A Strong Form of the Quantitative Wulff Inequality for Crystalline Norms

Kenneth DeMason

McMaster University

December 6, 2025

Outline

- Introduction
  - Anisotropy and crystals
  - Quantitative stability
- Main Result
- Overview of Argument

Anisotropy

Many physical phenomena can be explained in terms of energy minimization. E.g., soap bubbles are spheres because they need to minimize surface tension with a constrained volume.

The formation of crystals in the small mass regime can be explained similarly. Thermodynamically, crystals at equilibrium should minimize Gibbs free energy:

$$\Delta G := \sum_i \gamma_i A_i = \lambda \sum_i h_i A_i,$$

where γ_i is the surface energy per unit area and A_i is the area of the ith face. The equality is due to Wulff, where he interpreted the problem in terms of a Lagrange multiplier $\lambda > 0$, and h_i is the distance to each face.

Anisotropy

Many physical phenomena can be explained in terms of energy minimization. E.g., soap bubbles are spheres because they need to minimize surface tension with a constrained volume.

The formation of crystals in the small mass regime can be explained similarly. Thermodynamically, crystals at equilibrium should minimize Gibbs free energy:

$$\Delta G := \sum_i \gamma_i A_i = \lambda \sum_i h_i A_i,$$

where γ_i is the surface energy per unit area and A_i is the area of the ith face. The equality is due to Wulff, where he interpreted the problem in terms of a Lagrange multiplier $\lambda > 0$, and h_i is the distance to each face.

Anisotropy

Many physical phenomena can be explained in terms of energy minimization. E.g., soap bubbles are spheres because they need to minimize surface tension with a constrained volume.

The formation of crystals in the small mass regime can be explained similarly. Thermodynamically, crystals at equilibrium should minimize Gibbs free energy:

$$\Delta G := \sum_i \gamma_i A_i = \lambda \sum_i h_i A_i,$$

where γ_i is the surface energy per unit area and A_i is the area of the ith face. The equality is due to Wulff, where he interpreted the problem in terms of a Lagrange multiplier $\lambda > 0$, and h_i is the distance to each face.

Anisotropy: Wulff Shapes

These optimal configurations are called Wulff shapes. In general,

Definition

A *Wulff shape* is an open, bounded, convex set $K \subset \mathbb{R}^n$ containing the origin.

There are two important 1-homogeneous non-negative functions naturally associated to K:

- The *surface tension* $f : \mathbb{R}^n \to [0, \infty)$, for which $f(\nu)$ is the distance from the origin to the supporting hyperplane of K with normal ν. Typically view f as a function on S^{n-1}.
- The *gauge function* $f_* : \mathbb{R}^n \to [0, \infty)$, for which $K = \{f_* < 1\}$.

Anisotropy: Wulff Shapes

These optimal configurations are called Wulff shapes. In general,

Definition

A *Wulff shape* is an open, bounded, convex set $K \subset \mathbb{R}^n$ containing the origin.

There are two important 1-homogeneous non-negative functions naturally associated to K:

- The *surface tension* $f : \mathbb{R}^n \to [0, \infty)$, for which $f(\nu)$ is the distance from the origin to the supporting hyperplane of K with normal ν. Typically view f as a function on S^{n-1}.
- The *gauge function* $f_* : \mathbb{R}^n \to [0, \infty)$, for which $K = \{f_* < 1\}$.

Anisotropy: The Surface Tension and Gauge

The surface tension and gauge function are always semi-norms on $\mathbb{R}^n$, norms when K is symmetric about the origin. In this case, f and f_* are dual to each other.

For example, if $f_* = \ell^p$ then $f = \ell^q$, for p, q conjugate exponents.

In fact, we always have

$$f(\nu) = \sup\{\langle x, \nu\rangle \mid f_*(x) < 1\}$$
$$f_*(x) = \sup\{\langle x, \nu\rangle \mid f(\nu) < 1\},$$

so that for any $x \in \mathbb{R}^n$ and $\nu \in S^{n-1}$,

$$\langle x, \nu\rangle \leq f(\nu) f_*(x).$$

This is known as the *Fenchel inequality*.

Anisotropy: The Surface Tension and Gauge

The surface tension and gauge function are always semi-norms on $\mathbb{R}^n$, norms when K is symmetric about the origin. In this case, f and f_* are dual to each other.

For example, if $f_* = \ell^p$ then $f = \ell^q$, for p, q conjugate exponents.

In fact, we always have

$$f(\nu) = \sup\{\langle x, \nu\rangle \mid f_*(x) < 1\}$$
$$f_*(x) = \sup\{\langle x, \nu\rangle \mid f(\nu) < 1\},$$

so that for any $x \in \mathbb{R}^n$ and $\nu \in S^{n-1}$,

$$\langle x, \nu\rangle \leq f(\nu) f_*(x).$$

This is known as the *Fenchel inequality*.

Anisotropy: The Surface Tension and Gauge

The surface tension and gauge function are always semi-norms on $\mathbb{R}^n$, norms when K is symmetric about the origin. In this case, f and f_* are dual to each other.

For example, if $f_* = \ell^p$ then $f = \ell^q$, for p, q conjugate exponents.

In fact, we always have

$$f(\nu) = \sup\{\langle x, \nu\rangle \mid f_*(x) < 1\}$$
$$f_*(x) = \sup\{\langle x, \nu\rangle \mid f(\nu) < 1\},$$

so that for any $x \in \mathbb{R}^n$ and $\nu \in S^{n-1}$,

$$\langle x, \nu\rangle \leq f(\nu) f_*(x).$$

This is known as the *Fenchel inequality.*

Anisotropy: The Anisotropic Perimeter

Given a Wulff shape K we can ask the following question: For what energy functional Φ is K the volume-constrained minimizer? I.e., find Φ such that

$$E \in \arg\min\{\Phi(F) \mid |F| = v\} \quad \text{if and only if} \quad E = rK + x_0, |rK| = v.$$

It turns out the following energy is appropriate

Definition

The *anisotropic perimeter* (associated to K) is given by

$$\Phi(E) = \int_{\partial^* E} f(\nu_E(x)) \, d\mathcal{H}^{n-1}(x).$$

The isotropic perimeter is recovered when $f = \ell^2$, for which the Wulff shape is a ball.

Anisotropy: Crystalline Setting

When K is a polytope we say that Φ is crystalline. We denote by N the number of facets of K, by F_i a generic $n-1$ dimensional facet, and by ν_i the outer unit normal of this facet.

In this setting we have that $f(\nu_i) = h_i$, the distance from the origin to the supporting hyperplane of F_i. In particular,

$$\Phi(K) = \int_{\partial^* K} f(\nu_K(x)) \, d\mathcal{H}^{n-1}(x) = \sum_{i=1}^{N} h_i \mathcal{H}^{n-1}(F_i),$$

which is precisely the form the minimum Gibbs free energy takes for a crystal at equilibrium.

Anisotropy: Crystalline Setting

When K is a polytope we say that Φ is crystalline. We denote by N the number of facets of K, by F_i a generic $n-1$ dimensional facet, and by ν_i the outer unit normal of this facet.

In this setting we have that $f(\nu_i) = h_i$, the distance from the origin to the supporting hyperplane of F_i. In particular,

$$\Phi(K) = \int_{\partial^* K} f(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \sum_{i=1}^{N} h_i \mathcal{H}^{n-1}(F_i),$$

which is precisely the form the minimum Gibbs free energy takes for a crystal at equilibrium.

Anisotropy: The Wulff Inequality

Φ is the right generalization to use because we have an anisotropic version of the isoperimetric inequality known as the *Wulff inequality*:

$$\Phi(E) \geq n|K|^{1/n}|E|^{(n-1)/n}$$

with equality if and only if $|E\Delta(rK + x_0)| = 0$ for some $r > 0$ and $x_0 \in \mathbb{R}^n$.

This is the same as the isoperimetric inequality with Φ in place of P and $K = \{f_* < 1\}$ in place of B_1. We have a rigidity statement, so we can ask about stability.

Anisotropy: The Wulff Inequality

Φ is the right generalization to use because we have an anisotropic version of the isoperimetric inequality known as the *Wulff inequality*:

$$\Phi(E) \geq n|K|^{1/n}|E|^{(n-1)/n}$$

with equality if and only if $|E\Delta(rK + x_0)| = 0$ for some $r > 0$ and $x_0 \in \mathbb{R}^n$.

This is the same as the isoperimetric inequality with Φ in place of P and $K = \{f_* < 1\}$ in place of B_1. We have a rigidity statement, so we can ask about stability.

Quantitative Stability: The isotropic setting

To discuss quantitative stability we introduce the following scale invariant quantities.

- Closeness to equality: Define the *isoperimetric deficit* δ as

$$\delta(E) = \frac{P(E)}{n|B_1|^{1/n}|E|^{(n-1)/n}} - 1$$

  which is always non-negative owing to the isoperimetric inequality, and is zero precisely when E is essentially a ball.

- Closeness to a ball: Use an *asymmetry index* α. Supposed to capture the geometry and is also such that $\alpha(E) = 0$ iff E is essentially a ball.

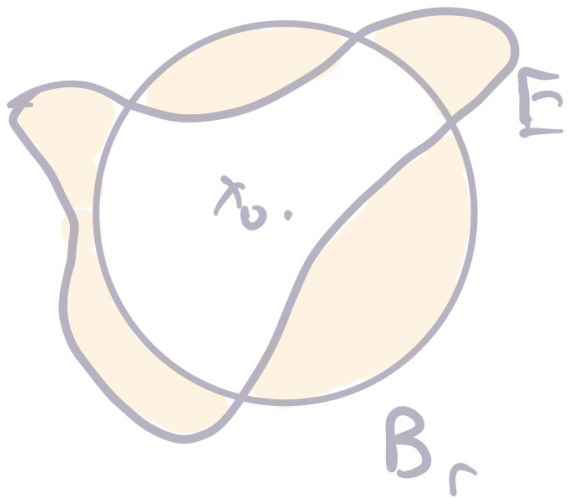
Qualitative stability says given $\{E_j\}_{j=1}^{\infty}$, if $\delta(E_j) \to 0$ then $\alpha(E_j) \to 0$. Quantitative stability quantifies this control, e.g. $\alpha(E)^p \leq \delta(E)$ for all sets of finite perimeter.

Quantitative Stability: Asymmetry Indexes

Many kinds, heuristically measure the distance to the set of minimizers $\{B_r(x_0) \mid x_0 \in \mathbb{R}^n, r > 0\}$. For ex. the Hausdorff distance.

The most common asymmetry index is the *Fraenkel asymmetry*

$$\alpha(E) = \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta B_r(x_0)|}{|E|} \,\middle|\, |B_r| = |E| \right\}.$$

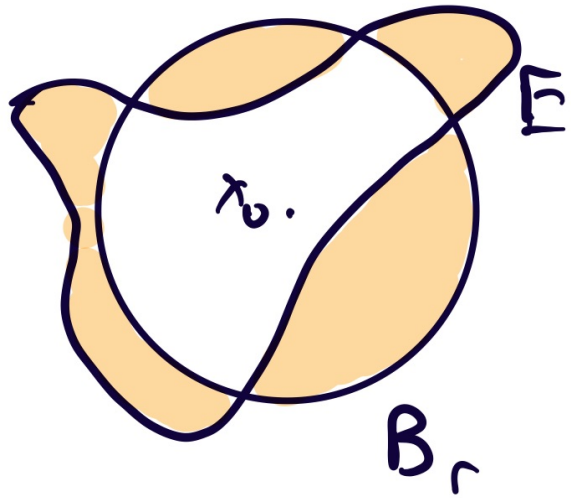


Quantitative Stability: Asymmetry Indexes

Many kinds, heuristically measure the distance to the set of minimizers $\{B_r(x_0) \mid x_0 \in \mathbb{R}^n, r > 0\}$. For ex. the Hausdorff distance.

The most common asymmetry index is the *Fraenkel asymmetry*

$$\alpha(E) = \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta B_r(x_0)|}{|E|} \;\middle|\; |B_r| = |E| \right\}.$$



Quantitative Stability: Previous Results

Theorem (Fusco-Maggi-Pratelli, '08)

There exists $C(n) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$,

$$\alpha(E)^2 \leq C(n)\delta(E). \qquad \text{(Q.S.)}$$

The power of 2 in (Q.S.) *is sharp.*

The proof exploits symmetrization techniques a la De Giorgi.

Previous results by Fuglede '89, Hall-Hayman-Weitsman '91, and Hall '92 prove (Q.S.) under various other hypotheses, e.g. if E is convex, nearly spherical, smooth, and/or axially symmetric.

Quantitative Stability: Previous Results

Theorem (Fusco-Maggi-Pratelli, '08)

There exists $C(n) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$,

$$\alpha(E)^2 \leq C(n)\delta(E). \tag{Q.S.}$$

The power of 2 in (Q.S.) *is sharp.*

The proof exploits symmetrization techniques a la De Giorgi.

Previous results by Fuglede '89, Hall-Hayman-Weitsman '91, and Hall '92 prove (Q.S.) under various other hypotheses, e.g. if E is convex, nearly spherical, smooth, and/or axially symmetric.

Quantitative Stability: Selection Principle

In '12 Cicalese-Leonardi showed sharp quantitative stability for the isoperimetric inequality by exploiting the regularity of almost minimizers. This technique became known as the *selection principle.*

Idea: proof by contradiction

- Use selection principle to replace original sequence with a new one with upgraded regularity, while still maintaining the contradictory hypothesis.
- Prove directly sharp stability with upgraded regularity.
- Derive a contradiction.

Does not use symmetrization!

Quantitative Stability: Selection Principle

In '12 Cicalese-Leonardi showed sharp quantitative stability for the isoperimetric inequality by exploiting the regularity of almost minimizers. This technique became known as the *selection principle.*

Idea: proof by contradiction

- Use selection principle to replace original sequence with a new one with upgraded regularity, while still maintaining the contradictory hypothesis.
- Prove directly sharp stability with upgraded regularity.
- Derive a contradiction.

Does not use symmetrization!

Quantitative Stability: A Strong Form

In '14 Fusco-Julin, using the selection principle, proved the following strong form of (Q.S.).

Theorem (Fusco-Julin)

There exists $C(n) > 0$ such that for any set of finite perimeter E with $0 < |E| < \infty$,

$$\alpha(E)^2 + \beta(E)^2 \leq C(n)\delta(E)$$

where $\beta(E)$ is the *oscillation index.*

Quantitative Stability: Oscillation Index

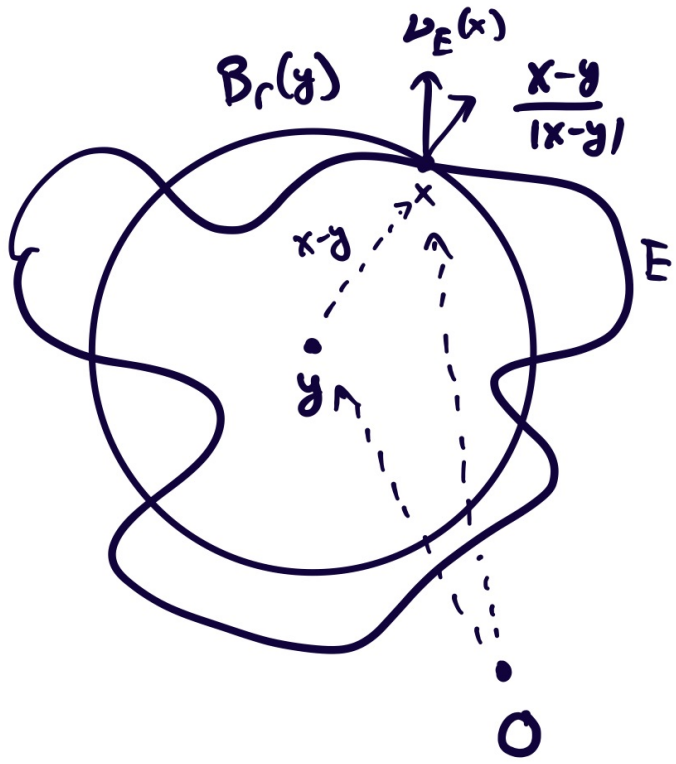
The oscillation index β is defined as

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left\{ c(n,E) \int_{\partial^* E} \left[1 - \frac{\langle x - y, \nu_E(x) \rangle}{|x-y|} \right] d\mathcal{H}^{n-1}(x) \right\}^{1/2}$$

where $c(n,E) = 1/(n|B_1|^{1/n}|E|^{(n-1)/n})$. It measures the deviation from equality in Cauchy-Schwarz:

$$\langle (x-y)/|x-y|, \nu_E(x) \rangle \leq 1,$$

with equality if and only if $(x-y)/|x-y| = \nu_E(x)$.



Quantitative Stability: Oscillation Index vs H^1

We suppose here that E is *nearly spherical*, i.e. $\partial E = \{x + u(x)x \mid x \in \partial B_1\}$ with $u \in C^1(\partial B_1)$ and $\|u\|_{W^{1,\infty}(\partial B_1)}$ small. In this case can parametrize ∂E in terms of ∂B_1 and compute

$$\beta(E)^2 \lesssim \|u\|^2_{H^1(\partial B_1)}.$$

On the other hand, Fuglede and Fusco-Julin show, respectively, that if E is nearly spherical then

$$\frac{1}{10}\|u\|^2_{H^1(\partial B_1)} \leq \delta(E).$$

and there exists $C(n) > 0$ such that (for any set of finite perimeter)

$$\alpha(E) + \delta(E)^{1/2} \leq C(n)\beta(E).$$

In particular since $\alpha(E) > 0$, $\delta(E) \leq C(n)\beta(E)^2$, we also have $\|u\|^2_{H^1(\partial B_1)} \lesssim \beta(E)^2$.

Anisotropy: Deficit, Asymmetry, and Oscillation

We define the anisotropic deficit, Fraenkel asymmetry, and oscillation index as

$$\delta_\Phi(E) := \frac{\Phi(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1$$

$$\alpha_\Phi(E) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta (rK + x_0)|}{|E|} \;\middle|\; |rK| = |E| \right\}$$

$$\beta_\Phi(E) := \inf_{y \in \mathbb{R}^n} \left\{ c_\Phi(n, E) \int_{\partial^* E} \left[f(\nu_E(x)) - \frac{\langle x - y, \nu_E(x) \rangle}{f_*(x - y)} \right] d\mathcal{H}^{n-1}(x) \right\}^{1/2}$$

where $c_\Phi(n, K) = 1/(n|K|^{1/n}|E|^{(n-1)/n})$. Notice the integrand for β_Φ comes from the Fenchel inequality

$$\langle x - y, \nu \rangle \leq f(\nu) f_*(x - y)$$

where equality occurs if and only if $\{\langle x - y, \nu \rangle = f(\nu)\}$ is a supporting hyperplane for K at $x - y$.

Anisotropy: Deficit, Asymmetry, and Oscillation

We define the anisotropic deficit, Fraenkel asymmetry, and oscillation index as

$$\delta_\Phi(E) := \frac{\Phi(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1$$

$$\alpha_\Phi(E) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta (rK + x_0)|}{|E|} \;\middle|\; |rK| = |E| \right\}$$

$$\beta_\Phi(E) := \inf_{y \in \mathbb{R}^n} \left\{ c_\Phi(n, E) \int_{\partial^* E} \left[f(\nu_E(x)) - \frac{\langle x - y, \nu_E(x) \rangle}{f_*(x - y)} \right] d\mathcal{H}^{n-1}(x) \right\}^{1/2}$$

where $c_\Phi(n, K) = 1/(n|K|^{1/n}|E|^{(n-1)/n})$. Notice the integrand for β_Φ comes from the Fenchel inequality

$$\langle x - y, \nu \rangle \leq f(\nu) f_*(x - y)$$

where equality occurs if and only if $\{\langle x - y, \nu \rangle = f(\nu)\}$ is a supporting hyperplane for K at $x - y$.

Anisotropy: Previous Results

Theorem (Figalli-Maggi-Pratelli, '12)

There exists $C(n) > 0$ such that for any set of finite perimeter with $0 < |E| < \infty$,

$$\alpha_\Phi(E)^2 \leq C(n)\delta_\Phi(E).$$

The power of 2 is sharp.

Theorem (Neumayer '16)

- *If K is uniformly convex there exists $C(n, ..., \|\nabla^2 f\|_{C^0(\partial K)}) > 0$ such that*

$$\alpha_\Phi(E)^2 + \beta_\Phi(E)^2 \leq C\delta_\Phi(E).$$

- *If instead $n = 2$ and K is a polygon (a crystalline case), there exists $C(K) > 0$ such that the above holds.*

Main Result

The following is the main result, a direct generalization of Neumayer's result in the crystalline $n = 2$ setting.

Theorem (D. '24)

Let K be a polytope. There exists $C(n, K) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$,

$$\alpha_\Phi(E)^2 + \beta_\Phi(E)^2 \leq C(n, K)\delta_\Phi(E).$$

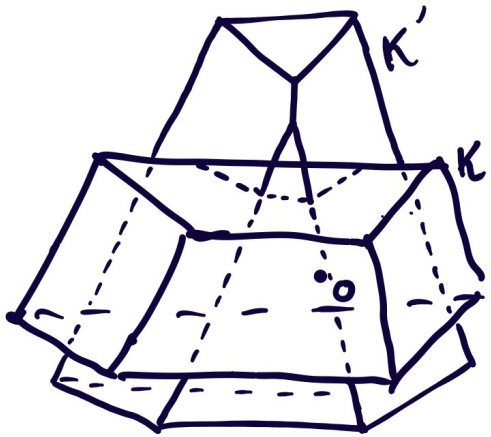
Comparison to Isotropic Case

Remark

- In the anisotropic setting we lack symmetry, so in particular we cannot appeal to symmetrization techniques as in the isotropic setting.
- The Figalli-Maggi-Pratelli result uses optimal transport methods, Neumayer uses the selection principle
- Only weak regularity theory is available. For a generic Wulff shape K can only conclude almost minimizers satisfy uniform density estimates, not (Λ, r_0)-minimizer.
- Need to pair uniform density estimates with L^1-closeness (by FMP) to get Hausdorff closeness.
- Further, in the crystalline setting $\nabla^2 f \equiv 0$ making the problem degenerate elliptic.

Overview of Argument: Parallel Polytopes

Step 1: Prove the result for *parallel* polytopes.



We say that K' is parallel to K if they share the same set of unit normals, and hence have the same amount of sides.

Overview of Argument: The Function γ_Φ

Recall that $\beta_\Phi(E)$ is defined as

$$\beta_\Phi(E) := \inf_{y\in\mathbb{R}^n}\left\{c_\Phi(n,E)\int_{\partial^* E}\left[f(\nu_E(x)) - \frac{\langle x-y,\nu_E(x)\rangle}{f_*(x-y)}\right]d\mathcal{H}^{n-1}(x)\right\}^{1/2}$$

where $c_\Phi(n,E) = 1/(n|K|^{1/n}|E|^{(n-1)/n})$. In practice, it is much more useful to rewrite this using the divergence theorem as

$$\beta_\Phi(E)^2 = \frac{\Phi(E) - (n-1)\gamma_\Phi(E)}{n|K|^{1/n}|E|^{(n-1)/n}}$$

where

$$\gamma_\Phi(E) := \sup_{y\in\mathbb{R}^n}\int_E \frac{1}{f_*(x-y)}\,dx.$$

Overview of Argument: The Function γ_Φ

Recall that $\beta_\Phi(E)$ is defined as

$$\beta_\Phi(E) := \inf_{y\in\mathbb{R}^n}\left\{c_\Phi(n,E)\int_{\partial^* E}\left[f(\nu_E(x)) - \frac{\langle x-y,\nu_E(x)\rangle}{f_*(x-y)}\right]d\mathcal{H}^{n-1}(x)\right\}^{1/2}$$

where $c_\Phi(n,E) = 1/(n|K|^{1/n}|E|^{(n-1)/n})$. In practice, it is much more useful to rewrite this using the divergence theorem as

$$\beta_\Phi(E)^2 = \frac{\Phi(E) - (n-1)\gamma_\Phi(E)}{n|K|^{1/n}|E|^{(n-1)/n}}$$

where

$$\gamma_\Phi(E) := \sup_{y\in\mathbb{R}^n}\int_E \frac{1}{f_*(x-y)}\,dx.$$

Overview of Argument: A Preliminary Estimate

In particular, it can be shown that

$$\Phi(K) = (n-1)\gamma_\Phi(K) = (n-1)\int_K \frac{1}{f_*(x)}\,dx.$$

Furthermore, by the Wulff inequality $\Phi(K) = n|K|$. Accordingly, if $|E| = |K|$ then by testing $\gamma_\Phi(E)$ at the origin,

$$\begin{aligned}
\beta_\Phi(E)^2 &\leq \frac{\Phi(E)}{n|K|} - \frac{(n-1)}{n|K|}\int_E \frac{1}{f_*(x)}\,dx \\
&= \delta_\Phi(E) + \frac{(n-1)}{n|K|}\left[\int_K \frac{1}{f_*(x)}\,dx - \int_E \frac{1}{f_*(x)}\,dx\right] \\
&= \delta_\Phi(E) + \frac{(n-1)}{n|K|}\left[\int_{K\setminus E} \frac{1}{f_*(x)}\,dx - \int_{E\setminus K} \frac{1}{f_*(x)}\,dx\right].
\end{aligned}$$

Overview of Argument: A Preliminary Estimate

In particular, it can be shown that

$$\Phi(K) = (n-1)\gamma_\Phi(K) = (n-1)\int_K \frac{1}{f_*(x)}\,dx.$$

Furthermore, by the Wulff inequality $\Phi(K) = n|K|$. Accordingly, if $|E| = |K|$ then by testing $\gamma_\Phi(E)$ at the origin,

$$\begin{aligned}
\beta_\Phi(E)^2 &\leq \frac{\Phi(E)}{n|K|} - \frac{(n-1)}{n|K|}\int_E \frac{1}{f_*(x)}\,dx \\
&= \delta_\Phi(E) + \frac{(n-1)}{n|K|}\left[\int_K \frac{1}{f_*(x)}\,dx - \int_E \frac{1}{f_*(x)}\,dx\right] \\
&= \delta_\Phi(E) + \frac{(n-1)}{n|K|}\left[\int_{K\setminus E} \frac{1}{f_*(x)}\,dx - \int_{E\setminus K} \frac{1}{f_*(x)}\,dx\right].
\end{aligned}$$

Overview of Argument: The Anisotropic Co-Area Formula

For any Borel $g : \mathbb{R} \to [0, \infty)$, Lipschitz $u : \mathbb{R}^n \to \mathbb{R}$, and open $\Omega \subset \mathbb{R}^n$ the (weighted) Anisotropic co-area formula states that

$$\int_\Omega f(\nabla u(x))\, g(f_*(x))\, dx = \int_0^\infty \Phi(\{u < r\}; \Omega)\, g(r)\, dr.$$

With $u(x) = f_*(x)$ and $g(r) = 1/r$ this reads

$$\int_\Omega \frac{1}{f_*(x)}\, dx = \int_0^\infty \frac{1}{r} \Phi(\{f_* < r\}; \Omega)\, dr = \int_0^\infty \frac{1}{r} \Phi(rK; \Omega)\, dr$$

since $f(\nabla f_*(x)) = 1$ for a.e. $x \in \mathbb{R}^n$ by duality.

Overview of Argument: Parallel Polytopes

Recall that $\Phi(K)$ takes the nice form

$$\Phi(K) = \sum_{i=1}^{N} f(\nu_i)\mathcal{H}^{n-1}(F_i).$$

So the computation simply involves bounding

$$\begin{aligned}
\int_{K\setminus K'} \frac{1}{f_*(x)}\, dx &= \int_0^\infty \frac{1}{r}\Phi(rK; K\setminus K')\, dr \\
&= \sum_{i=1}^{N} \int_0^1 \frac{f(\nu_i)}{r}\mathcal{H}^{n-1}(rF_i \cap (K\setminus K'))\, dr, \\
\int_{K'\setminus K} \frac{1}{f_*(x)}\, dx &= \int_0^\infty \frac{1}{r}\Phi(rK; K'\setminus K)\, dr \\
&= \sum_{i=1}^{N} \int_1^\infty \frac{f(\nu_i)}{r}\mathcal{H}^{n-1}(rF_i \cap (K'\setminus K))\, dr.
\end{aligned}$$

Overview of Argument: Uniform Density Estimates

Step 2: Prove the result for E satisfying uniform density estimates. Allows to upgrade L^1 control to Hausdorff.

Need to use the following projection theorem

Theorem (Figalli-Zhang '22)

There exists $\sigma(n,K) > 0$ and $\gamma(n,K) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $|E| = |K|$ and $|E \Delta K| \leq \sigma$, there exists a parallel polytope K' such that $|K'| = |K|$ and

$$\Phi(E) - \Phi(K') \geq \gamma |E \Delta K'|$$

We'll call the polytope obtained from this theorem K^*.

Overview of Argument: Uniform Density Estimates

Step 2: Prove the result for E satisfying uniform density estimates. Allows to upgrade L^1 control to Hausdorff.

Need to use the following projection theorem

Theorem (Figalli-Zhang '22)

There exists $\sigma(n,K) > 0$ and $\gamma(n,K) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $|E| = |K|$ and $|E \Delta K| \leq \sigma$, there exists a parallel polytope K' such that $|K'| = |K|$ and

$$\Phi(E) - \Phi(K') \geq \gamma |E \Delta K'|$$

We'll call the polytope obtained from this theorem K^*.

Overview of Argument: Uniform Density Estimates

Observe that since $\delta_\Phi(E) = \Phi(E)/(n|K|) - 1$ (and similarly for K^*), we have that

$$\delta_\Phi(E) - \delta_\Phi(K^*) = \frac{\Phi(E)}{n|K|} - \frac{\Phi(K^*)}{n|K|} \geq \frac{\gamma}{n|K|}|E\Delta K^*|.$$

In particular, this implies that $\delta_\Phi(K^*) \leq \delta_\Phi(E)$.

Use Hausdorff control to show that

$$|\gamma_\Phi(E) - \gamma_\Phi(K^*)| \leq \frac{1}{2}|E\Delta K^*|.$$

Overview of Argument: Uniform Density Estimates

Observe that since $\delta_\Phi(E) = \Phi(E)/(n|K|) - 1$ (and similarly for K^*), we have that

$$\delta_\Phi(E) - \delta_\Phi(K^*) = \frac{\Phi(E)}{n|K|} - \frac{\Phi(K^*)}{n|K|} \geq \frac{\gamma}{n|K|}|E\Delta K^*|.$$

In particular, this implies that $\delta_\Phi(K^*) \leq \delta_\Phi(E)$.

Use Hausdorff control to show that

$$|\gamma_\Phi(E) - \gamma_\Phi(K^*)| \leq \frac{1}{2}|E\Delta K^*|.$$

Overview of Argument: Uniform Density Estimates

Combining these with the identity

$$\beta_\Phi(E)^2 = \frac{\Phi(E) - (n-1)\gamma_\Phi(E)}{n|K|}$$

yields

$$\begin{aligned}\beta_\Phi(E)^2 \leq & \frac{\Phi(E)}{n|K|} - \frac{(n-1)\gamma_\Phi(K^*)}{n|K|} + C|E\Delta K^*| \\ &= \frac{1}{n|K|}\left[\Phi(E) - \Phi(K^*)\right] + \beta_\Phi(K^*)^2 + C|E\Delta K^*| \\ &\leq C[\delta_\Phi(E) - \delta_\Phi(K^*)] + \beta_\Phi(K^*)^2.\end{aligned}$$

Overview of Argument

Step 3: Selection Principle

- Aiming for a contradiction, generate a sequence $\{E_j\}_{j=1}^{\infty}$.
- Choose

$$F_j \in \arg\min\{\Phi(F) + C_1|\beta_\Phi(F)^2 - \beta_\Phi(E_j)^2| + C_2\big||F| - |K|\big|\}$$

  These are almost minimizers in the sense they minimize a perturbed volume-constrained problem.
- Need to show F_j satisfies same properties as E_j and control $\beta_\Phi(F_j)$ and $\Phi(F_j)$. Replace $\{E_j\}_{j=1}^{\infty}$ with almost minimizers $\{F_j\}_{j=1}^{\infty}$.
- Appeal to regularity theory of almost minimizers (uniform density estimates) to apply Step 2.

Overview of Argument

Step 3: Selection Principle

- Aiming for a contradiction, generate a sequence $\{E_j\}_{j=1}^{\infty}$.
- Choose

$$F_j \in \arg\min\{\Phi(F) + C_1|\beta_\Phi(F)^2 - \beta_\Phi(E_j)^2| + C_2\big||F| - |K|\big|\}$$

  These are almost minimizers in the sense they minimize a perturbed volume-constrained problem.

- Need to show F_j satisfies same properties as E_j and control $\beta_\Phi(F_j)$ and $\Phi(F_j)$. Replace $\{E_j\}_{j=1}^{\infty}$ with almost minimizers $\{F_j\}_{j=1}^{\infty}$.
- Appeal to regularity theory of almost minimizers (uniform density estimates) to apply Step 2.

Overview of Argument

Step 3: Selection Principle

- Aiming for a contradiction, generate a sequence $\{E_j\}_{j=1}^{\infty}$.
- Choose

$$F_j \in \arg\min\{\Phi(F) + C_1|\beta_\Phi(F)^2 - \beta_\Phi(E_j)^2| + C_2\big||F| - |K|\big|\}$$

  These are almost minimizers in the sense they minimize a perturbed volume-constrained problem.

- Need to show F_j satisfies same properties as E_j and control $\beta_\Phi(F_j)$ and $\Phi(F_j)$. Replace $\{E_j\}_{j=1}^{\infty}$ with almost minimizers $\{F_j\}_{j=1}^{\infty}$.
- Appeal to regularity theory of almost minimizers (uniform density estimates) to apply Step 2.

Overview of Argument

Step 3: Selection Principle

- Aiming for a contradiction, generate a sequence $\{E_j\}_{j=1}^{\infty}$.
- Choose

$$F_j \in \arg\min\{\Phi(F) + C_1|\beta_\Phi(F)^2 - \beta_\Phi(E_j)^2| + C_2\big||F| - |K|\big|\}$$

  These are almost minimizers in the sense they minimize a perturbed volume-constrained problem.

- Need to show F_j satisfies same properties as E_j and control $\beta_\Phi(F_j)$ and $\Phi(F_j)$. Replace $\{E_j\}_{j=1}^{\infty}$ with almost minimizers $\{F_j\}_{j=1}^{\infty}$.
- Appeal to regularity theory of almost minimizers (uniform density estimates) to apply Step 2.

Thanks for coming!