{3.45}$$

$$(\Sigma)'(V_i) = \kappa e^{-\kappa L(V_i)} L'(V_i), \tag{3.46}$$

are satisfied for $i = 1, 2, 3$. We then see that $\Sigma \in C^1(G) \cap C(\overline{G})$ and $\Sigma \in C^2(G \backslash \partial\mathcal{D})$ with locally bounded second order derivative near $\partial\mathcal{D}$, so conditions (i) and (v) are satisfied. Conditions (ii), (iii), (iv), (viii) and (ix) hold by construction of $\mathcal{D}$ and Σ and elementary properties of the process (V_t). Condition (vii) holds by observing that β defined in (3.6) is the positive root of the characteristic equation

$$\frac{1}{2}\eta^2 \beta(\beta - 1) + \left(\nu - \frac{\rho\eta\mu}{\sigma}\right)\beta = 0, \tag{3.47}$$

whereas condition (vi) follows from a direct calculation using the expression for L in (3.14). □

The obstacle $\Psi(v)$ and the function $\Sigma(v)$ are shown in Figure 9. We see that for symmetric firms, the optimal investment strategy for predetermined leader and follower are exactly the same as in the complete market, namely:

(i) For $0 \le v < V_1$ the leader waits to invest until the project value rises to V_1 and the follower waits to invest until it rises to V_F.
(ii) For $V_1 \le v \le V_2$ the leader invests immediately and the follower waits to invest until the project value rises to V_F.
(iii) For $V_2 < v < V_3$ the leader waits until either the project value drops to V_2, in which case the follower waits to invest until it rises to V_F, or rises to V_3, in which case the follower exercises immediately since $v \ge V_F$.
(iv) For $v \ge V_3$ both the leader and the follower invest immediately.

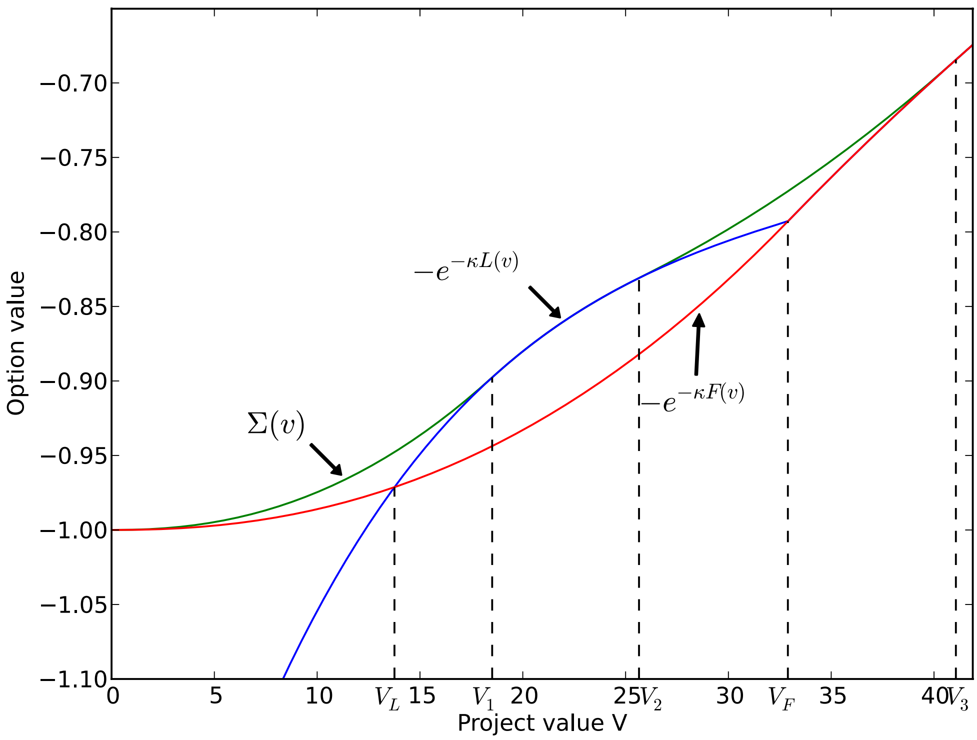

Fig. 9. Obstacle problem in an incomplete market with predetermined roles, showing the value function Σ for the predetermined leader, the payoff $-e^{-\kappa L}$ obtained upon investment, and the corresponding value function $-e^{-\kappa F}$ for the follower. Parameter values are the same as in Figure 1, with $\rho = 0.8$ and $\gamma = 0.1$, which results in $V_L = 13.76$, $V_1 = 18.51$, $V_2 = 25.65$, $V_F = 32.89$ and $V_3 = 41.06$.

Just as in the complete market case, we define the value of the priority option as the premium that a firm has to pay to acquire the right to be the leader. In the incomplete market setting, this is given by the value $\pi \equiv \pi(x,v)$ that makes a firm indifferent between having initial wealth $(x-\pi)$ and achieving the expected utility of a predetermined leader or having the initial wealth x and achieving the expected utility resulting from playing the game analyzed in Section 3.3. Using Proposition 11, we find that π is the solution to

$$\ell^{\pi}(x-\pi(x,v),v) = f(x,v). \tag{3.48}$$

Substituting the expressions for ℓ^{π} and f we obtain

$$\pi(x,v) \equiv \pi(v) = \frac{1}{\gamma}\log\left[\frac{e^{-\gamma F(v)}}{-\Sigma^{1/(1-\rho^2)}(v)}\right]. \tag{3.49}$$

The qualitative behavior of the priority option value is similar to the complete market case illustrated in Figure 5. In particular we see that $\pi(v) = L(v) - F(v)$ for $V_1 \leq v \leq V_2$ where $\ell^{\pi}(x,v) = \ell(x,v)$, and that $\pi(v) = 0$ for all $v \geq V_3$, since $L(v) = F(v) = (1-a)v - K$ in this region.

4. Concluding Remarks

We have analyzed competitive investment for two symmetric firms in both complete and incomplete markets. When neither firm has a predetermined role, they play a timing/coordination game with mixed strategies to decide when to invest in an underlying project subject to uncertainty. Depending on the outcome of the game, investment can occur either sequentially, with one firm emerging as the leader and the other as the follower, or simultaneously. When one of the firms has been preassigned as the leader, it extracts a larger value from the project than the expected value in the case of no preassigned roles. We call this difference the priority option and calculate its value.

In the complete market case, the underlying state variable is a stochastic demand (Y_t). Confirming the result obtained in [9], we find that when demand is below a threshold Y_L neither firm has an incentive to invest in the project and both firms should wait for it to rise, whereas when demand is above a higher threshold Y_F, it is optimal for both firms to invest immediately. In the intermediate region (Y_L, Y_F), both firms would prefer to be the first to invest, but simultaneous investment is strictly worse than being the second to invest, as illustrated in Figure 2. The unique symmetric Nash equilibrium in this region consists of each firm investing according to the optimal probability in (2.26). Our main contribution in this section is a rigorous analysis of the behaviour at the end points Y_L and Y_F: at Y_F both firms invest simultaneously, whereas at Y_L each becomes the leader with probability 1/2 and invests immediately, while the other becomes the follower and postpones investment until the demand rises to Y_F.

Preassigning the roles of of each firm corresponds to the situation analyzed in [2], with the leader value $L(y)$ previously computed in (2.19) appearing as a payoff in the optimal control problem (2.30) for the predetermined leader. The shape of this obstacle is illustrated in Figure 4 and yields the three-threshold form for the value function $L^{\pi}(y)$ in (2.40). As mentioned in the Introduction, this leads to the existence of the interval $[Y_2, Y_3]$, whereby the predetermined leader waits until demand either rises to Y_3 or *drops* to Y_2 before investing. The phenomenon of investment at falling demand (i.e, the existence of the threshold Y_2) has been linked to a "recession induced boom" in the context of real estate development analyzed in [8] in a model with delay in building time. Our results show that the same phenomenon can arise from the existence of a predetermined leader in the market. The actual advantage of being a leader is represented by the priority option value computed in Figure 5. We see that this advantage is not monotonic with respect to the underlying demand, being negligible for both very low and very high demand, but significant for intermediate values of demand.

Apart from technicalities, our analysis shows that the description above largely applies to incomplete markets as well. The first technical difficulty concerns the comparison of cash flows obtained at different times, which is straightforward in complete markets but requires subtle utility-indifference arguments in incomplete one. We accordingly treat project values as lump sums received at specific times, rather than accruing from continuous cash flows originated by instantaneous demand. This allowed us to obtain the incomplete market analogues for the leader, follower and simultaneous exercise values. Comparing these values, however, is significantly more complicated than in complete markets. Under the sufficient conditions presented in Proposition 9, the exact analogue of Figure 2 holds: when the project value left for the leader upon entrance of the follower is high enough, as measured by the constant b, there is a region $[V_L, V_F]$ where it is strictly better to be a leader than to be a follower. Nonetheless, when risk aversion increases these conditions become harder to satisfy and the existence of the lower threshold V_L cannot be guaranteed.

Assuming that the conditions of Proposition 9 are satisfied and that the firms are perfectly symmetric, including having the same risk aversion, we find that the Nash equilibrium for the game played in the region $[V_L, V_F]$ is entirely analogous to the case in complete markets. Finally, the case of preassigned roles also generalizes to incomplete markets, with the same type of value function for the predetermined leader given in terms of three distinct thresholds. As in complete markets, the value of the priority option can be defined as the premium that needs to be paid to acquire the right to be a predetermined leader, except that in incomplete markets this premium is computed as an indifference value.

Appendix A. Variational inequalities for optimal stopping problems

For convenience, we reproduce Theorem 10.4.1 in [12] adapted to our problem. Let

$$dY_t = b(Y_t) + \sigma(Y_t)dW_t \tag{A.1}$$

and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\mathbb{E}\left[\sup_{0\le t<\infty} |g(Y_t)|\right] < \infty. \tag{A.2}$$

Consider the problem

$$\Phi(y) := \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau} g(Y_\tau^{0,y})\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{A.3}$$

Define the generator of the diffusion Y as

$$\mathcal{L} = b(y)\frac{\partial}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2}{\partial y^2}. \tag{A.4}$$

We then have the following verification result:

Theorem 13. *Let $G = (0,\infty)$, $\phi : \overline{G} \to \mathbb{R}$ and $\mathcal{D} = \{x \in G : \phi(x) > g(x)\}$. Suppose that*

(i) $\phi \in C^1(G) \cap C(\overline{G})$,
(ii) $\phi \ge g$ on G and $\phi = g$ on ∂G,
(iii) Y_t spends 0 time on $\partial\mathcal{D}$ a.s.,

(iv) ∂D *is a Lipschitz surface,*
(v) $\phi \in C^2(G\backslash\partial\mathcal{D})$ *and the second order derivative of* ϕ *is locally bounded near* $\partial\mathcal{D}$*,*
(vi) $r\phi - L\phi \geq 0$ *on* $G\backslash\mathcal{D}$*,*
(vii) $r\phi - L\phi = 0$ *on* $\mathcal{D}$*,*
(viii) $\tau_{\mathcal{D}}(y) := \inf\{t > 0 : Y_t^{0,y} \notin \mathcal{D}\} < \infty$ *a.s. for all* $y \in G$*,*
(ix) the family $\{\phi(Y_\tau^{0,y}) : \tau \leq \tau_{\mathcal{D}}(y), \tau \in \mathcal{T}\}$ *is uniformly integrable for all* $y \in G$*.*

Then

$$\phi(y) = \Phi(y) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau} g(Y_\tau^{0,y})\mathbf{1}_{\{\tau<\infty\}}\right] \quad y \in G \tag{A.5}$$

and $\tau* = \tau_D$ *is an optimal stopping time for this problem.*

Appendix B. Proof of Proposition (3)

For $y \in (0, Y_F)$, define the function $d_1(y) := L(y) - F(y)$, that is

$$\begin{aligned} d_1(y) &= \frac{D_1 y}{\delta} - \frac{(D_1 - D_2)}{D_2}\frac{K\beta}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta - K - \frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta \\ &= \frac{D_1 y}{\delta} - \frac{\beta(D_1 - D_2) + D_2}{D_2}\frac{K}{(\beta-1)}\left(\frac{y}{Y_F}\right)^\beta - K. \end{aligned}$$

Therefore,

$$\begin{aligned} d_1'(Y) &= \frac{D_1}{\delta} - \frac{\beta(D_1 - D_2) + D_2}{\delta\beta}\left(\frac{y}{Y_F}\right)^{\beta-1}; \\ d_1''(Y) &= -\frac{[\beta(D_1 - D_2) + D_2](\beta-1)}{\delta\beta Y_F}\left(\frac{y}{Y_F}\right)^{\beta-2} < 0, \end{aligned}$$

so d_1 is strictly concave. Moreover we have that

$$\begin{aligned} d_1(0) &= -K < 0, \qquad d_1(Y_F) = 0, \\ d_1'(0) &= \frac{D_1}{\delta} > 0, \qquad d_1'(Y_F) = \frac{(1-\beta)}{\delta}(D_1 - D_2) < 0. \end{aligned}$$

Therefore, there is a unique value Y_L for which $d_1(Y_L) = 0$ and such that $d_1(y) < 0$ for $y \in (0, Y_L)$ and $d_1(y) > 0$ for $y \in (Y_L, Y_F)$, as shown in Figure 2.

To conclude, it follows from the definition of $S(y)$ and $L(y)$ that $S(y) - K < L(y) - K$ for $y < Y_F$. Define now $d_2(y) := S(y) - F(y)$, that is

$$d_2(y) = \frac{D_2 y}{\delta} - K - \frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^\beta. \tag{B.1}$$

We then have

$$\begin{aligned} d_2'(Y) &= \frac{D_2}{\delta}\left[1 - \left(\frac{y}{Y_F}\right)^{\beta-1}\right] \geq 0; \\ d_2''(Y) &= -\frac{D_2(\beta-1)}{\delta Y_F}\left(\frac{Y}{Y_F}\right)^{\beta-2} < 0, \end{aligned}$$

so that d_2 is also strictly concave. Moreover,

$$d_2(0) = -K, \qquad d_2(Y_F) = 0, \qquad d_2'(Y_F) = 0, \tag{B.2}$$

from which we can assert that $S(Y) < F(Y)$ for $Y < Y_F$.

Appendix C. An asymmetric example in incomplete markets

An example of asymmetric duopoly in a complete market has been extensively analyzed in [3], where the asymmetry arises from different investment costs for the two firms. We consider here an asymmetric case in incomplete markets where the risk-aversion of the two firms is not the same. More fundamentally, the asymmetry could arise from different discount rates and/or payoff functions, but the analysis is similar. Whichever the reason for asymmetry, all option payoffs and value functions are now firm-dependent and are denoted by $L_i(v)$, $F_i(v)$, etc, but it can be straightforwardly shown that there still exist thresholds V_L^i, V_F^i as before. We note that the leader value $L_i(v)$ depends on the follower threshold V_F^j of the other firm.

For concreteness, suppose that firm 1 is less risk-averse (smaller γ). As can be seen in Figure 7, lower γ leads to higher investment threshold, $V_F^1 > V_F^2$. Plugging-in the computed V_F^j into the equation defining $d_1(v)$ (recall that V_L is the resulting zero of this function), we numerically observe that $V_L^1 > V_L^2$ (note that are two effects here: changing γ and changing V_F. Our numerical experiments suggest that higher V_F decreases V_L and higher γ also decreases V_L, so both effects make V_L^2 to be less than V_L^1). Thus, the *more risk-averse* firm will in fact pre-empt. Overall, we have several cases for the outcome of the game depending on the initial condition $V_0 = v$:

$$\begin{aligned}
v < V_L^1: &\quad \text{both firms wait;}\\
V_L^2 \le v < V_L^1: &\quad \text{firm 2 becomes the Leader; firm 1 will invest at } V_F^1;\\
V_L^1 \le v < V_F^2: &\quad \text{a Nash equilibrium via a stage game;}\\
V_F^2 \le v < V_F^1: &\quad \text{firm 2 becomes the Leader; firm 1 will invest at } V_F^1;\\
V_F^1 \le v: &\quad \text{both firms invest immediately.}
\end{aligned}$$

We note that when $v \in [V_F^2, V_F^1]$ firm 2 is determined to invest immediately, and in light of this, firm 1 will delay investment since $S_1(v) < F_1(v)$, i.e. simultaneous investment is not preferred by firm 1. The most interesting region is $V_L^1 \le v < V_F^2$ where both firms wish to be Leader but do not want simultaneous investment. In that case, the payoff for firm 1 is (cf. (2.25))

$$E_1(v; p_1, p_2) = \frac{p_1(v)(1-p_2(v))L_1(v) + (1-p_1(v))p_2(v)F_1(v) + p_1(v)p_2(v)S_1(v)}{p_1(v) + p_2(v) - p_1(v)p_2(v)}.$$

Fixing momentarily p_2, and differentiating with respect to p_1 we obtain

$$\frac{\partial}{\partial p_1} E_1(v; p_1, p_2) = \frac{p_2(v)[L_1(v) - F_1(v)] + p_2^2(v)(S_1(v) - L_1(v))}{(p_1(v)(1-p_2(v)) + p_2(v))^2}.$$

We now observe that unless (i) $p_2(v) = \frac{L_1(v)-F_1(v)}{L_1(v)-S_1(v)}$ which leads to $E_1(v; p_1, p_2) = F_1(v)$ independent of p_1, (ii) the sign of $\frac{\partial}{\partial p_1} E_1(v; p_1, p_2)$ is constant on $[0,1] \ni p_1$. After checking similar expressions for firm 2 and its $E_2(v; p_1, p_2)$, it therefore follows that there are three Nash equilibria, namely the *pure* coordinated equilibria $(p_1(v), p_2(v)) = (0,1)$ and $(1,0)$, where one firm will become leader with certainty, and the mixed equilibrium

$$p_1^{mix}(v) = \frac{L_2(v) - F_2(v)}{L_2(v) - S_2(v)}, \qquad p_2^{mix}(v) = \frac{L_1(v) - F_1(v)}{L_1(v) - S_1(v)}. \tag{C.1}$$

An equilibrium refinement method is needed to pick among these possibilities; for instance we note that only the mixed Nash equilibrium is *perfect* in the sense of trembling-hand deviations. At the end-points of $[V_L^1, V_F^2]$ we find that

$$p_1^{mix}(V_L^1+) > 0,\ p_2^{mix}(V_L^1+) = 0, \quad \text{and} \quad p_1^{mix}(V_F^2-) = 1,\ p_2^{mix}(V_F^2-) < 1,$$

which shows that at both end-points, if the mixed strategies are used, firm *1* will emerge as leader. Hence, the mixed equilibrium generally preferences firm 1, in stark contrast to the neighboring regions $[V_L^2, V_L^1)$ and $[V_F^2, V_F^1)$ where firm *2* is guaranteed to invest first.

Appendix D. Solution to equation (3.39)

Let

$$h(x) := \kappa \left(x - \frac{1}{\kappa} \log \left[1 - (1 - e^{\kappa(1-b)V_F}) \left(\frac{x}{V_F} \right)^{\beta} \right] - K \right)$$
$$- \log \left(1 + \frac{\kappa}{\beta} x \cdot \left[1 + \frac{(1 - e^{\kappa(1-b)V_F}) \frac{\beta x^{\beta-1}}{V_F^{\beta}}}{\gamma(1-\rho^2)(1 - \left(1 - e^{\kappa(1-b)V_F}\right) \left(\frac{x}{V_F}\right)^{\beta})} \right] \right).$$

Then (3.39) is equivalent to the claim that $h(x)$ has a zero on $[0, V_F]$. We observe that

$$h(0) = -\kappa K < 0$$
$$h(V_F) = \kappa (bV_F - K) - \log \left[\frac{\gamma(1-\rho^2)V_F}{\beta} + e^{-\kappa(1-b)V_F} \right]$$

Taking $b = (1 - a)$ and using (3.7), the above is equivalent to

$$h(V_F) = \log \left(1 + \frac{\gamma(1-\rho^2)bV_F}{\beta} \right) - \log \left[\frac{\gamma(1-\rho^2)V_F}{\beta} + e^{-\kappa(1-b)V_F} \right].$$

It follows that $h(V_F) > 0$, whenever

$$1 - \frac{\kappa(1-b)V_F}{\beta} > e^{-\kappa(1-b)V_F},$$

which is in turn satisfied for any $b > b_3^*$, where b_3^* is defined by (3.40).

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