On the Lawrence–Doniach Model of Superconductivity: Magnetic Fields Parallel to the Axes

Stan Alama* Lia Bronsard* Etienne Sandier†

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Abstract

We consider periodic minimizers of the Lawrence-Doniach functional, which models highly anisotropic superconductors with layered structure, in the simultaneous limit as the layer thickness tends to zero and the Ginzburg-Landau parameter tends to infinity. In particular, we consider the properties of minimizers when the system is subjected to an external magnetic field applied either tangentially or normally to the superconducting planes. For normally applied fields, our results show that the resulting "pancake" vortices will be vertically aligned. In horizontal fields we show that there are two parameter regimes in which minimizers exhibit very different characteristics. The low-field regime resembles the Ginzburg-Landau model, while the high-field limit gives a "transparent state" described in the physical literature. To obtain our results we derive sharp matching upper and lower bounds on the global minimizers of the energy.

Keywords: Calculus of variations, elliptic equations and systems, superconductivity, vortices.

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*Dept. of Mathematics and Statistics, McMaster Univ., Hamilton, Ontario, Canada L8S 4K1. Supported by an NSERC Research Grant.

†Departement des Mathématiques, Université Paris XII, 64 avenue du Général de Gaulle, 94010 Créteil Cedex.

1 Introduction

In this paper we study minimizers of the Lawrence–Doniach energy functional in certain asymptotic limits. The Lawrence–Doniach (LD) energy was introduced [LaDo] to model highly anisotropic superconductors having a layered structure. Unlike the Ginzburg–Landau (GL) model, which represents a superconductor as a continuous three-dimensional solid, in the LD model the superconductor is idealized as a network of equally spaced, parallel superconducting planes. We will consider minimizers of the LD energy under *periodic* boundary conditions in three dimensions, with applied magnetic fields which are oriented perpendicularly or parallel to these superconducting planes. We study various asymptotic limits, with the distance between the planes and the radius of vortices both tending to zero, for applied magnetic fields which depend on these two parameters. By introducing periodicity we eliminate boundary pinning effects and concentrate on the lower critical field, the value H_{c1} of the applied field strength at which vortices appear in the superconductor, and how the nature of the vortex lattice is determined by the orientation of the applied field and the relationships between the physical parameters.

The superconductor occupies an infinite network of parallel planes orthogonal to $\vec{e_3}$, with equal spacing $s > 0$,

$$\mathcal{P} = \bigcup_{n\in\mathbb{Z}} \mathcal{P}_n, \qquad \mathcal{P}_n := \mathbb{R}^2 \times \{ns\}.$$

The most general periodic structure on this network fixes three independent vectors, $\vec{v_i}$, $i = 1, 2, 3$ for which $\mathcal{P} + k\vec{v_i} = \mathcal{P}$, $i = 1, 2, 3$. However, for our analysis we will assume that one vector lie along the applied field direction. We make this hypothesis for simplicity, but we conjecture that all the results below hold without regard to the geometry of the underlying period domain. Thus, we begin by assuming

$$\vec{v_3} = L_3\vec{e_3},$$

and define a fundamental period domain $\Omega \subset \mathbb{R}^3$ by fixing a basis $\{\vec{v_1}, \vec{v_2}\}$ of vectors in $\mathbb{R}^2 \subset \mathbb{R}^3$,

$$\Omega := \{t_1\vec{v_1} + t_2\vec{v_2} + t_3\vec{v_3} : \ t_j \in [0, 1], \ j = 1, 2, 3\}.$$

When convenient, we will abuse notation and write $\vec{v_1}, \vec{v_2}$ as elements of either $\mathbb{R}^2$ or $\mathbb{R}^3$. We denote the restriction of each plane $\mathcal{P}_n$ to Ω by P_n, that is

$$P_n = P + (0, 0, z_n), \quad z_n := ns, \quad P := \{t_1\vec{v_1} + t_2\vec{v_2} : \ t_j \in [0, 1] \ j = 1, 2\}, \quad n = 1, \ldots, N.$$

On each plane we define a complex-valued order parameter, $u_n : \ \mathcal{P}_n \simeq \mathbb{R}^2 \to \mathbb{C}$. As in the Ginzburg–Landau model, $|u_n| = 1$ represents the purely superconducting state, and $u_n = 0$

the normal state. The superconducting currents interact with a magnetic field, described by a vector potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ via $h = \nabla \times A$.

To write the LD energy we introduce some convenient notation. We denote by $\nabla' = (\partial_x, \partial_y)$, and $A' = (A_x(x,y,z), A_y(x,y,z))$, $A'_n = (A_x(x,y,z_n), A_y(x,y,z_n))$. We write (u_n, A) as a shorthand for $(\{u_n\}_{n\in\mathbb{Z}}, A)$. Then, the energy in the period domain Ω may be written as:

$$\begin{aligned}\mathcal{L}^\lambda_{\epsilon,s}(u_n, A) &= s\sum_{n=1}^{N(s)} \int_P \left[\frac{1}{2}\left|(\nabla' - iA'_n)\, u_n\right|^2 + \frac{1}{4\epsilon^2}(|u_n|^2-1)^2\right] dx\, dy \\ &\quad + s\sum_{n=1}^{N} \int_P \frac{1}{2\lambda^2 s^2}\left|u_n - u_{n-1} e^{i\int_{z_{n-1}}^{z_n} A_z(x,y,z)\, dz}\right|^2 dx\, dy \\ &\quad + \frac{1}{2}\int_\Omega \left|\nabla \times A - \vec{h}_{ex}\right|^2 dx\, dy\, dz.\end{aligned} \tag{1.1}$$

Here, $\epsilon > 0$ represents the reciprocal of the Ginzburg–Landau parameter; we will assume $\epsilon \ll 1$, which is typical for type-II superconducting materials. The constant $\lambda > 0$ represents the Josephson penetration depth, and will assumed to be fixed in this paper. (Results on the large λ limit of the LD minimizers may be found in [ABeB1, ABeB2].) The applied field $\vec{h}_{ex}$ is a given constant vector, which will in general depend on s or ϵ, and which we will assume to be either parallel to the planes P_n or to the x_3-axis throughout.

Next we must define a space of functions for $\mathcal{L}^\lambda_{\epsilon,s}$. We say $(u_n, A) \in \mathcal{H}$ if $u_n \in H^1_{loc}(\mathbb{R}^2; \mathbb{C})$ for all $n \in \mathbb{Z}$, $A \in H^1_{loc}(\mathbb{R}^3; \mathbb{R}^3)$, and there exist functions $\omega_j \in H^2_{loc}(\mathbb{R}^3)$, $j = 1,2,3$ such that:

$$\begin{cases} u_n(x' + \vec{v}_j) = u_n(x') e^{i\omega_j(x', z_n)}, \quad j = 1,2,\ n \in \mathbb{Z}; \\ u_{n+N}(x') = u_n(x') e^{i\omega_3(x', z_n)}, \quad n \in \mathbb{Z}; \\ A(x + v_j) = A(x) + \nabla\omega_j(x), \quad j = 1,2,3 \end{cases} \tag{1.2}$$

holds for all $x = (x,y,z) = (x', z) \in \mathbb{R}^3$. That is, the configuration (u_n, A) is Ω-periodic up to gauge transformation. In particular, the gauge-invariant quantities

$$j'_n(x,y) = \mathrm{Im}\,\{\overline{u_n}\,(\nabla' - iA'_n)u_n\}, \quad j_z(x,y) = -\mathrm{Im}\left\{\overline{u_n}\, u_{n-1} e^{i\int_{z_{n-1}}^{z_n} A_z(x,y,z)\, dz}\right\}$$
$$\rho_n(x,y) = |u_n|, \quad h = \nabla \times A,$$

are all Ω-periodic.

An important consequence of periodic boundary conditions is the exact quantization of the magnetic flux through period planes: see (4.24) in our later discussion of the minimizers in parallel fields.

Perpendicular fields

We first discuss the case where $\vec{h}_{ex}$ is *perpendicular* to the planes P_n. This is the setting in which the discrete nature of the model is the least apparent. Indeed, in this case we expect that the flux lines to be vertically aligned, penetrating each plane in a stack of "pancake" vortices ([Cl]) which resemble a two-dimensional GL vortex array in each plane.

First we recall the classical three dimensional Ginzburg–Landau functional, in our periodic setting. Let $\mathcal{H}_{GL}$ denote the space of functions (ψ, A) with $\psi \in H^1_{loc}(\mathbb{R}^3)$, $A \in H^1_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ for which there exist $\omega_j \in H^2_{loc}(\mathbb{R}^2)$, $j = 1,2,3$, with:

$$\psi(x+\vec{v_j}) = \psi(x)e^{i\omega_j(x)}, \quad A(x+\vec{v_j}) = A(x) + \nabla\omega_j(x), \quad x \in \mathbb{R}^3, \quad j = 1,2,3.$$

Write the Ginzburg–Landau energy as

$$G_\epsilon(\psi, A) = \int_\Omega \frac{1}{2}|(\nabla - iA)\psi|^2 + \frac{1}{4\epsilon^2}(|\psi|^2-1)^2 + \frac{1}{2}|\nabla \times A - h^\perp_{ex}\vec{e_3}|^2$$

We prove:

Theorem 1.1. *Let $\epsilon > 0$ and $\vec{h}_{ex} = h^\perp_{ex}\,\vec{e_3}$ be given.*

1. *For any $s = L_3/N$, any minimizer of $\mathcal{L}^\lambda_{\epsilon,s}$ in $\mathcal{H}$ is gauge equivalent to (u_n, A) with*

$$\left.\begin{aligned} u_n(x,y) = u_{n-1}(x,y), \quad A(x', z+s) = A(x',z), \\ A_x(x,y,s-z) = A_x(x,y,z),\ A_y(x,y,s-z) = A_x(x,y,z),\ A_z(x,y,z) = 0, \end{aligned}\right\} \tag{1.3}$$

   for all $(x,y) \in P$ and $n \in \mathbb{Z}$.

2. *If (ψ_{GL}, A_{GL}) are minimizers of G_ϵ, then*

$$G_\epsilon(\psi_{GL}, A_{GL}) \le G_\epsilon(u_n, A) \le \mathcal{L}^\lambda_{\epsilon,s}(u_n, A)(1+2s) \le G_\epsilon(\psi_{GL}, A_{GL})(1+2s), \tag{1.4}$$

3. *If in addition we assume the applied field $h^\perp_{ex} = H_{ex}|\ln\epsilon|$, we have*

$$\lim_{\epsilon,s\to 0} \frac{\mathcal{L}^\lambda_{\epsilon,s}(u_n, A)}{|\Omega||\ln\epsilon|^2} = \frac{1}{2}\min_{H\in\mathbb{R}}\left[|H| + (H - H_{ex})^2\right].$$

   Moreover, $\frac{h}{|\ln\epsilon|} \rightharpoonup H_*$ in L^2, where $H_* = (H_{ex} - \frac{1}{2})_+$ is the minimizer of the right-hand side above.

4. *If instead $|\ln\epsilon| \ll h_{ex} \ll \epsilon^{-2}$, then $\frac{h}{h_{ex}} \to 1$ in $L^2(\Omega)$ and*

$$\lim_{\epsilon,s\to 0} \frac{\mathcal{L}^\lambda_{\epsilon,s}(u_n, A)}{|\Omega| h^\perp_{ex} \ln\frac{1}{\epsilon\sqrt{h^\perp_{ex}}}} = \frac{1}{2}.$$

Corollary 1.2. *We keep the notation (u_n, A) for a minimizer of $\mathcal{L}^\lambda_{\epsilon,s}$ and (Ψ_{GL}, A_{GL}) for a minimizer of G_ϵ, and we let $h = \nabla \times A$ and $h_{GL} = \nabla \times A_{GL}$. Then*

1. *$h(x,y,z) = (-h_x(x,y,-z), -h_y(x,y,-z), h_z(x,y,-z))$, and $h(x,y,z+s) = h(x,y,s)$.*
2. *If $h^\perp_{ex} \ll \epsilon^{-2}$, then*
$$\lim_{s,\epsilon\to 0} \left(\frac{h}{h^\perp_{ex}} - \frac{h_{GL}}{h^\perp_{ex}} \right) = 0 \quad \text{in } L^2(\Omega).$$

Although the vortices will always be vertically aligned, the magnetic field $h = \nabla \times A$ will *not* be x_3-invariant. From the Euler–Lagrange equations (see Bauman & Ko [BaK]), the parallel components h_1, h_2 of the magnetic field are harmonic in the gaps between adjacent superconducting planes, and they satisfy jump conditions at the planes themselves. We expect that the magnetic flux lines will spread between the planes and be pinched together near the pancake vortices in each plane, forming hourglass shapes.

We provide the proofs of these results in section 3.

Parallel fields

Next, we discuss the situation when the applied field is parallel to the planes. For this part, we assume $\vec{v_j} = \vec{e_j}$, $j = 1,2,3$, the standard basis for $\mathbb{R}^3$, and $\vec{h}_{ex} = h^\parallel_{ex}\vec{e_2}$ is directed along the y-axis. As opposed to the perpendicular (and obliquely oriented) applied fields, when $\vec{h}_{ex}$ is parallel to the planes energy minimizers (u_n, A) will be two-dimensional: $u_n = u_n(x,z)$, $A = (A_x(x,z), A_z(x,z)$, with $\vec{h} = h(x,z)\vec{e_2}$ (this is proven in Lemma 4.1.) In section 4 we define a two-dimensional reduction $\mathcal{H}^\parallel$ of the space $\mathcal{H}$ of periodic functions, with fundamental domain $Q = [0, L_1] \times [0, L_3]$, for the functional $\mathcal{L}^\lambda_{\epsilon,s}$.

Despite this simplification, the case of parallel fields is the one in which the discreteness of the model is the most evident. The currents which flow in the xz-plane orthogonal to the magnetic field are extremely anisotropic: horizontal currents run along the planes (in the x-direction) but not in the gaps between the planes, and vertical z-direction currents (due to Josephson's tunneling effect) are measured in the gaps only, and can jump across the planes, as can the values of the magnetic field (see [ABeB1, ABeB2, ABS1] for the Euler–Lagrange equations in this case.) Since the order parameters u_n are not defined between the planes, magnetic flux may penetrate parallel to the planes without loss of superconductivity, and we expect $|u_n| \sim 1$ no matter how large the applied field and the induced currents. One may think of the vortex cores as being pinned between the planes, where they can carry flux without singularity. As a consequence, the usual Ginzburg–Landau parameter ϵ plays a

minor role in this setting, and the size of the vortices is determined by the interlayer spacing, s. We need only to make the hypothesis that

$$\epsilon \leq s/\alpha \tag{1.5}$$

for some $\alpha > 0$.

The behavior of minimizers in the parallel setting is determined by the asymptotic value of the quantity $h_{ex}s^2$. If $h_{ex}s^2 \ll 1$, we expect that the mean distance between vortices, which should be on the order of $h_{ex}^{-\frac{1}{2}}$, is much larger than the interplanar spacing s. Thus, the discrete nature of the problem can be seen as a small effect and the minimizers will resemble those of the Ginzburg–Landau model, but with s *playing the role of* ϵ.

Theorem 1.3. *Assume* (1.5) *and that* $s^2 h_{ex} \ll 1$, *and* $(u_n, A) \in \mathcal{H}^{\|}$ *are minimizers of* $\mathcal{L}^{\lambda}_{\epsilon,s}$, *and* $h_s = \operatorname{curl} A$.

1. *If* $h_{ex} \sim H_{ex}|\ln s|$ *for constant* $H_{ex} > 0$, *then*

$$\lim_{s\to 0} \frac{\mathcal{L}^{\lambda}_{\epsilon,s}(u_n, A)}{|Q|(\ln s)^2} = \lambda^{-1} \min_{H\in\mathbb{R}} \frac{1}{2}\left(|H| + (H - \lambda H_{ex})^2\right).$$

   Moreover, $\frac{h_s}{|\ln s|} \rightharpoonup H_*$ *in* L^2, *with* H_* *a minimizer of the right-hand side above. In particular, if* $H_{ex} \leq \frac{1}{2\lambda}$, *then* $H_* = 0$.

2. *If* $|\ln s| \ll h_{ex} \ll s^{-2}$, *we have*

$$\lim_{s\to 0} \frac{\mathcal{L}^{\lambda}_{\epsilon,s}(u_n, A)}{h_{ex} \ln\left(\frac{1}{s\sqrt{h_{ex}}}\right)} = \frac{|Q|}{2\lambda}, \qquad \text{and} \qquad \frac{h_s}{h_{ex}} \to 1 \quad \text{in } L^2.$$

In particular, the lower critical field $H^{\|}_{C1} = \frac{1}{2\lambda}|\ln s|$ for parallel fields. In fact, when $|\ln s| \ll h_{ex} \ll s^{-2}$ we show that the energy density of an auxilliary function $\mathcal{M}_s(\Psi, A)$ (see [ABS1]) associated to the minimizers converges weakly to a constant multiple of the Lebesgue measure.

The proof of Theorem 1.3 follows the same steps as the corresponding results for the two-dimensional Ginzburg–Landau functional (see [AyS, SS]), by means of an upper bound derived by constructing a configuration with a dense vortex lattice, and the vortex ball construction for the lower bound. The discreteness is eliminated by considering a gauge-invariant interpolation of u_n to the whole period domain Q and an auxiliary functional introduced in [ABS1].

When the applied field is very large, the minimizers of $\mathcal{L}^{\lambda}_{\epsilon,s}$ are qualitatively different from those of the Ginzburg–Landau model. Assuming $h_{ex}s^2 \gg 1$, the distance between vortex

cores is too small to ignore discretization, and indeed we expect vortices to lie in every gap between the planes. Roughly, minimizers will have magnetic field $h \simeq h_{ex}^{\parallel}$ everywhere and $|u_n| \simeq 1$, (a "transparent state"; see Bulaevskii & Clem [BuCm].) Since the planes are effectively one-dimensional, we may then solve for the phases ϕ_n, $u_n \simeq e^{i\phi_n(x)}$ exactly from the vector potential $A_x(x, z_n)$. As a result, the first and third terms in $\mathcal{L}^{\lambda}_{\epsilon,s}$ roughly vanish, leaving only the Josephson coupling term as the principal contribution to energy, at order $1/s^2$. The only complication with this plan is that the explicit solution to $h = \nabla \times A = h_{ex}$, $\phi_n' = A_x(x, z_n)$ may not satisfy the Floquet boundary conditions on Q. As was observed in a different limiting regime in [ABeB1, ABeB2], to obtain the minimum of energy among all periodic configurations, it is necessary to slightly modify either the horizontal period L_1 or the value of $h_{ex}^{\parallel}$ in order that the natural period of minimizers be commensurate with the imposed period:

$$h_{ex} \in \frac{2\pi N}{|Q|}\mathbb{Z}, \tag{1.6}$$

where N is the number of superconducting planes in the period domain Q. Since $h_{ex} \gg s^{-2} \sim N^2$, this demands a vanishingly small correction to either h_{ex} or L_1.

To state our results we make the following definitions:

$$\rho(x,z) = \sum_{n=1}^{N} |u_n(x)|\chi_{(z_{n-1},z_n]}(z), \quad J_x(x,z) = \sum_{n=1}^{N} (iu_n(x),\, u_n' - iA_x(x,z_n))\chi_{(z_{n-1},z_n]}(z). \tag{1.7}$$

In some sense, ρ measures the superconducting density, and J_x the horizontal currents, extended to the entire bulk. The interpolating function Ψ used in the previous regime cannot be easily related to the energy density, as the control on the error terms in the finite differences in x is lost in this highly oscillatory limit. We prove:

Theorem 1.4. *Let $(u_n, A) \in \mathcal{H}$ be minimizers of $\mathcal{L}^{\lambda}_{\epsilon,s}$ with $s^2 h_{ex} \gg 1$ and $\epsilon \ll s$. We have*

$$\frac{|Q|}{s^2} + o(s^{-2}) \leq \mathcal{L}^{\lambda}_{\epsilon,s}(u_n, A) \leq \frac{|Q|}{s^2}\left(1 + \frac{\pi^2}{2L_1^2}\right) + o(s^{-2}). \tag{1.8}$$

Moreover, if in addition we assume that L_1, h_{ex} are chosen such that (1.6) *holds, then:*

$$\lim_{s\to 0} s^2\, \mathcal{L}^{\lambda}_{\epsilon,s}(u_n, A) = |Q| \tag{1.9}$$

$$\rho \to 1, \quad s\, J_x \to 0 \quad \textit{strongly in } L^2, \tag{1.10}$$

$$s\, \|h_s - h_{ex}\|_{L^2} = s\, \|\nabla \times A - h_{ex}\|_{L^2} \to 0. \tag{1.11}$$

From the above we conclude that, to highest order in s^{-1}, the magnetic field associated to minimizers coincides with the applied field, h_{ex}, yet the mean density of superconducting

electrons does not diminish with increasing h_{ex}. The leading contribution to the energy comes from the Josephson coupling term in the energy, the energy of horizontal currents and of the magnetic field being of lower order in the expansion. As a consequence, the Lawrence–Doniach model has no upper critical field in this asymptotic regime in the parallel setting: $H_{C2}^{\parallel} = H_{C3}^{\parallel} = \infty$. That is, the minimizers for $h_{ex} \gg s^{-2}$ are never given by the normal solution, $u_n = 0$, since $\rho \to 1$ in L^2.

The relation (1.6) was also crucial to the results in [ABeB1]. In that paper a different limit is considered, that of the coupling constant $\lambda \to \infty$ with ϵ, sh_{ex} fixed. Nevertheless, (1.6) arises in a similar way, with the applied field and interlayer spacing s selecting a natural frequency of oscillation in x. If the artificially imposed period L_1 is not an exact multiple of the natural frequency $2\pi N/L_3 h_{ex}$ then the vortex lattice structure is frustrated, with a leading order energy cost. Since the periodic problem is intended to simulate the "bulk" properties of a very large material, it seems natural to preserve the condition (1.6) by allowing for a modification in the given period L_1. Note that for any fixed value L_1 which we might prescribe as a period, we may choose $\tilde{L}_1 \in 2\pi N/L_3 h_{ex}$ with $|\tilde{L}_1 - L_1| \leq 2\pi N/L_3 h_{ex} = 2\pi s/(s^2 h_{ex}) = o(s)$, and thus the change in the period domain is imperceptible even with respect to the vanishingly small distance s between the planes.

We believe that the transparent state described in the second part of Theorem 1.4 should resemble the minimizers in the limit $\lambda \to \infty$ as studied in [ABeB1, ABeB2]. To see the more detailed structure of minimizers, a sharper evaluation of the Josephson term is necessary, perhaps after some rescaling in s, as a sort of second Γ-limit of the functional.

Finally, the case where $s^2 h_{ex}$ is bounded above and below fits neither of these descriptions. In this case, we would expect that the vortices are separated by a finite number of planes, and so the discretization cannot be ignored, yet the external field should not yet penetrate the gaps completely. Other methods will be required to analyze this regime.

2 Preliminary results

In order to assert the existence of energy minimizers we must fix a gauge in which the Lawrence–Doniach energy is coercive. We include some details for completeness, as the three-dimensional periodic problem has not been treated in many papers.

For a *constant* vector field $\overline{h} = (\overline{h}_x, \overline{h}_y, \overline{h}_z)$ we choose our representative,

$$\overline{A} = \frac{1}{2}\,\overline{h} \times \vec{x} = \frac{1}{2}\left(z\overline{h}_y \;\; - y\overline{h}_z\,,\; x\overline{h}_z - z\overline{h}_x\,,\; y\overline{h}_x - x\overline{h}_y\right). \tag{2.1}$$

We then say $(u_n, A) \in \mathcal{H}_*$ if $u_n \in H^1_{loc}(\mathbb{R}^2; \mathbb{C})$ for all $n \in \mathbb{Z}$, $A \in H^1_{loc}(\mathbb{R}^3; \mathbb{R}^3)$, and there exist

a *constant* vector $\overline{h} \in \mathbb{R}^3$ such that:

$$\left.\begin{array}{c} A = \overline{A} + A_0, \quad \operatorname{div} A_0 = 0, \quad A_0(\vec{x} + \vec{v}_j) = A_0(\vec{x}), \quad j = 1,2,3,\ \vec{x} \in \mathbb{R}^3, \\ u_n(\vec{x}' + \vec{v}_j) = u_n(\vec{x}')e^{-i\overline{A}(\vec{x}', z_n)\cdot \vec{v}_j}, \qquad j = 1,2,\ \vec{x}' \in \mathbb{R}^2,\ n \in \mathbb{Z} \\ u_{n+N}(\vec{x}') = u_n(\vec{x}')e^{-i\overline{A}(\vec{x}', z_n)\cdot \vec{v}_3}, \quad \vec{x}' \in \mathbb{R}^2,\ n \in \mathbb{Z}, \end{array}\right\} \tag{2.2}$$

where $\overline{A}$ is associated to $\overline{h}$ as in (2.1). Note that $(u_n, A) \in \mathcal{H}_*$ satisfy (1.2) with $\omega_j(\vec{x}) = -\overline{A}(\vec{x}) \cdot \vec{v}_j$.

Lemma 2.1. **(a)** *For any $(u_n, A) \in \mathcal{H}$ there exists $\gamma \in H^2_{loc}(\mathbb{R}^3)$ so that*

$$(u_n e^{i\gamma(\cdot, z_n)},\ A + \nabla\gamma) \in \mathcal{H}_*.$$

Moreover, the constant vector $\overline{h}$ is the average value of $\nabla \times A$ in Ω.

(b) *There exists a constant $C_0 = C_0(\Omega)$ such that for any $(u_n, A) \in \mathcal{H}_*$*

$$\|A\|_{H^1(\Omega)} \le C_0 \|\nabla \times A\|_{L^2(\Omega)}. \tag{2.3}$$

Proof. First assume $(u_n, A) \in \mathcal{H}$. Let $\overline{h} = \frac{1}{|\Omega|}\int_\Omega h$, the (component-wise) average of $h = \nabla \times A$ over the period Ω. Now let ψ be the solution of the periodic problem,

$$\left\{\begin{array}{c} -\Delta\psi = h - \overline{h} \quad \text{in } \mathbb{R}^3, \\ \psi(\vec{x} + \vec{v}_j) = \psi(\vec{x}), \quad \text{in } \mathbb{R}^3,\ j = 1,2,3, \\ \displaystyle\int_\Omega \psi = 0. \end{array}\right.$$

By standard elliptic theory, ψ exists and is unique, $\psi \in H^2_\Omega$, and there exists a constant $C_1 = C_1(\Omega)$ such that

$$\|\psi\|_{H^2(\Omega)} \le C_1 \|h - \overline{h}\|_{L^2(\Omega)} \le C_1 \|h\|_{L^2(\Omega)}. \tag{2.4}$$

Set $A_0 := \nabla \times \psi \in H^1_\Omega$. Then,

$$\nabla \times A_0 = -\Delta\psi = h - \overline{h}.$$

Set $\tilde{A} = \overline{A} + A_0$; then $\nabla \times \tilde{A} = h$ in $\mathbb{R}^3$.

Since $\nabla \times (\tilde{A} - A) = 0$ in $\mathbb{R}^3$ there exists γ with $\tilde{A} - A = \nabla\gamma$ in $\mathbb{R}^3$. Note that *a fortiori*, $\gamma \in H^2_{loc}(\mathbb{R}^3)$. We have

$$\begin{aligned} \tilde{A}(\vec{x} + \vec{v}_j) &= A(\vec{x} + \vec{v}_j) + \nabla\gamma(\vec{x} + \vec{v}_j) \\ &= \tilde{A}(\vec{x}) + \nabla\left(\gamma(\vec{x} + \vec{v}_j) - \lambda(\vec{x}) + \omega_j(\vec{x})\right) \\ &= \tilde{A}(\vec{x}) + \nabla\xi_j(\vec{x}), \end{aligned}$$

with $\xi_j(\vec{x}) = \gamma(\vec{x}+\vec{v}_j) - \gamma(\vec{x}) + \omega_j(\vec{x}) \in H^2_{loc}(\mathbb{R}^3)$. On the other hand, from the explicit form of $\tilde{A} = \overline{A} + A_0$ we easily see that $\nabla\xi_j(\vec{x}) = \frac{1}{2}\overline{h} \times \vec{v}_j$, and therefore we conclude that

$$\xi_j(\vec{x}) = \frac{1}{2}\overline{h} \times \vec{v}_j \cdot \vec{x} = -\overline{A}(\vec{x}) \cdot \vec{v}_j, \quad j = 1, 2, 3.$$

Now define $\tilde{u}_n := u_n e^{i\gamma(\cdot, z_n)}$ for $n \in \mathbb{Z}$. We then have:

$$\begin{aligned}
\tilde{u}_n(\vec{x'} + \vec{v'}_j) &= u_n(\vec{x'} + \vec{v'}_j) e^{i\gamma(\vec{x'}+\vec{v'}_j, z_n)} \\
&= u_n(\vec{x'}) e^{i(\lambda(\vec{x'}+\vec{v'}_j, z_n) + \omega_j(\cdot, z_n))} \\
&= \tilde{u}_n e^{i\xi_j(\vec{x'}, z_n)}, \qquad j = 1, 2,
\end{aligned}$$

and when $j = 3$,

$$\begin{aligned}
\tilde{u}_{n+N}(\vec{x'}) &= u_{n+N}(\vec{x'}) e^{i\gamma(\vec{x'}, z_{n+N})} \\
&= u_n(\vec{x'}) e^{i(\gamma(\vec{x'}, z_{n+N}) + \omega_3(\vec{x'}, z_n))} \\
&= \tilde{u}_n e^{i\xi_3(\vec{x'}, z_n)}.
\end{aligned}$$

In conclusion, $(\tilde{u}_n, \tilde{A}) \in \mathcal{H}_*$ and is gauge-equivalent to (u_n, A). This proves part (a) of the lemma.

The proof of part (b) follows from the decomposition of $A = \overline{A} + A_0$ above and the elliptic estimate (2.4). □

Given the estimate (2.3) the existence of minimizers for $\mathcal{L}^\lambda_{\epsilon,s}$ in $\mathcal{H}_*$ follows from the direct method in the calculus of variations.

Proposition 2.2. *For any $s, \epsilon > 0$ the minimum of $\mathcal{L}^\lambda_{\epsilon,s}$ is attained in $\mathcal{H}_*$. Moreover, the minimizer satisfies $|u_n| \leq 1$ for all $(x, y) \in \mathbb{R}^2$ and $n \in \mathbb{Z}$.*

The Euler–Lagrange equations and regularity for solutions may be found in Bauman & Ko [BaK], which implies the estimate $|u_n| \leq 1$ via the maximum principle.

Finally, we note that analogous spaces for the Ginzburg–Landau functional in the three-dimensional periodic setting may be defined as in (1.2), (2.2). For instance, the Floquet boundary conditions may be stated as

$$\begin{cases} u(\vec{x} + \vec{v}_j) = u(\vec{x}) e^{i\omega_j(\vec{x})}, \quad j = 1, 2, 3; \\ A(\vec{x} + \vec{v}_j) = A(\vec{x}) + \nabla\omega_j(\vec{x}), \quad j = 1, 2, 3, \end{cases} \tag{2.5}$$

for $\omega_j \in H^2_{loc}(\mathbb{R}^3; \mathbb{R})$, $j = 1, 2, 3$.

3 Applied field orthogonal to the planes

We assume now that

$$\vec{h}_{ex} = h_{ex}^{\perp}\, \vec{e_3}.$$

Our goal is to show that energy minimization chooses an s-periodic configuration in z, that is, the vortices are vertically aligned and have identical profiles in each plane and gap.

To prove Theorem 1.1 we introduce a functional defined on only one superconducting plane P and in half of a gap, $T^+ = P \times [0, s/2]$. We consider configurations which are periodic with respect to the plane P, and free in z. Later on, the minimizers in the entire gap will be obtained by reflection. We denote by H^1_{loc+} the space of functions which are in $H^1([-R,R]^2 \times [0,s/2])$ for every $R \geq 1$. Then we define the space $\mathcal{H}^+$ to consist of those pairs (v, A) with $v \in H^1_{loc}(\mathbb{R}^2;\mathbb{C})$ and $A \in H^1_{loc+}$ such that there exist $\omega_1, \omega_2 \in H^2_{loc+}$ with

$$\left.\begin{aligned} v(x' + \vec{v'}_1) = v(x')e^{i\omega_1}, \qquad & v(x' + \vec{v'}_2) = v(x')e^{i\omega_2} \\ A(x + \vec{v}_1) = A(x) + \nabla\omega_1(x), \qquad & A(x + \vec{v}_2) = A(x) + \nabla\omega_2(x) \end{aligned}\right\} \tag{3.6}$$

for all $x' \in \mathbb{R}^2$, $x \in \mathbb{R}^2 \times [0, s/2]$. We define a functional F^+ for $(v, A) \in \mathcal{H}^+$,

$$\begin{aligned} F^+(v,A) = \frac{s}{2}\int_P \left[\frac{1}{2}\left|(\nabla' - iA'(x,y,0))\, v\right|^2 + \frac{1}{4\epsilon^2}(|v|^2-1)^2\right] dx\, dy \\ + \frac{1}{2}\int_{T^+} \left|\nabla \times A - \vec{h}_{ex}\right|^2 dx\, dy\, dz \end{aligned} \tag{3.7}$$

As we did for the fully periodic problem we define a subspace $\mathcal{H}^+_*$ which fixes a Coulomb-type gauge in $\mathcal{H}^+$. Given a constant $\overline{h}_z \in \mathbb{R}$, let

$$\overline{A} = \frac{\overline{h}_z}{2}(-y, x, 0).$$

We say $(v, A) \in \mathcal{H}^+_*$ if $v \in H^1_{loc}(\mathbb{R}^2;\mathbb{C})$ and $A \in H^1_{loc+}$ such that there exist a constant $\overline{h}_z \in \mathbb{R}$ so that (recall $\vec{v}_3 = \vec{e}_3$)

$$\left.\begin{aligned} A(\vec{x}) = \overline{A} + A_0, \quad \operatorname{div} A_0 = 0, \quad A_0(\vec{x} + \vec{v}_j) = A_0(\vec{x}), \quad \text{in } T^+,\ j = 1,2, \\ A \cdot \vec{e}_3 = 0 \quad \text{on } \partial T^+, \\ v(x' + \vec{v'}_j) = v(x')e^{-i\overline{A}\cdot\vec{v'}_j}, \quad j = 1,2,\ x' \in \mathbb{R}^2. \end{aligned}\right\} \tag{3.8}$$

Lemma 3.1. **(a)** *For any $(v, A) \in \mathcal{H}^+$ there exists $\gamma \in H^2_{loc}(T^+)$ so that*

$$(ve^{i\gamma(\cdot,z_n)},\ A + \nabla\lambda) \in \mathcal{H}^+_*.$$

(b) *There exists a constant C_0 such that for any $(u_n, A) \in \mathcal{H}^+_*$*

$$\|A\|_{H^1(T^+)} \leq C_0 \|\nabla \times A\|_{L^2(T^+)}. \tag{3.9}$$

Proof. The proof follows the same lines as that of Lemma 2.1, except that ψ should solve

$$\begin{cases} -\Delta\psi = h - \overline{h}\vec{e}_3 \quad \text{in } \mathbb{R}^2 \times [0, s/2], \\ \psi(\vec{x} + \vec{v}_j) = \psi(\vec{x}), \quad \text{in } \mathbb{R}^3, \ j = 1, 2, \\ \psi_1 = \psi_2 = 0, \quad \text{for } x_3 = 0 \text{ or } x_3 = s/2, \\ \dfrac{\partial \psi_3}{\partial x_3} = 0 \quad \text{for } x_3 = 0 \text{ or } x_3 = s/2, \\ \displaystyle\int_Q \psi_3 = 0. \end{cases}$$

This is essentially a mixed periodic and Dirichlet problem for ψ_1, ψ_2, and a mixed periodic and Neumann problem for ψ_3; see the appendix of [BBO] for a general theory of existence and uniqueness of such problems. In this case the geometry of the problem is simple, and existence is a straightforward application of the Riesz representation theorem in the space of H^1 vector fields with $\psi_1 = \psi_2 = 0$ on the lateral boundaries of the strip and periodic in x, y. As before, we set $A_0 = \nabla \times \psi$ and note that $A_0 \cdot \nu = \pm \nabla \times \psi \cdot \vec{e}_3 = 0$ on the lateral boundaries by the Dirichlet condition. The rest of the proof follows exactly as in Lemma 2.1 and is omitted. □

The space $\mathcal{H}^+$ fixes a Coulomb gauge for A and hence the existence of minimizers $(v^+, A^+) \in \mathcal{H}^+$ of F^+ follows from the direct method. By the regularity theory proved by Bauman & Ko for the Lawrence–Doniach system we have v^+ smooth on P, $A^+ \in C^{1,\alpha}(T^+)$, and in fact (note that $\Delta h^+ = 0$ in the interior of T^+), A^+ is smooth on $P \times (0, s/2]$.

We now show that the minimizers of F^+ on $\mathcal{H}^+$ may be reflected to obtain (symmetric) energy minimizers in each (full) gap. To see this, we define a space $\mathcal{H}^-$ of configurations in the lower half-gap, $T^- = P \times [-s/2, 0]$. The analogous functional $F^-(v, A)$ is defined as (3.7), with the integral now over T^-. Again, the minimizer of F^- over $\mathcal{H}^-$ is attained. In fact, for any $(v, A) \in \mathcal{H}^+$, define $(\tilde{v}, \tilde{A})$ in T^- by $\tilde{v} = v$ and

$$\tilde{A}(x, y, z) = (A_x(x, y, -z), A_y(x, y, -z), -A_z(x, y, -z)), \qquad z \in [-s/2, 0] \tag{3.10}$$

It is easy to see that $(\tilde{v}, \tilde{A}) \in \mathcal{H}^-$. Moreover, we have

$$\tilde{h}(x, y, z) = \nabla \times \tilde{A}(x, y, z) = (-h_x(x, y, -z), -h_y(x, y, -z), h_z(x, y, -z)), \quad z \in [-s/2, 0]$$

and hence

$$\begin{aligned}|\nabla \times \tilde{A}(x,y,z) - \vec{h}_{ex}|^2 &= |\nabla \times \tilde{A} - h_{ex}^{\perp}\vec{k}|^2 \\ &= |h_x|^2 + |h_y|^2 + |h_z - h_{ex}^{\perp}|^2 \\ &= |\nabla \times A(x,y,-z) - \vec{h}_{ex}|^2.\end{aligned}$$

In particular, $F^-(\tilde{v},\tilde{A}) = F^+(v,A)$, and therefore

$$I := \min_{(v,A)\in\mathcal{H}^+} F^+(v,A) = \min_{(v,A)\in\mathcal{H}^-} F^-(v,A).$$

Next, we use the minimizers of $F^{\pm}$ as building blocks for a configuration (u_n, A). Let $(v^+, A^+) \in \mathcal{H}^+_*$ be a minimizer of F^+. By the above remarks, $(v^+, \tilde{A}^+) \in \mathcal{H}^-_*$ (with $\tilde{A}^+$ constructed as in (3.10)) are minimizers of F^-. Moreover, by odd/even symmetry and the boundary condition of (3.8), we note that the extension $\hat{A}$ defined by

$$\hat{A}(x,y,z) = \begin{cases} A^+(x,y,z), & \text{for } (x,y,z) \in T^+, \\ \tilde{A}^+(x,y,z) & \text{for } (x,y,z) \in T^-, \end{cases}$$

is continuous across the plane P, and therefore $\hat{A} \in H^1_{per}(P \times [-s/2, s/2]; \mathbb{R}^2)$. We remark that $\hat{A}_z$ is odd in z, and hence

$$\int_{-s/2}^{s/2} \hat{A}_z \, dz = 0. \tag{3.11}$$

Continuing, the symmetries of $\hat{A}$ permit us to extend $\hat{A}$ to all of $\mathbb{R}^3$ as an s-periodic H^1 (in fact, continuous) function A^* in z,

$$A^*(x,y,z+ns) = \hat{A}(x,y,z), \qquad (x,y) \in P,\ z \in [-s/2, s/2],\ n \in \mathbb{Z}.$$

From our observation (3.11) we conclude that

$$\int_{z_{n-1}}^{z_n} A^*_z \, dz = 0, \qquad \forall n. \tag{3.12}$$

We also define $u^*_n = v^+$ for all planes $n = 0, \ldots, N(s)-1$. It is easy to see that $(u^*_n, A^*) \in \mathcal{H}$.

The configuration $(u^*_n, A^*) \in \mathcal{H}$ gives us an upper bound on the Lawrence–Doniach energy. Indeed, notice that since $u^*_n = u^*_m$ for all n, m and by (3.12) the Josephson interaction term is identically zero,

$$u^*_n(x,y) - u^*_{n-1}(x,y) e^{i\int_{z_{n-1}}^{z_n} A^*_z(x,y,z)\,dz} \equiv 0, \qquad \forall (x,y) \in P,\ n = 1, \ldots, N(s). \tag{3.13}$$

Therefore, the energy of this configuration exactly decouples into $N(s)$ copies of F^+ and F^-, and this gives the upper bound,

$$\begin{aligned}\min_{\mathcal{H}_*} \mathcal{L}^\lambda_{\epsilon,s} \leq \mathcal{L}^\lambda_{\epsilon,s}(u_n^*, A^*) &= \sum_{n=1}^{N(s)} \left(F^+(u_n^*, A^*(\cdot,\cdot,z-ns)) + F^-(u_n^*, A^*(\cdot,\cdot,z-ns))\right) \\ &= 2N(s)\, I.\end{aligned}$$

We now show that the Lawrence–Doniach energy is also bounded below by this same quantity. Indeed, for any configuration $(u_n, A) \in \mathcal{H}$, the restriction to each half-gap $T^\pm$ gives an element of $\mathcal{H}^\pm$. We then observe that

$$\mathcal{L}^\lambda_{\epsilon,s}(u_n, A) \geq \sum_{n=1}^{N(s)} \left(F^+(u_n, A(\cdot,\cdot,z-ns)) + F^-(u_n, A(\cdot,\cdot,z-ns))\right) \geq 2N(s)\, I.$$

Therefore, $\min_{\mathcal{H}} \mathcal{L}^\lambda_{\epsilon,s} = 2N(s)\, I = \mathcal{L}^\lambda_{\epsilon,s}(u_n^*, A^*)$. In particular, the configuration (u_n^*, A^*) with vertically aligned vortices is a global minimizer of $\mathcal{L}^\lambda_{\epsilon,s}$ for all values of the parameters.

Now we claim that any minimizer (u_n, A) of $\mathcal{L}^\lambda_{\epsilon,s}$ is of the above form, with vertically aligned vortices and with the desired symmetries. Indeed, the tightness of the upper and lower bounds on the energy imposes two conclusions. First, the Josephson coupling term must vanish, and hence (3.13) holds for any minimizer (u_n, A). By making the gauge transformation with $\varphi(x,y,z) = \int_0^z A_z(x,y,z')\, dz'$, we may assume our minimizers satisfy

$$u_n = u_{n-1} =: u, \quad \text{for all } n \in \mathbb{Z}, \quad A_z(x,y,z) = 0. \tag{3.14}$$

In particular, the vortices are vertically aligned, and the currents are identical in each superconducting plane. To arrive at a second conclusion from the energy identity, define

$$A_n^\pm(x,y,z) := A(x,y,z+ns)|_{T^\pm}, \quad n \in \mathbb{Z},$$

the restriction of the minimizing potential A to a half-gap around P_n. Then, $(u, A_n^\pm) \in \mathcal{H}_*^\pm$, and minimizes $F^\pm$ for each $n \in \mathbb{Z}$. Thus, we observe that $A_n^\pm$ also minimizes the functional $f^\pm(B) = F^\pm(u, B)$, among B obeying the conditions (3.8), with *given* $u = u_n$. Since $f^\pm(B)$ is quadratic and coercive on such B, the minimizer is *unique*, and thus $A_n^\pm$ is the same for each $n \in \mathbb{Z}$, that is, $A(x,y,z+s) = A(x,y,z)$ in this gauge. To recover the other symmetries of the minimizer, we again define the reflection $\tilde{A}$ as in (3.10). As above, we have $f^-(\tilde{A}) = f^+(A)$ uniquely attains the minimum value among vector potentials obeying (3.8). Thus, $\tilde{A} = A$ in the strip $z \in [-s/2, 0]$. In particular, the magnetic field satisfies

$$h(x,y,z) = (-h_x(x,y,-z), -h_y(x,y,-z), h_z(x,y,-z), \qquad z \in [-s/2, 0].$$

This completes the proof of part *1.* of Theorem 1.1.

Next we compare the energy of minimizers of $\mathcal{L}^\lambda_{\epsilon,s}$ with those of the Ginzburg–Landau model. Assume (ψ_{GL}, A_{GL}) is a Ω-periodic minimizer of the Ginzburg–Landau functional with vertical applied field,

$$G_\epsilon(\psi, A) = \int_\Omega \frac{1}{2}|(\nabla - iA)\psi|^2 + \frac{1}{4\epsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{2}|h - h^\perp_{ex}\vec{e}_3|^2,$$

on the space $\mathcal{H}_{GL}$ incorporating the Floquet boundary conditions (see (2.5).) From (REF?) the minimizers in this geometry are two-dimensional, that is, they are gauge equivalent to $\psi_{GL} = \psi_{GL}(x, y)$, $A_{GL} = A'_{GL}(x, y)$. Choosing this special gauge, we define a test configuration for Lawrence–Doniach, $\hat{u}_n = \psi_{GL}(x, y)$, $n = 1, \dots, N$ and $\hat{A} = A_{GL}$. We then clearly have

$$\min \mathcal{L}^\lambda_{\epsilon,s} \le \mathcal{L}^\lambda_{\epsilon,s}(\hat{u}_n, \hat{A}) = G_\epsilon(\psi_{GL}, A_{GL}), \tag{3.15}$$

valid for all $s, \epsilon > 0$.

To produce a matching lower bound, let (u_n, A) be minimizers of $\mathcal{L}^\lambda_{\epsilon,s}$. By the preceeding analysis, $u_n = u_{n-1} = u$ for each n. In each gap we define

$$\Psi(x, y, z) = u(x, y)e^{-i\int_z^{z_n} A_z(x,y,z')\,dz'}, \qquad z_{n-1} \le z < z_n.$$

By (3.12), the exponent vanishes at each plane $z = z_n$, and so Ψ is continuous across the planes. Furthermore, the configuration $(\Psi, A) \in \mathcal{H}_{GL}$ satisfies Floquet boundary conditions on the solid domain Ω (with the same ω_j as u_n.) Clearly we have

$$|(\partial_z - iA_z(x, y, z))\Psi|^2 = 0.$$

For the other derivatives we calculate:

$$\begin{aligned}
\int_P \int_{z_{n-1}}^{z_n} |(\partial_x - iA_x(x, y, z))\Psi|^2 &= \int_P \int_{z_{n-1}}^{z_n} \left|(\partial_x - iA_x(x, y, z))u - iu\int_z^{z_n} \partial_x A_z\,dz'\right|^2 \\
&= \int_P \int_{z_{n-1}}^{z_n} \left|(\partial_x - iA_x(x, y, z))u - iu\int_z^{z_n} [\partial_z A_x - h_y]\,dz'\right|^2 \\
&= \int_P \int_{z_{n-1}}^{z_n} \left|(\partial_x - iA_x(x, y, z_n))u + iu\int_z^{z_n} h_y\,dz'\right|^2 \\
&\le (1 + s)\int_P \int_{z_{n-1}}^{z_n} |(\partial_x - iA_x(x, y, z_n))u|^2 \\
&\qquad + (1 + s^{-1})s\int_P \int_{z_{n-1}}^{z_n} \left[\int_{z_{n-1}}^{z_n} h_y^2\,dz'\right] \\
&= (1 + s)\int_P \int_{z_{n-1}}^{z_n} |(\partial_x - iA_x(x, y, z_n))u|^2 + 2s\int_P \int_{z_{n-1}}^{z_n} h_y^2.
\end{aligned}$$

An analogous estimate holds for $(\partial_y - iA_y)\Psi$, in terms of the L^2 norm of h_x in the gap. Summing over the gaps we have

$$\int_\Omega |(\nabla' - iA'(x,y,z))\Psi|^2 \leq (1+s)s\sum_{n=1}^N \int_{P_n} |(\nabla' - iA')u_n|^2 + 2s\int_\Omega \left(h_x^2 + h_y^2\right).$$

As each of these terms on the right-hand side appear in the Lawrence–Doniach energy, we arrive at:

$$G_\epsilon(\psi_{GL}, A_{GL}) \leq G_\epsilon(\Psi, A) \leq (1+2s)\mathcal{L}^\lambda_{\epsilon,s}(u_n, A) \leq (1+2s)G_\epsilon(\psi_{GL}, A_{GL}) \tag{3.16}$$

via (3.15). This proves part *2.* of Theorem 1.1.

To verify part *3.*, we assume $h_{ex}^\perp = H_{ex}^\perp |\ln\epsilon|$. From Aydi [AyS] (see also [ABS2] for the three-dimensional setting,) in this regime the energy of periodic GL minimizers satisfies

$$\lim_{\epsilon\to 0} \frac{G_\epsilon(\psi_{GL}, A_{GL})}{|\Omega||\ln\epsilon|^2} = \min_{H\in\mathbb{R}} \frac{1}{2}(|H| + (H - H_{ex}^\perp)^2).$$

Since (3.16) holds uniformly in ϵ, we conclude that

$$\lim_{\epsilon,s\to 0} \frac{\mathcal{L}^\lambda_{\epsilon,s}(u_n, A)}{|\Omega||\ln\epsilon|^2} = \min_{H\in\mathbb{R}} \frac{1}{2}(|H| + (H - H_{ex}^\perp)^2).$$

We conclude that, in the simultaneous limit $\epsilon, s \to 0$, the two models give rise to the same lower critical field, $H_{c1} = H_{c1}^\perp = \frac{1}{2}|\ln\epsilon|$.

Next consider $h_{ex}^\perp$ with $|\ln\epsilon| \ll h_{ex}^\perp \ll \epsilon^{-2}$. Since the minimizers of G_ϵ are two-dimensional, we may apply Theorem 8.1 of [SS] which gives the asymptotic limit of minimizers of the Ginzburg–Landau functional in this regime. Again using the estimate (3.16) we obtain part *4.* of Theorem 1.1.

This concludes the proof of Theorem 1.1. The statements in Corollary 1.2 follow immediately.

♢

4 Applied field parallel to the planes

We now consider applied fields parallel to the planes,

$$\vec{h}_{ex} = h_{ex}\,\vec{e}_2 \tag{4.1}$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the standard basis of $\mathbb{R}^3$. We first observe that for parallel fields the problem of minimizing $\mathcal{L}^\lambda_{\epsilon,s}$ reduces to a two-dimensional one. Indeed, define the two-dimensional

LD energy of $v_n = v_n(x)$, $B = (B_x(x,z), B_z(x,z))$ as:

$$\begin{aligned}\mathcal{L}^{\lambda,2D}_{\epsilon,s}(v_n, B) &= s\sum_{n=1}^{N}\int_0^{L_1}\left[\frac{1}{2}\left|(\partial_x - iB_x(x,z_n))\,v_n\right|^2 + \frac{1}{4\epsilon^2}(|v_n|^2-1)^2\right]dx \\ &\quad + s\sum_{n=1}^{N}\int_0^{L_1}\frac{1}{2\lambda^2 s^2}\left|v_n - v_{n-1}e^{i\int_{z_{n-1}}^{z_n}B_z(x,z)\,dz}\right|^2 dx \\ &\quad + \frac{1}{2}\int_0^{L_3}\int_0^{L_1}\left|\partial_z B_x - \partial_x B_z - h_{ex}\right|^2 dx\,dz.\end{aligned} \tag{4.2}$$

Lemma 4.1. *Assume $\epsilon, s > 0$ are fixed, and $\vec{h}_{ex} = h_{ex}\,\vec{e}_2$ is any fixed parallel external field. If $(u_n, A) \in \mathcal{H}$ is a minimizer of $\mathcal{L}^{\lambda}_{\epsilon,s}$ then it is gauge-equivalent to $(v_n, B) \in \mathcal{H}$ with $v_n = v_n(x)$, $B = (B_x, 0, B_z)(x,z)$ a minimizer of $\mathcal{L}^{\lambda,2D}_{\epsilon,s}$.*

Proof. Let (u_n, A) be minimizers of $\mathcal{L}^{\lambda}_{\epsilon,s}$ (for s, ϵ fixed and $\vec{h}_{ex}$ satisfying (4.1) also fixed.) For any $\delta > 0$ choose y_0 so that

$$\mathcal{L}^{\lambda,2D}_{\epsilon,s}(u_n(\cdot,y_0), A(\cdot,y_0,\cdot)) \le \inf_y \mathcal{L}^{\lambda}_{\epsilon,s}(u_n(\cdot,y), A(\cdot,y,\cdot)) + \delta.$$

Setting $v_n = u_n(x,y_0)$, $B(x,z) = (A_x(x,y_0,z), 0, A_z(x,y_0,z))$ we have

$$\mathcal{L}^{\lambda,2D}_{\epsilon,s}(v_n, B) - \delta \le \mathcal{L}^{\lambda}_{\epsilon,s}(u_n, A) = \min \mathcal{L}^{\lambda}_{\epsilon,s}.$$

Since the minimizers $(\tilde{v}_n, \tilde{B})$ of $\mathcal{L}^{\lambda,2D}_{\epsilon,s}$ (with periodic boundary conditions) are admissible for the three-dimensional energy, we clearly have

$$\min \mathcal{L}^{\lambda}_{\epsilon,s} \le \mathcal{L}^{\lambda}_{\epsilon,s}(\tilde{v}_n, \tilde{B}) = \mathcal{L}^{\lambda,2D}_{\epsilon,s}(\tilde{v}_n, \tilde{B}) \le \mathcal{L}^{\lambda,2D}_{\epsilon,s}(v_n, B).$$

As $\delta > 0$ is arbitrary, we conclude that (v_n, B) minimizes both $\mathcal{L}^{\lambda}_{\epsilon,s}$ and $\mathcal{L}^{\lambda,2D}_{\epsilon,s}$, and that any minimizer of $\mathcal{L}^{\lambda}_{\epsilon,s}$ has $h_x, h_z \equiv 0$ and $(\partial_y - iA_y)u_n \equiv 0$, that is, the minimizers must be two dimensional. □

In the remainder of the paper we denote $\mathcal{L}^{\lambda,2D}_{\epsilon,s} = \mathcal{L}^{\lambda}_{\epsilon,s}$ for simplicity.

As was noted in [ABS1], in a purely two-dimensional setting the anisotropy λ may be removed by an appropriate change of variables. This is not the case in the fully three-dimensional setting, where anisotropy plays an essential role in the geometry of minimizers: see [ABS2]. Here we make the simple rescaling,

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{z} = z, \quad \tilde{A}_x = \lambda A_x, \quad \tilde{A}_z = A_z.$$

Then, for any configuration $(u_n, A) \in \mathcal{H}^{\|}(Q)$, we have $(\tilde{u}_n, \tilde{A}) \in \mathcal{H}^{\|}([0, \frac{L_1}{\lambda}] \times [0, L_3])$ with

$$\mathcal{LD}^{\lambda=1}_{\frac{\epsilon}{\lambda},s}(\tilde{u}_n, \tilde{A}; [0, \frac{L_1}{\lambda}] \times [0, L_3]; \lambda h_{ex}) = \lambda \mathcal{LD}^{\lambda}_{\epsilon,s}(u_n, A; Q; h_{ex}). \tag{4.3}$$

In the following we may then suppose $\lambda = 1$, and adjust our results for the general case accordingly.

We now distinguish two different parameter regimes, corresponding to two very different behaviors for minimizers.

Case $s^2 h_{ex} \ll 1$

In this case, the distance between vortices $h_{ex}^{-\frac{1}{2}} \gg s$, the interlayer spacing, and we will see that the vortices resemble Ginzburg–Landau vortices to leading order but with the rôle of ϵ replaced by s. In this regime we will prove Theorem 1.3 from the introduction.

We begin by deriving an upper bound for the energy of minimizers. This upper bound is a modification of the more standard upper bound for periodic two-dimensional configuration for the 2D Ginzburg-Landau energy. We provide some details for the standard case since we will need to modify slightly the construction and also obtain more precise estimates. Similar constructions can be found in earlier papers (see for example Section 3 in [AyS] or Proposition 8.1 of [SS] for case 2 below.) Here we present a proof that includes both the cases where $h_{ex} = H_{ex}|\ln\epsilon|$ and where $|\ln\epsilon| \ll h_{ex} \ll \epsilon^{-2}$.

An upper bound construction via Ginzburg–Landau

We begin by constructing a periodic two-dimensional configuration which gives a good highest-order approximation to the Ginzburg–Landau energy,

$$G_\eta(v, A) = \int_Q \left\{ \frac{1}{2}|(\nabla - iA)v|^2 + \frac{1}{4\eta^2}(|v|^2 - 1)^2 + \frac{1}{2}(\nabla \times A - h_{ex})^2 \right\} d\vec{x},$$

for $v \in H^1_{loc}(\mathbb{R}^2; \mathbb{C})$, $A \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ satisfying two-dimensional Floquet boundary conditions with respect to the unit square $Q = [0, 1]^2$,

$$\begin{cases} v(\vec{x} + \vec{e}_j) = v(\vec{x}) e^{i\omega_j(\vec{x})}, & j = 1, 2, \\ A(\vec{x} + \vec{e}_j) = A(\vec{x}) + \nabla\omega_j(\vec{x}), & j = 1, 2, \end{cases} \tag{4.4}$$

for $\omega_j \in H^2_{loc}(\mathbb{R}^2)$, $j = 1, 2$. The choice of period domain $Q = [0, 1]^2$ is made for simplicity, and any other parallelogram may be substituted with identical result for $\frac{1}{|Q|} G_\eta$. We refer to Proposition 8.1 in [SS] for the modification needed to handle a general domain Ω.

Proposition 4.2. *Assume that $\eta > 0$ and $h_{ex} \ll \eta^{-2}$.*

1. *If $h_{ex} = H_{ex}|\ln\eta|$ for constant $H_{ex} > 0$, then for any given constant $H \in \mathbb{R}$ there exists a configuration (u_η, A_η) satisfying (4.4) such that*

$$G_\eta(u_\eta, A_\eta) \le \frac{1}{2}\left[|H| + (H - H_{ex})^2\right] |\ln\eta|^2 + o(|\ln\eta|^2). \tag{4.5}$$

2. *If $|\ln\eta| \ll h_{ex} \ll \eta^{-2}$, then there exists a configuration (u_η, A_η) satisfying (4.4) such that*

$$G_\eta(u_\eta, A_\eta) \le \frac{1}{2} h_{ex} \ln\left(\frac{1}{\eta\sqrt{h_{ex}}}\right) + O(h_{ex}). \tag{4.6}$$

Proof. Let $K = [-\frac{1}{2}, \frac{1}{2}]^2$, and consider the solution of the boundary-value problem

$$\begin{cases} -\Delta f = 2\pi(\delta_0 - 1) & \text{in } K, \\ \partial_\nu f = 0 & \text{on } \partial K, \\ \displaystyle\int_K f = 0. \end{cases} \tag{4.7}$$

By standard elliptic theory the solution exists and is unique, and $f(x) + \ln|x|$ is smooth in $K \setminus \{0\}$, where we let $x = \vec{x}$. As a result we have

$$\int_{K\setminus B_r(0)} \frac{1}{2}|\nabla f|^2\, dx = \pi \ln\left(\frac{1}{r}\right) + O(1) \tag{4.8}$$

as $r \to 0$. Note that f is even in both directions, and hence it may be extended periodically to all $\mathbb{R}^2$.

We now construct our configuration. First, we choose a potential B with $\nabla \times B = 2\pi$. To fix ideas, let $B(x_1, x_2) = \pi(-x_2, x_1)$. Then define $v = e^{i\phi_1}$ in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ by

$$\nabla\phi_1 = \nabla^\perp f + B$$

This is well-defined, since for any closed curve $\gamma \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, $\gamma = \partial\omega$,

$$\begin{aligned} \int_\gamma \left(\nabla^\perp f + B\right) \cdot d\vec{s} &= \int_\omega \nabla \times (\nabla^\perp f + B) \\ &= \int_\omega (-\Delta f + 2\pi) \in 2\pi\mathbb{Z}. \end{aligned}$$

Now we rescale and truncate (v, B) to create a test configuration with the correct leading order energy. Choose a smooth cut-off $\rho_t \in C^\infty(K)$,

$$\rho_t(x) = \begin{cases} 0, & |x| \le t, \\ 1, & |x| \ge 2t, \end{cases}$$

and extend ρ_t to $\mathbb{R}^2$ periodically. Then, for $0 < \eta < \delta$ define

$$u_{\eta,\delta}(x) = \left[\rho_{\frac{\eta}{\delta}}\, v\right]\left(\frac{x}{\delta}\right), \qquad A_\delta(x) = \delta^{-1}\, B\left(\frac{x}{\delta}\right). \tag{4.9}$$

In order that $(u_{\eta,\delta}, A_\delta)$ be gauge-periodic with respect to Q we require that $\delta^{-1} \in \mathbf{Z}$, a condition which will be met by approximation (since we will always take $h_{ex} \to \infty$.) Assuming this compatibility condition on δ, we then calculate

$$(\nabla - iA_\delta)u_{\eta,\delta}(x) = \delta^{-1}(\nabla - iB)[\rho_{\frac{\eta}{\delta}} v](y),$$

where $y = x/\delta$, and

$$\nabla \times A_\delta(x) = \delta^{-2}\nabla \times B(y) = 2\pi\delta^{-2}.$$

We then have

$$\begin{aligned}
\int_Q |\nabla \times A_\delta - h_{ex}|^2 &= \delta^{-2} \int_{Q_{\delta^{-1}}} |\nabla \times B - \delta^2 h_{ex}|^2 \\
&= \delta^{-4} \int_K |\nabla \times B - \delta^2 h_{ex}|^2 \\
&= |2\pi\delta^{-2} - h_{ex}|^2,
\end{aligned} \tag{4.10}$$

where $Q_{\delta^{-1}} = [0, \delta^{-1}]^2$.

Next, we have

$$\begin{aligned}
\int_Q \left\{\frac{1}{2}|(\nabla - iA)v|^2 + \frac{1}{4\eta^2}(|v|^2 - 1)^2\right\} &= \delta^{-2} \int_{Q_{1/\delta}} \left[\frac{1}{2}|(\nabla - iA_\delta)u_{\eta,\delta}^2 + \frac{1}{4\eta^2}(|u_{\eta,\delta}|^2 - 1)^2\right] \\
&= \delta^{-2} \int_K \left[\frac{1}{2\delta^2}|(\nabla - iB)(\rho_{\frac{\eta}{\delta}} v)|^2 + \frac{1}{4\eta^2}(|v|^2 - 1)^2\right] \delta^2\, dy \\
&\leq \delta^{-2} \int_{K \setminus B_{\frac{\eta}{\delta}}} \frac{1}{2}|\nabla\phi - B|^2 \\
&\qquad + \delta^{-2} \int_{B_{2\frac{\eta}{\delta}} \setminus B_{\frac{\eta}{\delta}}} \left[\frac{1}{2}|\nabla \rho_{\frac{\eta}{\delta}}|^2 + \frac{\delta^2}{4\eta^2}(\rho_{\frac{\eta}{\delta}}^2 - 1)^2\right] \\
&\leq \delta^{-2} \int_{K \setminus B_{\frac{\eta}{\delta}}} \frac{1}{2}|\nabla f|^2 + O(\delta^{-2}) \\
&= \pi\delta^{-2} \ln\left(\frac{\delta}{\eta}\right) + O(\delta^{-2})
\end{aligned}$$

Putting the above estimate and (4.10) together we obtain the upper bound,

$$G_\eta(u_{\eta,\delta}, A_\delta) \leq \pi\delta^{-2} \ln\left(\frac{\delta}{\eta}\right) + |2\pi\delta^{-2} - h_{ex}|^2 + O(\delta^{-2}). \tag{4.11}$$

To conclude we must choose δ appropriately for each case. In case $h_{ex} = H_{ex}|\ln\eta|$, given any constant H, we choose δ so that $\delta^{-1} \in \mathbb{Z}$ is the closest integer to $\sqrt{H|\ln\eta|/2\pi}$. In particular, we then have $|2\pi\delta^{-2} - h_{ex}| = |H|\ln\eta| - h_{ex}| + O(1)$, and hence the upper bound (4.11) implies

$$G_\eta(u, A) \leq \frac{1}{2}\left[|H| + (H - H_{ex})^2\right] |\ln\eta|^2 + O(|\ln\eta| \ln|\ln\eta|).$$

In case $h_{ex} \gg |\ln\eta|$, we instead make $\delta^{-1} \in \mathbb{Z}$ the closest integer to $\sqrt{h_{ex}/2\pi}$. In that case we see that

$$\int_Q |\nabla \times A_\delta - h_{ex}|^2 = O(1),$$

and (4.11) now yields

$$G_\eta(u, A) \leq \frac{1}{2} h_{ex} \ln\left(\frac{1}{\eta\sqrt{h_{ex}}}\right) + O(h_{ex}).$$

□

We plan to obtain the upper bound for the Lawrence–Doniach energy using the configuration constructed above, with Ginzburg–Landau parameter $\eta = s/\sigma$ (with $\sigma = \sigma(s)$ to be chosen below,) giving a periodic vortex lattice with lattice spacing $O(\delta) = O(h_{ex}^{-1/2})$ and vortices of core size $O(\eta)$. In principle, the minimizer of the Lawrence–Doniach functional should roughly resemble this configuration, but with u_η discretized to take values only on the superconducting planes. Here is where the details of the model play a subtle role. Indeed, if a row of vortices lies across a plane P_n, the contribution to the Lawrence–Doniach energy will be much larger than desired, on the order of $\frac{s^2}{\sigma^2\epsilon^2}$ per vortex. A much smaller energy will be obtained by fitting the rows of vortices between adjacent planes, so that the order parameter u_η never vanishes on the planes P_n. Unfortunately, the intervortex distance δ may not be an exact multiple of the interplanar distance s, and so we will need to make a small perturbation in the periodic vortex lattice so as to locate the vortices between the planes where they normally should be observed.

Lemma 4.3. *Let $\eta > 0$, h_{ex} be as in Proposition 4.2, and assume $\eta \lesssim s$. There exists a configuration $(\tilde{u}_{s,\eta,\delta}, \tilde{A}_{s,\eta,\delta})$ such that the set*

$$Z_{s,\eta,\delta} = \{z : \ |\tilde{u}_{s,\eta,\delta}(x, z)| < 1 \text{ for some } x \in [0, 1]\}$$

consists of δ^{-1} strips and is contained in the set $\bigcup_{n=1}^N \left[\left(n + \frac{1}{2}\right) s - \eta, \left(n + \frac{1}{2}\right) s + \eta\right]$, and

$$G_\eta(\tilde{u}_{s,\eta,\delta}, \tilde{A}_{s,\eta,\delta}) \leq \begin{cases} \frac{1}{2}\left[|H| + (H - H_{ex})^2\right] |\ln\eta|^2 + o\left(|\ln\eta|^2\right), & \text{if } h_{ex} \leq H_{ex}|\ln\eta|, \\ \frac{1}{2} h_{ex} \ln\left(\frac{1}{\eta\sqrt{h_{ex}}}\right)[1 + o(1)], & \text{if } \ln\eta \ll h_{ex} \ll \eta^{-2}. \end{cases}$$

Proof. We return to the construction in the proof of Proposition 4.2. Choose two bands,

$$E^+ = [-1/2, 1/2] \times [1/8, 3/8],\ E^- = [-1/2, 1/2] \times [-3/8, -1/8] \subset Q.$$

Since $f(x)$ (defined in (4.7)) is smooth in $E^\pm$, we may conclude $|\nabla f(x)| \leq C$ for $x \in E^\pm$, with constant C. Following the construction in the proof we replicate $E^+ \cup E^-$ periodically to form a series of horizontal strips in $\mathbb{R}^2$, and rescale by δ^{-1} to obtain a set of horizontal strips of width $\delta/4$. Restricting to the basic period domain Q, we obtain a set E_δ consisting of $2\delta^{-1}$ horizontal strips, each of width $\delta/4$,

$$\begin{aligned} E_\delta &= \bigcup_{m=1}^{\delta^{-1}} \left(E^-_{\delta,m} \cup E^+_{\delta,m} \right), \\ E^-_{\delta,m} &= \left\{ (x,z) \in Q :\ \zeta_m - \frac{3\delta}{8} \leq z \leq \zeta_m - \frac{\delta}{8} \right\}, \\ E^+_{\delta,m} &= \left\{ (x,z) \in Q :\ \zeta_m + \frac{\delta}{8} \leq z \leq \zeta_m + \frac{3\delta}{8} \right\}, \end{aligned}$$

where $\zeta_m = (m + \frac{1}{2})\delta$, $m = 1, \ldots, \delta^{-1}$, the z-coordinate of the vortex locations in the construction of u_η. Using the scaling and cutoff (4.9) we conclude that

$$|(\nabla - iA_\eta)u_\eta| \leq C\delta^{-1} \tag{4.12}$$

for all $(x, z) \in E_\delta$.

We now describe how to move the row of vortices located at $z = \zeta_m$ so that they be located exactly at the midpoint between adjacent pairs of planes. For each m, we identify the planes which lie around the vortex row: choose n such that $z_{n-1} < \zeta_m \leq z_n$. If ζ_m lies below (respectively, above) the midpoint between the planes $z = z_{n-1}$ and $z = z_n$, we slightly expand (resp., contract) the strip $E^-_{\delta,m}$ below the row, and slightly contract (resp., expand) the strip $E^+_{\delta,m}$ above the row, so as to move the row ζ_m to the midpoint $z = \frac{1}{2}(z_{n-1} + z_n)$. Since the distance between the rows $\delta \gg s$, the distortion is small, and the change in energy will be neglibible.

Following this plan, for each m we choose $t_m \in (-s/2, s/2]$ so that $\zeta_m + t_m = (n + \frac{1}{2})s$. Define a piecewise linear transformation $\Phi^m_{s,\delta}(x, z)$ which maps each strip $\{\zeta_m - \frac{\delta}{2} \leq z \leq \zeta_m + \frac{\delta}{2}\}$ to itself, as follows:

$$\Phi^m_{s,\delta}(x,z) = \begin{cases} (x,z), & \text{if } \zeta_m - \frac{\delta}{2} \leq z \leq \zeta_m - \frac{\delta}{4} - s, \\ (x, z + t_m), & \text{if } \zeta_m - \frac{\delta}{4} + s \leq z \leq \zeta_m + \frac{\delta}{4} - s, \\ (x,z), & \text{if } \zeta_m + \frac{\delta}{4} + s \leq z \leq \zeta_m + \frac{\delta}{2}, \end{cases}$$

and $\Phi^m_{s,\delta}$ is linear (in z, identity in x) in the transition regions. Next, define $\Phi_{s,\delta}: Q \to Q$ to agree with $\Phi^m_{s,\delta}$ in each horizontal strip of height δ. In this way, $\Phi_{s,\delta}$ is piecewise C^1, and since it is the identity along the top edges of Q (and independent of x throughout) it extends periodically to $\mathbb{R}^2$. Now, let $\Psi_{s,\delta} = \Phi^{-1}_{s,\delta}$. By the construction, we note that $\Phi_{s,\delta}$ is invertible, and both $|D\Phi_{s,\delta}| = |\det D\Phi_{s,\delta}| \leq 2$, $|D\Psi_{s,\delta}| = |\det D\Psi_{s,\delta}| \leq 2$.

We then define our modified configuration, $(\tilde{u}_{s,\eta,\delta}, \tilde{A}_{s,\eta,\delta})$,

$$\tilde{u}_{s,\eta,\delta}(x,z) = u_\eta(\Psi_{s,\delta}(x,z)), \quad \tilde{A}_{s,\eta,\delta} = \Psi^*_{s,\delta} A_\eta,$$

where we treat A_η as a 1-form. Since $\Psi_{s,\delta}$ is a rigid motion except for the $2\delta^{-1}$ strips of width $2s$ (where the linear deformation takes place), the energy of $(\tilde{u}_{s,\eta,\delta}, \tilde{A}_{s,\eta,\delta})$ is unchanged except for these strips. These transition strips are contained within the set E_δ above, and therefore the energy density in these strips is uniformly bounded by $C\delta^{-2}$, by (4.12). Thus, the difference in energy,

$$|G_\eta(u_\eta, A_\eta) - G_\eta(\tilde{u}_{s,\eta,\delta}, \tilde{A}_{s,\eta,\delta})| \leq 2s \cdot 2\delta^{-1} \cdot C\delta^{-2} = C'\, s\sqrt{h_{ex}}\, h_{ex} = o(1) h_{ex}.$$

Thus, the claim is established. □

We now prove our upper bound on the Lawrence–Doniach energy:

Proposition 4.4. *For any $H \in \mathbb{R}$, there exists $(v_{s,n}, B_s)$ satisfying (1.2) with*

$$\frac{1}{|Q|}\mathcal{L}_{\epsilon,s}(v_{s,n}, B_s) \leq \begin{cases} \frac{1}{2}\left[|H| + (H - H_{ex})^2\right] |\ln s|^2 + o(|\ln s|^2), & \text{if } h_{ex} \leq H_{ex} |\ln s|, \\ \frac{1}{2} h_{ex} \ln\left(\frac{1}{s\sqrt{h_{ex}}}\right)[1 + o(1)], & \text{if } \ln s \ll h_{ex} \ll s^{-2}. \end{cases} \tag{4.13}$$

Proof. Define σ so that $1 \ll \ln\sigma \ll \ln\frac{1}{s\sqrt{h_{ex}}}$, which is possible under the hypothesis $s^2 h_{ex} \ll 1$. Then, we apply Lemma 4.3 with $\eta = s/\sigma$ to get (u, A) satisfying the bounds

$$\begin{aligned} G_\eta(u,A) &\leq \begin{cases} \frac{1}{2}\left[|H| + (H - H_{ex})^2\right] \left|\ln\frac{s}{\sigma}\right|^2 + o\left(\left|\ln\frac{s}{\sigma}\right|^2\right), & \text{if } h_{ex} \leq H_{ex} |\ln\frac{s}{\sigma}|, \\ \frac{1}{2} h_{ex} \ln\left(\frac{\sigma}{s\sqrt{h_{ex}}}\right)[1 + o(1)], & \text{if } \ln\frac{s}{\sigma} \ll h_{ex} \ll \frac{\sigma^2}{s^2}, \end{cases} \\ &\leq \begin{cases} \frac{1}{2}\left[|H| + (H - H_{ex})^2\right] |\ln s|^2 + o(|\ln s|^2), & \text{if } h_{ex} \leq H_{ex} |\ln s|, \\ \frac{1}{2} h_{ex} \ln\left(\frac{1}{s\sqrt{h_{ex}}}\right)[1 + o(1)], & \text{if } \ln s \ll h_{ex} \ll s^{-2}, \end{cases} \end{aligned}$$

and whose vortices lie in the set $\bigcup_{n=1}^N \left[\left(n + \frac{1}{2}\right) s - \eta, \left(n + \frac{1}{2}\right) s + \eta\right]$.

The general idea is to use the mean-value theorem and periodicity to choose a vertical translation of (u, A), which when restricted to the planes P_n gives the same energy for

Lawrence–Doniach. However, we must exclude translations which move the rows of vortices to within distance η of planes P_n. We define the set of "good" translations,

$$\begin{aligned}\mathcal{T} &= \left\{ t \in \left[-\frac{s}{2}, \frac{s}{2}\right) : \ |u| = 1 \quad \text{for all } (x, z_n) \in P_n, \text{ and } n = 1, \dots, N \right\} \\ &= \left[-\frac{s}{2} + \frac{s}{\sigma}, \frac{s}{2} - \frac{s}{\sigma}\right),\end{aligned}$$

for which none of the vortex balls cross the planes. Having chosen $\sigma \gg 1$, the excluded translations are very small compared with the layer spacing s, and will not contribute to the energy estimate we obtain.

Applying Fubini's theorem with

$$f = |(\partial_x - iA_x)u|^2 + \frac{\sigma^2}{2s^2}\left(1 - |u|^2\right)^2,$$

we observe that there exists $t_* \in \mathcal{T}$ so that:

$$\begin{aligned}\int_Q f(x,z)\,dx\,dz &= \int_{t=-\frac{s}{2}}^{\frac{s}{2}} \left(\sum_{n=1}^N \int_0^1 f(x, z_n + t)\,dx \right) dt \\ &\geq \int_{\mathcal{T}} \left(\sum_{n=1}^N \int_0^1 f(x, z_n + t)\,dx \right) dt \\ &\geq |\mathcal{T}| \sum_{n=1}^N \int_0^1 f(x, z_n + t_*)\,dx \\ &\geq (1 - \sigma^{-1}) s \sum_{n=1}^N \int_0^1 f(x, z_n + t_*)\,dx.\end{aligned}$$

Define $v_{s,n}(x) = u(x, z_n + t_*)$ and $B_s(x,z) = A(x, z + t_*)$. Since $t_* \in \mathcal{T}$, $|v_{\epsilon,n}(x)| = 1$ for all n, x, and hence

$$\begin{aligned}&s \sum_{n=1}^N \int_0^1 |(\partial_x - iB_{s,x})v_{s,n}|^2 + \frac{1}{2\epsilon^2}\left(1 - |v_{s,n}|^2\right)^2 dx \\ &\qquad = s \sum_{n=1}^N \int_0^1 |(\partial_x - iB_{s,x})v_{s,n}|^2 + \frac{\sigma^2}{2s^2}\left(1 - |v_{s,n}|^2\right)^2 dx \\ &\qquad\qquad \leq (1 + o(1)) \int_Q |(\partial_x - iA_{,x})u|^2 + \frac{\sigma^2}{2s^2}\left(1 - |u|^2\right)^2 dx\,dz. \quad (4.14)\end{aligned}$$

On the other hand, dropping the subsript ϵ,

$$\begin{aligned}\frac{v_n - v_{n-1} e^{i\int_{z_{n-1}}^{z_n} B_{s,z}(x,z')\,dz'}}{s} &= \frac{1}{s} \int_{z_{n-1}+t}^{z_n+t} \partial_z \left(u(x,z) e^{i\int_z^{z_n+t} A_z(x,z')\,dz'} \right) dz \\ &= \frac{1}{s} \int_{z_{n-1}+t}^{z_n+t} \left(\partial_z u - iA_z u\right) e^{i\int_z^{z_n+t} A_z(x,z')\,dz'} dz.\end{aligned}$$

Thus, using Cauchy-Schwarz and summing,

$$\sum_{n=1}^{N}\int_0^1 \left\{ \frac{1}{s^2}\left|v_n - v_{n-1}e^{i\int_{z_{n-1}}^{z_n} B_{s,z}(x,z')\,dz'}\right|^2 \right\} dx \le \int_Q |(\partial_z - iA_z)u|^2\,dx\,dz. \tag{4.15}$$

From (4.14) and (4.15) we deduce that $\mathcal{L}_{\epsilon,s}(v_{s,n}, B_s) \le G_\eta(u,A)(1+o(1))$. □

Lower bounds

We will use the tools developed in [ABS1] to derive lower bounds for the energy in this regime. Let (u_n, A) be minimizers of $\mathcal{L}_{\epsilon,s}$. We introduce an interpolated order parameter by letting for $z \in (z_{n-1}, z_n)$,

$$\Psi(x,z) := \left[\frac{z - z_{n-1}}{s}\, u_n(x) + \frac{z_n - z}{s}\, u_{n-1}(x) e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')\,dz'} \right] e^{-i\int_z^{z_n} A_z(x,z')\,dz'}. \tag{4.16}$$

We again note that Ψ satisfies the Floquet boundary conditions (2.5) (now with $j = 1, 2$ only.)

As in [ABS1] we have the following comparison bounds between terms in $\mathcal{L}_{\epsilon,s}$ and G_s:

Lemma 4.5. *Assume (u_n, A) are minimizers of $\mathcal{L}_{\epsilon,s}$, and Ψ is defined as in (4.16). Then,*

$$\begin{aligned}
&\int_Q |(\nabla - i\vec{A})\Psi|^2\,dx\,dz \qquad (4.17)\\
&\quad \le s\sum_{n=1}^{N}\int_0^1 \left\{ |(\partial_x - iA_x(x,z_n))u_n|^2 + \frac{1}{s^2}\left| u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right|^2 \right\} dx\\
&\qquad\quad + o\left(\mathcal{L}_{\epsilon,s}(u_n,A)\right),\\
&\int_Q \frac{1}{s^2}\left(1-|\Psi|^2\right)^2\,dx\,dz \qquad (4.18)\\
&\quad \le s\sum_{n=1}^{N}\left[\frac{2}{s^2}\int_0^1 \left(1-|u_n|^2\right)^2\,dx + \frac{1}{2s^2}\int_0^1 \left| u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right|^2 dx \right],
\end{aligned}$$

Proof. Many of the computations are unchanged from [ABS1]. Estimate (4.18) follows exactly as in Lemma 4.2 of [ABS1]. An explicit calculation shows

$$(\partial_z - iA_z(x,z))\Psi(x,z) = \frac{1}{s}\left[u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right].$$

It remains to estimate the x-derivative term. Here we use an observation which is specific to the parallel case, in which the problem is two-dimensional. The Euler–Lagrange equations for

$\mathcal{L}_{\epsilon,s}$ in this case (see [ABS1]) imply that the magnetic field $h(x,z) = h_y(x,z) = \partial_z A_x - \partial_x A_z$ is *constant in* z in each of the gaps between the planes:

$$h(x,z) = h^{(n)}(x), \quad z_{n-1} < z < z_n.$$

With this simplification, and letting $t = \frac{z - z_{n-1}}{s}$, we may calculate:

$$\begin{aligned}
&|(\partial_x - iA_x(x,z))\Psi|^2 \\
&\quad = \Big| t(\partial_x - iA_x(x,z_n))u_n + (1-t)e^{i\int_{z_{n-1}}^{z_n} \tilde{A}_z(x,z')\,dz'}(\partial_x - iA_x(x,z_{n-1}))u_{n-1} \\
&\qquad -ist(1-t)\,h^{(n)}(x)\left[u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right]\Big|^2 \\
&\quad \le (1+\gamma)\big|t(\partial_x - iA_x(x,z_n))u_n + (1-t)e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')\,dz'}(\partial_x - iA_x(x,z_{n-1}))u_{n-1}\big|^2 \\
&\qquad (1+\gamma^{-1})\left|st(1-t)\,h^{(n)}(x)\left[u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right]\right|^2 \\
&\quad \le (1+\gamma)\left\{t\big|(\partial_x - iA_x(x,z_n))u_n\big|^2 + (1-t)\big|(\partial_x - iA_x(x,z_{n-1}))u_{n-1}\big|^2\right\} \\
&\qquad (1+\gamma^{-1})\,\frac{1}{16}s^2\,(h^{(n)}(x))^2\left|u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right|^2, \qquad (4.19)
\end{aligned}$$

with $\gamma > 0$ (to be determined), and where we have used Jensen's inequality to estimate the square of the convex combination in the last line. We now choose

$$\gamma = s^2 h_{ex} \ll 1,$$

and show that, with this choice, the last line is negligible. We use $(h^{(n)})^2 \le 2h_{ex}^2 + 2(h^{(n)} - h_{ex})^2$ and estimate the last line for each. First, we note

$$\begin{aligned}
&s\sum_{n=1}^{N}\int_{Q_n}(1+\gamma^{-1})s^2h_{ex}^2\left|u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right|^2 \\
&\quad \le s\sum_{n=1}^{N}\int_{Q_n}(1+\gamma^{-1})(s^2h_{ex})^2\frac{1}{s^2}\left|u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right|^2 \\
&\quad \le 2(s^2h_{ex})\,s\sum_{n=1}^{N}\int_{Q_n}\frac{1}{s^2}\left|u_n - u_{n-1}e^{i\int_{z_{n-1}}^{z_n} A_z(x,z')dz'}\right|^2 \\
&\quad \le 2s^2h_{ex}\,Ch_{ex}\ln\left(\frac{1}{s^2h_{ex}}\right) = o(\mathcal{L}_{\epsilon,s}(u_n,A)),
\end{aligned}$$

which is indeed a term of smaller order of energy. Finally, we calculate:

$$\begin{aligned} & s\sum_{n=1}^{N}\int_{Q_n}(1+\gamma^{-1})s^2(h^{(n)}-h_{ex})^2\left|u_n-u_{n-1}e^{i\int_{z_{n-1}}^{z_n}A_z(x,z')dz'}\right|^2 \\ &\quad \le s\sum_{n=1}^{N}\int_{Q_n}(1+\gamma^{-1})4s^2\left|h(x,z)-h_{ex}\right|^2 \\ &\quad \le 8(s^2h_{ex})^{-1}\,s^2\int_{Q}\left|h-h_{ex}\right|^2 \\ &\quad \le \frac{4}{h_{ex}}\,Ch_{ex}\ln\left(\frac{1}{s^2h_{ex}}\right). \end{aligned}$$

Putting these two estimates together, we integrate (4.19) to obtain

$$\begin{aligned} & \int_Q\left|(\partial_x-iA_x(x,z))\Psi\right|^2 \\ &\quad \le s\sum_{n=1}^{N}\int_{P_n}(1+\gamma)\left\{t\left|(\partial_x-iA_x(x,z_n))u_n\right|^2+(1-t)\left|(\partial_x-iA_x(x,z_{n-1}))u_{n-1}\right|^2\right\} \\ &\qquad + o\left(Ch_{ex}\ln\left(\frac{1}{s^2h_{ex}}\right)\right) \\ &\quad \le s\sum_{n=1}^{N}\int_{P_n}\left|(\partial_x-iA_x(x,z_n))u_n\right|^2+o\left(h_{ex}\ln\left(\frac{1}{s^2h_{ex}}\right)\right), \end{aligned}$$

using the periodicity in n and the definition of $\gamma = s^2h_{ex} \ll 1$. This proves (4.17). □

As the Lawrence–Doniach energy is roughly but not exactly bounded below by the Ginzburg–Landau energy of the interpolation Ψ, we need to introduce a modified functional as in [ABS1],

$$\begin{aligned} & \mathcal{M}_s(\Psi,A,Q) = \\ & \frac{1}{2}\int_Q\left\{|\Psi|^2|(\nabla-iA)\Psi|^2+\frac{1}{2}(1-|\Psi|^2)\left(|\nabla|\Psi||^2+\frac{\alpha}{s^2}(1-|\Psi|^2)^2\right)+(h-h_{ex})^2\right\}. \end{aligned} \tag{4.20}$$

We recall that α comes from our hypothesis $\epsilon \le s/\alpha$. From Lemma 4.4 of [ABS1] we have for energy minimizers (u_n, A),

$$\mathcal{L}_{\epsilon,s}(u_n,A) \ge \mathcal{M}_s(\Psi,A)+o\left(h_{ex}\ln\left(\frac{1}{s^2h_{ex}}\right)\right). \tag{4.21}$$

When $h_{ex} \le C|\log s|^k \ll s^{-1}$ for some $k > 0$, we may derive a lower bound on the energy via the ball-construction method [Sa],[Je]:

Proposition 4.6. *Choose any $k, l > 0$.*

Assume that $h_{ex} \le C|\log s|^k$, $\mathcal{L}_{\epsilon,s}(u_n, A) \le C|\log s|^k$ and Ψ is defined as in (4.16)*. Then, for any small enough s, there exists a finite collection of disjoint closed balls $\mathcal{B} = \{B_i\}_{i\in I}$, such that*

1. $r(\mathcal{B}) = |\log s|^{-l}$.
2. $\{x \in \Omega : \ |1 - |\Psi|| \ge |\log s|^{-l}\} \subset \bigcup_{i\in I} B_i$;
3. *With $d_i = \deg(\Psi, \partial B_i)$, $i \in I$, there is a constant C such that,*

$$\mathcal{M}_s(\Psi, A, B_i) \ge \pi d_i \left(|\ln s| - C \log|\log s|\right) \tag{4.22}$$

Proof. From the hypotheses and the lower bounds in Lemma 4.5 we immediately deduce the following bound for the classical Ginzburg–Landau energy (but with parameter s!):

$$G_s(\Psi, A) \le 4\mathcal{L}_{\epsilon,s}(u_n, A) \le C|\log s|^k.$$

In particular (see for instance Proposition 4.8 of [SS]) there exists a collection of closed disjoint vortex balls, $\mathcal{B}_0 = \{B_i^0\}_{i\in I}$ such that item 2. of Proposition 4.6 is satisfied and $r_0 := r(\mathcal{B}_0) \le Cs|\log s|^m$, with $m = k + 2l$. Then from Proposition 4.8 of [ABS1] there exists a collection of balls $\mathcal{B}$ of total radius $r = |\log s|^{-l}$ (s being small enough so that $|\log s|^{-l} > Cs|\log s|^m$) and such that the union of the balls in $\mathcal{B}$ contains the balls in $\mathcal{B}_0$. Moreover, if $B_i \in \mathcal{B}$ and $v = \Psi/|\Psi|$ then

$$\int_{B_i\setminus\mathcal{B}_0} |(\nabla - i\vec{A})v|^2 + (\operatorname{curl}\vec{A})^2 \, dx\, dy \ge 2\pi|d_i| \left(\log\frac{r}{r_0} - \frac{1}{2}\right), \tag{4.23}$$

where d_i is the degree of v on ∂B_i. Using the fact that outside $\mathcal{B}_0$ we have $|\Psi| \ge 1 - |\log s|^{-l}$, we have

$$|\Psi|^2|(\nabla - iA)\Psi|^2 \ge \left(1 - 2|\log s|^{-l}\right)|(\nabla - i\vec{A})v|^2$$

on $Q \setminus \mathcal{B}_0$. Together with (4.20) and replacing r and r_0 by their values in (4.23) yields the desired result. □

Among other things, Proposition 4.6 gives a sense of what it means for the configuration (u_n, A) to have vortices, since the discrete nature of the planes allows circulation of supercurrents along and between the planes without the vanishing of any of the u_n. As in the periodic Ginzburg–Landau model (see [ABS1, ABS2]) we may also use the Floquet boundary

conditions to connect the total degree of vortices $\sum_i d_i$ with the magnetic flux through the cross-section Q,

$$\int_Q h = 2\pi \sum_i d_i. \tag{4.24}$$

Indeed, since there are only finitely many vortex balls for any fixed s, by translating the period square Q if necessary, we may guarantee that none of the balls in $\mathcal{B}$ intersect ∂Q. Thus, $|\Psi| \geq \frac{1}{2}$ on ∂Q and the total degree may be calculated by integration of the phase of $\Phi = |\Phi| e^{i\phi}$ along ∂Q. Using the Floquet conditions (2.5), we observe that $\nabla\phi$ and A satisfy the same relations with respect to the auxilliary functions ω_1, ω_2, and hence

$$\begin{aligned}\sum_i d_i = \deg(\Psi, \partial Q) &= \frac{1}{2\pi}\int_{\partial Q} \nabla\phi \cdot \tau \, ds \\ &= \frac{1}{2\pi}\int_{\partial Q} A \cdot \tau \, ds = \frac{1}{2\pi}\int_Q h,\end{aligned}$$

by Stokes' Theorem. The claim is thus established.

We are now ready to give the lower bound for minimizers of the Lawrence–Doniach functional in case h_{ex} is of the order of $|\ln s|$:

Proposition 4.7. *Assume $h_{ex} = H_{ex}\,|\ln s|$ for constant H, and $\epsilon \leq s/\alpha$ for $\alpha > 0$ constant. If (u_n, A) are minimizers of $\mathcal{L}_{\epsilon,s}$ in $\mathcal{H}$, then for all $s > 0$ sufficently small,*

$$\frac{1}{|Q|}\mathcal{L}_{\epsilon,s}(u_n, A) \geq \frac{1}{2}\left|\int_Q h\right| + \frac{1}{2}\int_Q (h - h_{ex})^2 + o(|\ln s|^2).$$

Proof. From (4.21) it suffices to prove the estimate for $\mathcal{M}_s(\Psi, A)$ with Ψ given by (4.16). By Lemma 4.4 we may apply Proposition 4.6 with (say) $r = |\ln s|^{-10}$ to obtain for each small $s > 0$ a family $\mathcal{B} = \{B_i\}$ of balls and degrees $d_i = \deg(\Psi, \partial B_i)$ such that the three conclusions of Proposition 4.6 hold. In particular, we have the lower estimate inside the balls,

$$\begin{aligned}&\int_{\cup_i B_i}\left[\frac{1}{2}|\Psi|^2\,|\nabla\Psi|^2 + r^2|\nabla\times A|^2 + \frac{1}{2}(1-|\Psi|^2)\left(|\nabla|\Psi||^2 + \frac{\alpha}{s^2}(1-|\Psi|^2)^2\right)\right] \\ &\quad\geq \pi\sum_i d_i\,(|\ln s| - C\log|\log s|) \\ &\quad\geq \frac{1}{2}\left|\int_Q h\right|\,|\ln s|\,(1-o(1))\,,\end{aligned}$$

where we have used (4.24) to connect the total degree to the total magnetic flux through Q. Note that by the upper bound on the energy and the choice of $r = |\ln s|^{-10}$, we have

$$r^2\int_{\cup_i B_i}(\nabla\times A)^2 \leq 2r^2\int_Q\left((\nabla\times A - h_{ex})^2 + h_{ex}^2\right) = o(1).$$

Thus, we obtain:

$$\begin{aligned}\mathcal{M}_s(\Psi, A) &\geq \int_{\cup_i B_i} \left[\frac{1}{2}|\Psi|^2\,|\nabla\Psi|^2 + r^2|\nabla\times A|^2 + \frac{1}{2}(1-|\Psi|^2)\left(|\nabla|\Psi||^2 + \frac{\alpha}{s^2}(1-|\Psi|^2)^2\right)\right.\\ &\qquad \left. + \frac{1}{2}\left(\nabla\times A - h_{ex}\right)^2\right] + o(1)\\ &\geq \frac{1}{2}\left|\int_Q h\right| |\ln s| + \frac{1}{2}\int_Q (\nabla\times A - h_{ex})^2 + o(|\ln s|^2). \end{aligned} \tag{4.25}$$

□

We may now resolve the minimization problem for parallel applied fields of magnitude $h_{ex} = H_{ex}|\ln s|$. After normalizing the magnetic fields $h_s = \nabla\times A$ by letting

$$H_s = \frac{h_s}{|\ln s|},$$

and applying the upper bound (4.5), we conclude that $(H_s - H_{ex})$ is uniformly bounded in L^2 and hence a subsequence $H_{s_k} \rightharpoonup H_*$ is weakly convergent. Passing to the limit using the upper bound (4.5) and (4.25) we have

$$\begin{aligned}\Phi(H) := \frac{1}{2}\left[|H| + (H-H_{ex})^2\right] &\geq \liminf_{s\to 0} \frac{\mathcal{L}_{\epsilon,s}(u_n, A)}{|Q||\ln s|^2}\\ &\geq \frac{1}{2}\frac{1}{|Q|}\left[\left|\int_Q H_*\right| + \int_Q (H_* - H_{ex})^2\right], \end{aligned} \tag{4.26}$$

which holds for every constant $H \in \mathbb{R}$. Note that the right-hand side is strictly convex, and thus it may be bounded below by replacing H_* by its average value over the square Q, $\overline{H}_* = \frac{1}{|Q|}\int_Q H_*$. Choosing the constant H to minimize the left-hand expression, we have,

$$\min_{H\in\mathbb{R}} \Phi(H) \geq \frac{1}{2}\left[|\overline{H}_*| + (\overline{H}_* - H_{ex})^2\right] = \Phi(\overline{H}_*).$$

By the strict convexity of $\Phi:\ \mathbb{R}\to\mathbb{R}$ we conclude that $\overline{H}_*$ minimizes Φ, and since there is equality in (4.26) we further conclude that $H_* = \overline{H}_*$ is constant and minimizes Φ. A simple calculation shows that this minimum is given by $H_* = 0$ if $0 \leq H_{ex} \leq \frac{1}{2}$ and $H_* = H_{ex} - \frac{1}{2}$ if $H_{ex} \geq \frac{1}{2}$. Thus, the lower critical field in the parallel case is (to highest order in s,) $H_{c1} = \frac{1}{2}|\ln s|$. This concludes the proof of Theorem 1.3 in the case $h_{ex} = O(|\ln s|)$.

We now consider the case, $|\ln s| \ll h_{ex} \ll s^{-2}$. For strong applied fields the vortex ball construction may not be applied directly, and we require a different approach, as in Chapter 8 of [SS].

Proposition 4.8. *Assume* $|\ln s| \ll h_{ex} \ll s^{-2}$, (u_n, A) *minimizes* $\mathcal{L}_{\epsilon,s}$, *and* Ψ *is given as in* (4.16). *Then, for every constant* $K > 0$, *there exists* λ *with* $1 \ll \lambda \ll 1/s$ *and* $\beta_K > 0$ *such that for every ball* $B_y^\lambda := B(y, \lambda^{-1})$ *we have, as* $s \to 0$,

$$\mathcal{M}_s(\Psi, A; B_y^\lambda) \geq \frac{\beta_K |B_y^\lambda|}{2} h_{ex} \ln \frac{1}{s\sqrt{h_{ex}}} (1 - o(1)),$$

and $\beta_K \to 1$ *as* $K \to \infty$.

Proof. We provide a sketch, as the steps follow those in Proposition 8.2 of [SS]. Without loss, we take the center of the ball to be the origin, $y = 0$. First, for any fixed positive λ we rescale, $\Psi_\lambda(\lambda x) = \Psi(x)$, $\lambda A_\lambda(\lambda x) = A(x)$, $s' = \lambda s$, and $h'_{ex} = h_{ex}\lambda^{-2}$. Introducing a rescaled energy, we have

$$\begin{aligned}
\mathcal{M}_s(\Psi, A; B_0^\lambda) &= \mathcal{M}_{s'}^\lambda(\Psi_\lambda, A_\lambda; B_0^1) \\
&= \frac{1}{2} \int_{B_0^1} |\Psi_\lambda|^2 \left|\nabla_{A_\lambda} \Psi_\lambda\right|^2 + \lambda^2 \left(\operatorname{curl} A_\lambda - h'_{ex}\right)^2 \\
&\qquad + \frac{1}{2}(1 - |\Psi_\lambda|^2) \left[\left|\nabla |\Psi_\lambda|\right|^2 + \frac{\alpha}{(s')^2}(1 - |\Psi_\lambda|^2)^2 \right].
\end{aligned}$$

As in (8.16) of [SS], it is possible to choose λ such that

$$h'_{ex} = K|\ln s'|, \qquad 1 \ll \lambda \ll s^{-1}, \qquad s' \ll 1, \qquad \ln \frac{1}{s\sqrt{h_{ex}}} \sim |\ln s'|. \tag{4.27}$$

Thus, in the new coordinates we return to the small-field case above, in which the vortex ball construction is valid, and we seek to show that, as $s' \to 0$,

$$\mathcal{M}_{s'}^\lambda(\Psi_\lambda, A_\lambda; B_0^1) \geq \frac{\beta_K |B_0^1|}{2} h'_{ex} |\ln s'| (1 - o(1)), \tag{4.28}$$

for some $\beta_K > 0$. We may assume that, for some constant $C > 0$,

$$\mathcal{M}_{s'}^\lambda(\Psi_\lambda, A_\lambda; B_0^1) \leq C(h'_{ex})^2,$$

for otherwise (4.28) certainly holds for any constant β_K.

The next step is to bound $\mathcal{M}_{s'}^\lambda(\Psi_\lambda, A_\lambda; B_0^1)$ from below by the minimum of $\mathcal{M}_s$ in $H^1(B_0^1)$ (that is, with natural boundary conditions on ∂B_0^1, rather than periodic.) This follows the analogous argument for the Ginzburg–Landau functional, using the ball construction (Theorem 4.6) and weak convergence of the rescaled fields and currents outside of the vortex balls, as in the proof of item 1 of Theorem 7.1 in [SS]. The crucial observation is that outside

of the vortex balls, $\mathcal{M}_s$ agrees with Ginzburg–Landau (with s replacing ϵ) to highest order, and so the same lower bound may be obtained:

$$\liminf_{s'\to 0} \frac{\mathcal{M}_{s'}(\Psi_\lambda, A_\lambda; B_0^1)}{(h'_{ex})^2} \geq \min_\mu E_K(\mu),$$

with

$$E_K(\mu) = \frac{\|\mu\|}{2K} + \frac{1}{2}\int_{B_0^1} |\nabla h_\mu|^2 + |h_\mu - 1|^2,$$

defined for measures $\mu \in H^{-1}$, where h_μ solves

$$-\Delta h_\mu + h_\mu = \mu, \qquad h_\mu|_{\partial B_0^1} = 1.$$

The minimizers correspond to the solutions of an obstacle problem, and for the ball an explicit solution is given in (7.15) of [SS]. In particular,

$$\min_\mu E_K \geq \frac{1}{2K}\left(1 - \frac{1}{2K}\right)|\omega_K|,$$

with coincidence set $\omega_K \to B_0^1$ as $K \to \infty$. Using the relations (4.27), we obtain (4.28) with $\beta_K = \left(1 - \frac{1}{2K}\right)|\omega_K|/|B_0^1| \to 1$ as $K \to \infty$. □

We may now complete the proof of the second part of Theorem 1.3. Consider the energy density associated to $\mathcal{M}_s(\Psi, A)$ as a measure,

$$d\nu_s = \frac{dx}{h_{ex}\ln\left[\frac{1}{s\sqrt{h_{ex}}}\right]}\left\{|\Psi|^2|(\nabla_A)\Psi|^2 + \frac{1}{2}(1-|\Psi|^2)\left(|\nabla|\Psi||^2 + \frac{\alpha}{s^2}(1-|\Psi|^2)^2\right) + (h-h_{ex})^2\right\}.$$

By the upper bound (Lemma 4.4,) ν_s is a bounded family and along a subsequence converges weakly in the sense of measures, $\nu_s \rightharpoonup \nu$. Following as in [SS], given any open set U, integrating the lower bound (from Proposition 4.8) over the centers $y \in U$ we obtain

$$\liminf_{s\to 0} \nu_s(U) = \liminf_{s\to 0} \frac{\mathcal{M}_s(\Psi, A; U)}{h_{ex}\ln\left[\frac{1}{s\sqrt{h_{ex}}}\right]} \geq \frac{|U|}{2}.$$

In consequence, $\nu \geq \frac{1}{2}dx$, and from the upper bound we must have equality. This proves the first assertion of item 2. To conclude, note that

$$\|\frac{h_s}{h_{ex}} - 1\|_{L^2}^2 \leq \frac{\mathcal{M}_s(\Psi, A)}{h_{ex}^2} \to 0$$

since $\ln\left[\frac{1}{s\sqrt{h_{ex}}}\right] \ll h_{ex}$.

Case $s^2 h_{ex} \gg 1$

In case $s^2 h_{ex} \gg 1$ the minimizers of the Lawrence–Doniach functional in a parallel applied field behave quite differently from those of the Ginzburg–Landau model. As we will see, there is no "upper critical field" at which superconductivity is diminished and eventually extinguished, but instead a "transparent phase" in which the applied field penetrates the sample completely with no apparent loss of superconductivity in the planes.

It will be convenient to permit some variation in the period domain, $Q = [0, L_1] \times [0, L_3]$, and in fact we will see that energy minimization reveals a subtle relationship between the period and the applied field h_{ex}.

We assume throughout that $(u_n, A) \in \mathcal{H}$ are minimizers of $\mathcal{L}_{\epsilon,s}$. Recall the definitions of ρ, J_x from (1.7) in the introduction.

$$\rho(x,z) = \sum_{n=1}^{N} |u_n(x)| \chi_{(z_{n-1},z_n]}(z), \qquad J_x(x,z) = \sum_{n=1}^{N} (i u_n(x),\, u_n' - i A_x(x,z_n)) \chi_{(z_{n-1},z_n]}(z).$$

In some sense, ρ measures the superconducting density, and J_x the horizontal currents, extended to the entire bulk. The interpolating function Ψ used in the previous regime cannot be easily related to the energy density, as the control on the error terms in the finite differences in x is lost in this highly oscillatory limit.

Proof of Theorem 1.4. We first recall some facts about fixing a gauge in $\mathcal{H}$ in the parallel field setting. For a constant field $\overline{h} \in \mathbb{R}$, we fix the representative vector potential,

$$\overline{A} = (\overline{A}_x, \overline{A}_z) = \frac{\overline{h}}{2}(z, -x). \tag{4.29}$$

As in (2.1),(2.2) we say $(u_n, A) \in \mathcal{H}_*$ if $u_n \in H^1_{loc}(\mathbb{R}; \mathbb{C})$ for all $n \in \mathbb{Z}$, $A \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$, and there exists a constant number $\overline{h} \in \mathbb{R}$ so that

$$\left.\begin{aligned} &A = \overline{A} + \hat{A}, \qquad \operatorname{div} \hat{A} = 0, \qquad \hat{A}(x + L_1, z) = \hat{A}(x,z) = \hat{A}(x, z+1), \\ &u_n(x + L_1) = u_n(x) e^{-i\overline{h} z_n L_1/2}, \qquad u_{n+N}(x) = u_n(x) e^{i\overline{h} x L_3/2}, \quad \forall n \in \mathbb{Z} \end{aligned}\right\} \tag{4.30}$$

Since by Green's theorem we must have $\int_Q \operatorname{curl} \hat{A} = 0$, it follows that the constant $\overline{h}$ is the mean value of $h = \operatorname{curl} A$ over the period domain Q. By Lemma 2.1, for any configuration $(\tilde{u}_n, \tilde{A}) \in \mathcal{H}$ there exists $\lambda \in H^2_{loc}$ so that $(u_n, A) = (\tilde{u}_n e^{i\lambda(\cdot, z_n)}, \tilde{A} + \nabla\lambda) \in \mathcal{H}_*$. Furthermore, from (2.3) we conclude that there exists a constant C_0 with

$$\|\hat{A}\|_{H^1(Q)} \le C_0 \|\operatorname{curl} \hat{A}\|_{L^2(Q)} \tag{4.31}$$

for any $(u_n, A) \in \mathcal{H}_*$.

Upper bound. We obtain an upper bound on the energy by choosing a configuration $(v_n, \overline{A})$ with $\overline{A} = \frac{\overline{h}}{2}(z, -x)$, $\overline{h}$ constant, and $v_n = e^{i\phi_n}$, with $\phi_n(x) = \frac{\overline{h}}{2} z_n x$, $n \in \mathbb{Z}$. Note that with this choice, $v_n' - i\overline{A}_x(x, z_n) = 0$. A simple calculation shows that $(v_n, \overline{h}) \in \mathcal{H}_*$ provided $\overline{h} s L_1 \in 2\pi\mathbb{Z}$. Hence, recalling that $sN = L_3$,

$$\begin{aligned}\min_{\mathcal{H}_*} \mathcal{L}_{\epsilon,s} &\leq \frac{1}{2s} \sum_{n=1}^{N} \int_0^{L_1} 2\left[1 - \cos\left(\overline{h} s x\right)\right] \, dx + \frac{1}{2}|Q|(\overline{h} - h_{ex})^2 \\ &= \frac{|Q|}{s^2} + \frac{1}{2}|Q|(\overline{h} - h_{ex})^2.\end{aligned}$$

By minimizing the right-hand side over $\overline{h} \in \frac{2\pi}{sL_1}\mathbb{Z}$ we obtain the upper bound in the generic case,

$$\min_{\mathcal{H}_*} \mathcal{L}_{\epsilon,s} \leq \frac{|Q|}{s^2} + \frac{1}{2} \min_{\overline{h} \in \frac{2\pi}{sL_1}\mathbb{Z}} |Q|(\tilde{h} - h_{ex})^2 \leq \frac{|Q|}{s^2}\left(1 + \frac{\pi^2}{2{L_1}^2}\right) + o(s^{-2}). \tag{4.32}$$

Of course, in the special case where $h_{ex} \in \frac{2\pi}{sL_1}\mathbb{Z}$, the sharper bound prevails,

$$\min_{\mathcal{H}_*} \mathcal{L}_{\epsilon,s} \leq \frac{|Q|}{s^2} + o(s^{-2}).$$

Lower bound. We next derive a lower bound on minimizers $(u_n, A) \in \mathcal{H}_*$. The idea is to try to mimic the test function in the upper bound as closely as possible. Decompose $A = \overline{A} + \hat{A}$ as above with $\overline{A} = \frac{h_{ex}}{2}(-z, x)$ and define a family of phases $\phi_n(x) = \frac{h_{ex}}{2} z_n x$, $n \in \mathbb{Z}$. Now define

$$v_n = u_n e^{-i\phi_n}.$$

We observe that

$$|v_n| = |u_n|, \qquad |v_n' - i\hat{A}_x(x, z_n)v_n| = |u_n' - iA_x(x, z_n)u_n|, \tag{4.33}$$

$$(iu_n, u_n' - iA_x(x, z_n)u_n) = (iv_n, v_n' - i\hat{A}_x(x, z_n)v_n). \tag{4.34}$$

Written in terms of v_n and $\hat{A}$, we have:

$$\begin{aligned}\mathcal{L}_{\epsilon,s}(u_n, A) = s \sum_{n=1}^{N} \int_0^{L_1} \left(\frac{1}{2}|v_n' - i\hat{A}_x(x, z_n)v_n|^2 + \frac{1}{4\epsilon^2}(|v_n|^2 - 1)^2\right) \\ + \frac{1}{2s} \sum_{n=1}^{N} \int_0^{L_1} |v_n - v_{n-1} \exp\{i\Phi_n(x)\}|^2 + \frac{1}{2}\int_Q \hat{h}^2,\end{aligned} \tag{4.35}$$

where $\hat{h} := \operatorname{curl}(\hat{A})$ and

$$\Phi_n(x) := -h_{ex} s x + \int_{z_{n-1}}^{z_n} \hat{A}_z(x, z)\, dz.$$

The Josephson coupling term will provide the desired lower bound. We expand it:

$$\begin{aligned}\left|v_n - v_{n-1}e^{i\Phi_n}\right|^2 &= |v_n|^2 + |v_{n-1}|^2 - 2\mathrm{Re}\left\{v_n\overline{v_{n-1}}e^{-i\Phi_n}\right\}\\ &= 2 - (1-|v_n|^2) - (1-|v_{n-1}|^2) - 2\mathrm{Re}\left\{v_n\overline{v_{n-1}}e^{-i\Phi_n}\right\}\\ &\geq 2 - \frac{s^2}{8\epsilon^2}\left[(1-|v_n|^2) + (1-|v_{n-1}|^2)\right] - \frac{4\epsilon^2}{s^2} - 2\mathrm{Re}\left\{v_n\overline{v_{n-1}}e^{-i\Phi_n}\right\},\end{aligned}$$

applying the inequality $x \leq \delta x^2 + 1/4\delta$ with $\delta = \frac{s^2}{8\epsilon^2}$ to bound $x = (1-|v_n|^2)$. It follows that

$$\begin{aligned}\mathcal{L}_{\epsilon,s}(u_n, A) \geq\ & s\sum_{n=1}^{N}\int_0^{L_1}\left(\frac{1}{2}|v_n' - i\hat{A}_x(x,z_n)v_n|^2 + \frac{1}{8\epsilon^2}(|v_n|^2-1)^2\right)\\ &+ s\sum_{n=1}^{N}\int_0^{L_1}\frac{1}{s^2}\,dx + \frac{1}{2}\int_Q \hat{h}^2 - s\sum_{n=1}^{N}\int_0^{L_1}\frac{\mathrm{Re}\left\{v_n\overline{v_{n-1}}e^{-i\Phi_n}\right\}}{s^2}.\end{aligned} \tag{4.36}$$

We use integration by parts to estimate the last term, which is the only potentially negative one:

$$\begin{aligned}&\int_0^{L_1} v_n\overline{v_{n-1}}e^{-i\Phi_n} = \int_0^{L_1}\left[v_n\overline{v_{n-1}}e^{-i\int_{z_{n-1}}^{z_n}\hat{A}_z\,dz}\right]e^{ih_{ex}sx}\,dx\\ &= \frac{1}{h_{ex}s}\left[v_n\overline{v_{n-1}}e^{-i\Phi_n}\right]_0^{L_1} - \frac{1}{h_{ex}s}\int_0^{L_1}\left(v_n'\overline{v_{n-1}} + v_n\overline{v_{n-1}'} - iv_n\overline{v_{n-1}}\int_{z_{n-1}}^{z_n}\partial_x\hat{A}_z\,dz\right)e^{-i\Phi_n}\,dx.\end{aligned}$$

It follows, using $|v_n| \leq 1$ from Proposition 2.2 and the equality

$$\int_{z_{n-1}}^{z_n}\partial_x\hat{A}_z\,dz = \int_{z_{n-1}}^{z_n}(\hat{h} + \partial_z\hat{A}_x) = \hat{A}_x(z_n) - \hat{A}_z(z_{n-1}) + \int_{z_{n-1}}^{z_n}\hat{h},$$

that

$$\begin{aligned}\left|\int_0^{L_1} v_n\overline{v_{n-1}}e^{-i\Phi_n}\right| &\leq \frac{2}{h_{ex}s}\\ &+ \frac{1}{h_{ex}s}\int_0^{L_1}\left(\left|v_n' - i\hat{A}_x(x,z_n)v_n\right| + \left|v_{n-1}' - i\hat{A}_x(x,z_{n-1})v_{n-1}\right| + \int_{z_{n-1}}^{z_n}|\hat{h}|\,dz\right)dx.\end{aligned}$$

In particular,

$$\begin{aligned}\frac{1}{s^2}\left|\int_0^{L_1} v_n\overline{v_{n-1}}e^{-i\Phi_n}\right| \leq \frac{2}{h_{ex}s^3} + \int_0^{L_1}\Bigg[&\left(\frac{\left|v_n' - i\hat{A}_x(x,z_n)v_n\right|^2}{8} + \frac{8}{h_{ex}^2s^6}\right) +\\ &\left(\frac{\left|v_{n-1}' - i\hat{A}_x(x,z_{n-1})v_{n-1}\right|^2}{8} + \frac{8}{h_{ex}^2s^6}\right) + \left(\frac{1}{4}\int_{z_{n-1}}^{z_n}\hat{h}^2\,dz + \frac{s}{h_{ex}^2s^6}\right)\Bigg]\,dx.\end{aligned}$$

Together with (4.36) we find

$$\mathcal{L}_{\epsilon,s}(u_n,A) \geq s\sum_{n=1}^{N}\int_0^{L_1}\frac{1}{s^2}\,dx + s\sum_{n=1}^{N}\int_0^{L_1}\frac{\left|v_n' - i\hat{A}_x(x,z_n)v_n\right|^2}{4} + \frac{1}{8\epsilon^2}(1-|v_n|^2)^2\,dx + \frac{1}{4}\int_Q \hat{h}^2 - \frac{2L_3}{h_{ex}s^3} - \frac{16L_1L_3}{h_{ex}^2s^6} - \frac{sL_3}{h_{ex}^2s^6}.$$

The second line is $o(1/s^2)$, using the fact that $h_{ex}s^2 = o(1)$ and $s = o(1)$.

Then, with no assumption on h_{ex}, L_1 we obtain (1.8), while if $h_{ex} \in \frac{2\pi N}{|Q|}\mathbb{Z}$ holds, then the matching upper bound $\mathcal{L}_{\epsilon,s}(u_n,A) \leq |Q|/s^2 + o(1/s^2)$ holds and therefore, using (4.33) and the notation (1.7),

$$\int_Q \left(J_x^2 + \frac{1}{4\epsilon^2}(\rho^2-1)^2 + \frac{1}{2}(h-h_{ex})^2\right) = o(s^{-2}).$$

This concludes the proof of Theorem 1.4. □

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