Math 3GR3, Tutorial 8

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Topics: Internal direct products. Normal subgroups.

Question 1. True or false? Justify your answers.

(a) $U(20) \cong U(24)$.
(b) Any subgroup of S_3 is normal.
(c) A_n is always normal in S_n.
(d) Every subgroup of a cyclic group is normal.
(e) Every group has at least 2 distinct normal subgroups.

Recall: Theorem 10.3. Let N be a subgroup of G. The following are equivalent:

(a) N is normal in G,
(b) $gNg^{-1} = N$,
(c) $gNg^{-1} \subseteq N$.

Question 2. Let $T = \{z \in \mathbb{C}^* \mid |z| = 1\}$ be the multiplicative subgroup of complex numbers lying on the unit circle and let $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Show that $\mathbb{C}^* \cong \mathbb{R}^+ \times T$.

Question 3 (Dummit–Foote 3.1.34)**.** Consider the dihedral group D_n. Fix an integer k dividing n. Show that the cyclic subgroup $\langle r^k \rangle$ is a normal subgroup of D_n.

Question 4. Suppose N is a subgroup of G such that if $g \in G$, then $g^2 \in N$. Show that N is normal.

Question 5. Prove or disprove: if a group G has normal subgroups N and K such that $N \cong K$, then $G/N \cong G/K$.