Γ–CONVERGENCE OF 2D GINZBURG-LANDAU FUNCTIONALS WITH VORTEX CONCENTRATION ALONG CURVES

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Abstract. We study the variational convergence of a family of two-dimensional Ginzburg-Landau functionals arising in the study of superfluidity or thin-film superconductivity as the Ginzburg-Landau parameter ε tends to 0. In this regime and for large enough applied rotations (for superfluids) or magnetic fields (for superconductors), the minimizers acquire quantized point singularities (vortices). We focus on situations in which an unbounded number of vortices accumulate along a prescribed Jordan curve or a simple arc in the domain. This is known to occur in a circular annulus under uniform rotation, or in a simply connected domain with an appropriately chosen rotational vector field. We prove that if suitably normalized, the energy functionals Γ-converge to a classical energy from potential theory. Applied to global minimizers, our results describe the limiting distribution of vortices along the curve in terms of Green equilibrium measures.

1 Introduction

The Ginzburg-Landau theories have had an enormous influence on both physics and mathematics. Physicists employ Ginzburg-Landau models in modeling superconductivity, superfluidity, and, more recently, rotating Bose-Einstein condensates (BECs), all systems which present quantized defects commonly known as vortices. In mathematics, starting with the work by Bethuel, Brezis & Hélein [7], many powerful methods have been developed to study the physical London limit, i.e., the limit as the Ginzburg-Landau parameter ε tends to 0. This limit corresponds to the Thomas-Fermi regime in BEC and to an analogous regime in superfluids, where the characteristic length scale ε is very small. In a two-dimensional setting, vortices are essentially characterized as isolated zeroes of the order parameter carrying a winding number and, in the London limit, as point defects where energy concentration occurs. The questions of whether energy minimizers develop vortices, where they appear in the domain, and how many there should be (for given

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boundary conditions, constant applied fields, or angular velocities) has been analyzed in many contexts and parameter regimes.

In this paper, we focus on the Ginzburg-Landau energy arising, for instance, in the physical context of a rotating superfluid. Considering a bounded simply connected domain $\mathcal{D} \subset \mathbb{R}^2$, a smooth vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$, $\Omega > 0$, and $\varepsilon > 0$, we define the functional

$$u \in H^1(\mathcal{D};\mathbb{C}) \mapsto \mathcal{F}_\varepsilon(u) := \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 - \Omega\, V(x)\cdot j(u) \right\} dx.$$

Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we denote by $j(u) := u\wedge\nabla u \in L^1(\mathcal{D};\mathbb{R}^2)$ the *pre-Jacobian* of u. The L^1-vector field $j(u)$ may also be written as $j(u) = (iu, \nabla u)$, where $(\cdot,\cdot)$ is the standard inner product of two complex numbers viewed as vectors in $\mathbb{R}^2$. We choose this particular model because it is the simplest setting in which to analyze vortex concentration on arbitrary sets. Nevertheless, these same results may be extended to other physical models; see Remark 1.6 below.

Serfaty [19] studied minimizers of a closely related functional in the case of uniform rotation, that is, $V(x) = x^\perp = (-x_2, x_1)$ and $\mathcal{D}$ is a disk, to determine the critical value $\Omega_1 = \Omega_1(\varepsilon)$ of the angular speed Ω at which vortices first appear; see Remark 1.6 and for BECs, [11, 12]. She discovered that minimizers acquire vorticity at $\Omega_1 = k(\mathcal{D})|\ln\varepsilon| + O(\ln|\ln\varepsilon|)$ for an explicitly determined constant $k(\mathcal{D})$. In a series of papers culminating with the publication of the research monograph [17], Sandier and Serfaty developed powerful tools for studying vortices in Ginzburg-Landau models. Although they primarily worked with the full Ginzburg-Landau model with magnetic field, their methods apply to the functional $\mathcal{F}_\varepsilon$ above as well. In particular, their results apply to the near critical regime in *simply connected* domains.

In our setting, their results show that for any simply connected domain $\mathcal{D}$, the first order expansion of the critical value Ω_1 for vortex existence in minimizing configurations is also given by $k(\mathcal{D})|\ln\varepsilon|$ for some constant $k(\mathcal{D})$. Moreover, the locus of concentration of vortices for $\Omega = \Omega_1 + o(|\ln\varepsilon|)$ is given by the set of maxima of $|\zeta|$, where ζ the solution of the boundary-value problem

$$\begin{cases} -\Delta\zeta(x) = \operatorname{curl} V(x) & \text{if } x \in \mathcal{D}, \\ \zeta(x) = 0 & \text{if } x \in \partial\mathcal{D}. \end{cases} \tag{1.1}$$

The constant $k(\mathcal{D})$ is then determined by $k(\mathcal{D}) = 1/(2|\zeta|_{\max})$, where $|\zeta|_{\max}$ denotes the maximum value of $|\zeta|$. If, for instance, V is real-analytic and $\operatorname{curl} V$ is non-negative, then so is the solution ζ, and the maximum is generically attained at a

finite number of points in $\mathcal{D}$; see e.g. [8]. In this situation, if $\mathbf{\Omega} = \mathbf{\Omega}_1 + o(|\ln\varepsilon|)$, minimizers exhibit *vortex concentration at isolated points*, and the number of vortices remains uniformly bounded whenever $\mathbf{\Omega} - \mathbf{\Omega}_1$ is of order $O(\ln|\ln\varepsilon|)$; see [17, 19, 11, 12].

The case of a multiply connected domain provides a slightly different qualitative picture. In a work on rotating Bose-Einstein condensates, Aftalion, Alama, and Bronsard [1] first considered a similar functional in a domain given by a circular annulus $\mathcal{A}$ (centered at the origin) and then with uniform rotation $V(x) = x^{\perp}$; see Remark 1.6. Unlike in the simply connected case, minimizers in the annulus may have vorticity without vortices, as the hole acquires positive winding at bounded rotation $\mathbf{\Omega}$. Then point vortices are nucleated inside the interior of $\mathcal{A}$ at a critical value $\mathbf{\Omega}_1$, again of leading order $|\ln\varepsilon|$. Solving equation (1.1) in the annulus $\mathcal{A}$, one discovers that the set of maxima of the function ζ is given by a circle inside $\mathcal{A}$ (see Example 5.1). Hence, one can expect that in the limit as ε tends to zero, rather than accumulating at isolated points, vortices concentrate along this circle. The main feature proved in [1] is that if $\mathbf{\Omega} \sim \mathbf{\Omega}_1 + O(\ln|\ln\varepsilon|)$, then vortices are indeed essentially supported by a circle Σ and the total degree of these vortices is of order $\ln|\ln\varepsilon|$. In other words, in the limit $\varepsilon \to 0$, infinitely many vortices concentrate on Σ, a phenomenon that we call **vortex concentration along a curve**.

However, the question of the distribution of the limiting vorticity around the circle was left open. Subsequent results of Alama and Bronsard [2, 3] extended the result of [1] to multiply connected domains and to the full Ginzburg-Landau model with magnetic field and pinning potential. As opposed to in a simply connected domain, in a general multiply onnected domain, concentration on curves might not be a generic phenomenon for the Ginzburg-Landau model with magnetic field. Indeed, in the multiply connected domain setting. the vector field V represents the electromagnetic potential and is an unknown of the problem.

The results in of Alma and Bronsard [2, 3] show that for near critical external applied fields, the concentration set of vortices is also given by the set of maxima of a certain potential related to V. This set may contain finitely many points and/or closed loops. The authors prove, assuming that this set contains closed loops, that vortex concentration along a curve occurs. However, the determination of the limiting vorticity was again left open.

To separate the question of the nature of the concentration set from the question of localizing vortices effectively, we instead start with a simply connected domain $\mathcal{D}$ and prescribe a function $\zeta : \mathcal{D} \to \mathbb{R}$, $\zeta \geq 0$, and $\zeta|_{\partial\mathcal{D}} = 0$ in such a way that ζ

is maximized on a *prescribed curve* $\Sigma \subset\subset \mathcal{D}$. We then define the vector field

$$V(x) := -\nabla^{\perp}\zeta(x) = \left(\frac{\partial \zeta}{\partial x_2}, -\frac{\partial \zeta}{\partial x_1}\right).$$

We prove that vortices accumulate on Σ as ε tends to 0. We allow Σ to be either a smooth Jordan curve or a smooth embedded simple arc whose closure is compact and contained in $\mathcal{D}$. We resolve the problem of distribution of vortices along curves, both for minimizers and in the more general setting of Γ-convergence in this setting. While this arbitrary choice of concentration set Σ may seem unphysical, Ginzburg–Landau functionals of this form appear, in fact, naturally in thin film limits for superconductors in strong constant magnetic fields; see Remark 1.6 below, and [4].

In the last section. we show that in a multiply connected domain, and for more general vector fields V, the problem does not differ too much in nature, and that a similar analysis can be performed.

To state our main result, we must give more specific hypotheses on ζ and the angular speed $\mathbf{\Omega}$. We assume that ζ satisfies the following assumptions:

(H1) $\zeta \in C_0^{0,1}(\mathcal{D})$, $\zeta \geq 0$ in $\overline{\mathcal{D}}$, and $\zeta_{\max} := \max_{x\in\overline{\mathcal{D}}} \zeta(x) > 0$;

(H2) $\Sigma := \{x \in \mathcal{D} : \zeta(x) = \zeta_{\max}\} \subset\subset \mathcal{D}$ is a Jordan curve or a simple embedded arc of class C^2.

We further assume that $\mathbf{\Omega} = \mathbf{\Omega}(\varepsilon)$ is near to the critical value needed for the presence of vortices. More precisely,

$$\mathbf{\Omega}_\varepsilon := \frac{|\ln \varepsilon|}{2\zeta_{\max}} + \omega(\varepsilon) \tag{1.2}$$

for some $\omega : (0, +\infty) \to (0, +\infty)$ satisfying $\omega(\varepsilon) \to +\infty$, $|\ln \varepsilon|^{-1}\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For $u \in H^1(\mathcal{D}; \mathbb{C})$, we consider the rescaled functional

$$F_\varepsilon(u) := \frac{1}{\omega^2(\varepsilon)} \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 + \mathbf{\Omega}_\varepsilon \nabla^{\perp}\zeta \cdot j(u) \right\} dx,$$

and for a nonnegative Radon measure μ on $\mathcal{D}$, we define

$$I(\mu) := \frac{1}{2} \iint_{\mathcal{D}\times\mathcal{D}} G(x, y)\, d\mu(x)\, d\mu(y),$$

where G denotes the Dirichlet Green's function of the domain $\mathcal{D}$; i.e., for every $y \in \mathcal{D}$, $G(\cdot, y)$ is the solution of

$$\begin{cases} -\Delta G(x, y) = \delta_y(x) & \text{if } x \in \mathscr{D}'(\mathcal{D}), \\ G(x, y) = 0 & \text{if } x \in \partial\mathcal{D}. \end{cases} \tag{1.3}$$

Our main result concerns the Γ-convergence of the family of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$ and is stated (as usual) in terms of the vorticity distribution given by the weak Jacobian, that is, half the distributional curl of the pre-Jacobian; see e.g. [17].

Theorem 1.1. Assume (H1), (H2), and (1.2). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. Then,

(i) for any $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ satisfying $\sup_n F_{\varepsilon_n}(u_n) < +\infty$, there exist a subsequence (not relabelled) and a nonnegative Radon measure μ in $H^{-1}(\mathcal{D})$ supported by Σ such that

$$\frac{1}{\omega(\varepsilon_n)}\,\mathrm{curl}\; j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^*; \tag{1.4}$$

(ii) for any $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ such that (1.4) holds for some nonnegative Radon measure μ in $H^{-1}(\mathcal{D})$ supported by Σ,

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq I(\mu) - \zeta_{\max}\,\mu(\mathcal{D});$$

(iii) for any nonnegative Radon measure μ in $H^{-1}(\mathcal{D})$ supported by Σ, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ such that (1.4) holds and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\,\mu(\mathcal{D}).$$

As is well known, the Γ-convergence theory is well suited to the study of asymptotics in minimization problems; see e.g. [10]. In this context, we derive from Theorem 1.1 the following convergence result for the vorticity of global minimizers, and hence solve the problem of the limiting distribution of vortices along Σ; see Remark 1.3 below.

Corollary 1.1. Assume (H1), (H2) and (1.2). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. For each integer $n \in \mathbb{N}$, let $u_n \in H^1(\mathcal{D};\mathbb{C})$ be a minimizer of F_{ε_n}. Then

$$\frac{1}{\omega(\varepsilon_n)}\,\mathrm{curl}\; j(u_n) \underset{n\to+\infty}{\longrightarrow} \frac{\zeta_{\max}}{2I_*}\,\mu_* \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^*,$$

where μ_* is the unique minimizer of I amongst all probability measures supported on Σ. Moreover, $I_* := I(\mu_*)$.

Remark 1.1 (Renormalized energy)**.** As a direct application of the results in [17] (see also [13]), we show, in Section 2, that for configurations $\{u_\varepsilon\}$ with F_ε-energy uniformly bounded from above, the vorticity distribution curl $j(u_\varepsilon)$ can be approximated (with respect to the $(C_0^{0,1}(\mathcal{D}))^*$–topology) by a measure of the form

$2\pi \sum_{i\in I_\varepsilon} d_i \delta_{a_i}$ for some finite set of points $\{a_i\}_{i\in I_\varepsilon} \subset \mathcal{D}$ and integers $\{d_i\}_{i\in I_\varepsilon} \subset \mathbb{Z}$. In other words, each point a_i can be viewed as an approximate vortex with winding number d_i. Thus $D_\varepsilon = \sum_{i\in I_\varepsilon} |d_i|$ may be refered to as the **approximate total vorticity** of the configuration u_ε. It is well known that approximate vortices carry a kinetic energy essentially no less than $\pi D_\varepsilon |\ln \varepsilon|$; for more details, see Section 2. With such an estimate in hand and using the arguments of Section 3, we obtain a more refined lower bound for the energy than the one given by Theorem 1.1(ii). More precisely, we obtain

$$\liminf_{\varepsilon\to 0} \frac{1}{\omega^2(\varepsilon)} \left(\int_{\mathcal{D}} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\, dx - \pi D_\varepsilon |\ln \varepsilon| \right) \geq I(\mu)$$

and

$$\liminf_{\varepsilon\to 0} \frac{1}{\omega^2(\varepsilon)} \left(\Omega_\varepsilon \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_\varepsilon)\, dx + \pi D_\varepsilon |\ln \varepsilon| \right) \geq -\zeta_{\max}\, \mu(\mathcal{D}).$$

As a consequence, if $\{u_\varepsilon\}$ is any recovery sequence (in the sense of Theorem 1.1(iii)), then the lim inf's above become limits, and equality holds in each case. In analogy with [7], we see that $I(\mu)$ plays the role of renormalized energy.

Remark 1.2 (Minimizers)**.** From Corollary 1.1 and Remark 1.1, we deduce that if u_ε is energy minimizing, then

$$D_\varepsilon = \frac{\zeta_{\max}}{4\pi I_*}\, \omega(\varepsilon) + o(\omega(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

and, from Theorem 1.1, the minimal value of the energy expands as

$$\min_{H^1(\mathcal{D};\mathbb{C})} \omega^2(\varepsilon) F_\varepsilon = -\frac{\zeta_{\max}^2}{4I_*}\, \omega^2(\varepsilon) + o(\omega^2(\varepsilon)).$$

Remark 1.3 (Equilibrium measures)**.** The value $I(\mu)$ gives the electrostatic energy of a positive charge distribution μ on the set $\Sigma \subset\subset \mathcal{D}$. The minimizer μ_* of I over all probability measures on Σ is called the **Green equilibrium measure** in $\mathcal{D}$ associated to the set Σ. It gives the equilibrium charge distribution of a charged conductor inside of a neutral conducting shell represented by $\partial\mathcal{D}$. The value $1/I_*$ is referred to as the **capacity of the condenser** $(\Sigma, \partial\mathcal{D})$.

The interested reader can find many results on the existence and general (regularity) properties of equilibrium measures as well as some examples in [15]. For example, if $\mathcal{D}$ is a disk and Σ is a concentric circle, then the equilibrium measure μ_* is the normalized arclength measure on Σ (see [15, Example II.5.13]) and thus, as $\varepsilon \to 0$, vortices are asymptotically equidistributed along Σ. However, in general, for an arbitrary curve Σ, the distribution is, of course, nonuniform. In the case that Σ is an embedded arc, it is even singular at the endpoints; see [15, Example II.5.14].

Remark 1.4 (Regularity of Σ)**.** The structure and regularity assumptions on the set Σ given in (H2) are mainly motivated by the physical context of [1, 2, 3]. However, as we make clear later, (H2) can be relaxed. More precisely, the full force of (H2) is required only for the $\Gamma-\limsup$ inequality Theorem 1.1(iii). The recovery sequence (see Section 4) can be recovered with minor modifications if, for instance, the set Σ is a finite union of piecewise C^2 arcs/Jordan curves. In fact, Σ can even have a more general structure, such as a nonempty interior. In this case, we assume that $\partial\Sigma$ consists of finitely many arcs and Jordan curves of class C^2. Given a nonnegative Radon measure $\mu \in H^1(\mathcal{D})$ supported by Σ, we can construct a recovery sequence for μ by applying the approximation techniques of Section 4 to $\mu \llcorner \partial\Sigma$ and a more standard regularization procedure for $\mu \llcorner \mathrm{int}(\Sigma)$ as in [17].

Remark 1.5 (Multiply connected domains)**.** In Section 5, we show how to apply the method to a multiply connected domain and more general vector field V. For simplicity, we consider for domains $\mathcal{A}$ only topological annuli, i.e., $\mathcal{A} = \mathcal{D} \setminus \overline{\mathcal{B}}$, where $\mathcal{D}$ and $\mathcal{B}$ are simply connected and $\mathcal{B} \subset\subset \mathcal{D}$. For a multiply connected domain and a general field V, an extra step is required in the analysis. Indeed, the leading order term in the minimal energy (of order $|\ln\varepsilon|^2$) is caused by the curl-free part of V and the vorticity in the hole $\mathcal{B}$. The curl-free part of V induces a diverging phase in any minimal configuration; the vorticity in the hole $\mathcal{B}$ acts as a sort of giant vortex (by analogy to [1]). In other words, to get information on the internal vorticity, one has to perform a second order Γ-convergence analysis.

As in [2], we first describe minimizing vortexless configurations, i.e., energy minimizers over $\mathbb{S}^1$-valued maps, which nonetheless have vorticity around the hole. We then show that for arbitrary configurations, the energy due to the curl free part of V and the hole decouples nearly exactly; see Proposition 5.2. The residual energy functional resembles F_ε above, and the Γ-convergence analysis can be done in much the same way, with some care taken to control the residual vorticity around the hole.

The result is stated in Theorem 5.2. The Γ-limit again involves a Green energy for measures supported by a prescribed set.

Remark 1.6 (Related funtionals)**.** As mentioned earlier, the methods here can also be applied to other Ginzburg-Landau functionals that have the same structure as $\mathcal{F}_\varepsilon$. A simple variant is

$$\tilde{\mathcal{F}}_\varepsilon(u) = \int_{\mathcal{D}} \left\{ \frac{1}{2}|(\nabla - i\Omega_\varepsilon V)u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \right\} dx.$$

This energy has been studied in more than one context. Serfaty [19] considered the

minimization of this energy for rotating superfluids with $V = x^\perp$ under a Dirichlet boundary condition $u|_{\partial\mathcal{D}} = 0$. The the same energy under natural (Neumann) boundary conditions also appears in a simplified model of thin-film superconductors; see Chapman, Du, and Gunzburger [9]). In this setting, $\mathbf{\Omega}_\varepsilon V(x) = A_\varepsilon(x)$ represents an effective magnetic vector potential of the externally applied magnetic field $h_{ex} = \mathbf{\Omega}_\varepsilon \mathrm{curl}\, V$.

As observed by Alama, Bronsard, and Galvão-Sousa [4], the shape of a thin superconducting film modifies the effective normal magnetic field acting in the limiting two dimensional model, resulting in variable fields $h_{ex}(x_1, x_2)$ of the form we study here. Indeed, any ζ or Σ may be obtained in the limiting functional by varying the contour of the film surface appropriately.

If $\mathbf{\Omega}_\varepsilon$ satisfies (1.2), and the sequence $\{u_\varepsilon\}$ is in the energy regime of Theorem 1.1 (which holds for minimizers under a homogeneous Neumann boundary condition), then the two energies are related by

$$\tilde{\mathcal{F}}_\varepsilon(u_\varepsilon) = \mathcal{F}_\varepsilon(u_\varepsilon) + \frac{\mathbf{\Omega}_\varepsilon^2}{2} \int_{\mathcal{D}} |V|^2 \, dx + o(1) \quad \text{as } \varepsilon \to 0.$$

In the case of the homogeneous Dirichlet boundary condition, the result of Theorem 1.1 remains essentially the same; only some care must be taken since a singular boundary layer arises near $\partial\mathcal{D}$ as $\varepsilon \to 0$ (see [19]).

For annular domains, the following functional, which resembles $\mathcal{F}_\varepsilon$, has been used in modeling rotating BECs in certain anharmonic traps (see [1]):

$$\mathcal{G}_\varepsilon(u) = \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(a(x) - |u|^2)^2 - \mathbf{\Omega}_\varepsilon \, V(x) \cdot j(u) \right\} dx.$$

Here, $V(x) = x^\perp$ is the velocity field of uniform rotation, and the function $a(x)$ (positive in $\mathcal{A}$) gives the trapping potential which contains the condensate. It was shown in [1] that for sufficently high rotation $\mathbf{\Omega}_\varepsilon$, $\mathcal{G}_\varepsilon$-minimizers develop infinitely many vortices concentrated along a circle as $\varepsilon \to 0$. Although we believe that results similar to the ones in Section 5 hold, our method is not directly applicable to this functional. Indeed, our analysis is based on "global energy estimates" of [17] (see (2.1) in Proposition 2.1), and the presence of the inhomogeneity $a(x)$ requires local estimates. Moreover, our analysis of vortices for the energy $\mathcal{G}_\varepsilon$ is complicated by the fact that $a(x)$ vanishes on the boundary. Thus, some delicate estimates are required, so as not to lose too much information near the boundary.

One may also consider the full Ginzburg-Landau model of superconductivity for complex order parameter $u : \mathcal{D} \to \mathbb{C}$ and magnetic vector potential $A : \mathcal{D} \to \mathbb{R}^2$. For the Ginzburg-Landau model in a simply connected domain, with constant applied magnetic field, Sandier and Serfaty [17, Theorem 9.1] proved a

Γ-convergence theorem of the form of Theorem 1.1 for applied fields of the form $h_{ex} = H_{C1} + \omega(\varepsilon)$. Here, it is appropriate to take $\ln|\ln\varepsilon| \lesssim \omega(\varepsilon) \ll |\ln\varepsilon|$. This problem exhibits vortex concentration at points, and the limiting energy is obtained by rescaling around the points of concentration. The rescaled vorticity measures of minimizers converge to an equilibrium measure associated to a different problem in potential theory, a "Gauss variation" problem whereby charges subject to an applied electric field are to be placed optimally in $\mathbb{R}^2$; see [15].

Concentration on curves is possible in multiply connected regions (see [2, 3]). The methods described here should apply in this situation once adapted to the magnetic setting, although the energy obtained as a Γ-limit should be for the Helmholtz (and not the Laplace) Green's function, $-\Delta_x H(x,y) + H(x,y) = \delta_y(x)$. However, it is an interesting open problem in PDE to determine which (if any) nonsymmetric multiply connected domains exhibit vortex concentration on curves for the full Ginzburg-Landau functional with magnetic field in general domains.

Remark 1.7 (Related results)**.** The concentration of a divergent number of vortices on curves has been studied by numerous authors. Sandier and Soret [18] considered a sequence of $\mathbb{S}^1$-valued harmonic maps with Dirichlet boundary values having increasing degree. Under a monotonicity hypothesis on the boundary data, they proved that the associated renormalized energies converge to the electrostatic equilibrium measure supported on the boundary of the domain. Aydi [6] constructed (nonminimizing) critical points of the Ginzburg-Landau energy which concentrate along straight lines (in a periodic setting) or concentric circles (in the disk). Finally, we mention a recent paper by Kachmar [14], which exhibits concentration of vortices on a circle for a magnetic Ginzburg-Landau functional in the disk with an inhomogeneity $a(x)$ as above described by a radial step function (modeling, for instance, a superconducting body made up of different species). Here again, we believe that similar results should hold, but, because of the inhomogeneity of the Ginzburg-Landau energy density, our method does not apply directly.

The outline of the paper is as follows. In Section 2, we prove statement (i) of Theorem 1.1. Section 3 tackles statement (ii) of the theorem. The upper bound of statement (iii) is derived in Section 4, completing the proof of Theorem 1.1. The proof of Corollary 1.1 is presented at the end of Section 4. Section 5 sketches how the preceeding arguments must be modified to treat annular domains.

Notation. For any open set $B \subset \mathcal{D}$ and any admissible map u, we denote by

$$E_\varepsilon(u,B) := \int_B \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \right\} dx$$

the so called Ginzburg-Landau energy of u in B. Given $\varepsilon > 0$, we define

$$\mathcal{D}_\varepsilon := \{x \in \mathcal{D} : \ \mathrm{dist}\,(x, \partial\mathcal{D}) > \varepsilon\},$$

and, for a sequence $\varepsilon_n \to 0^+$, we write $\omega_n := \omega(\varepsilon_n)$ and $\Omega_n := \Omega_{\varepsilon_n}$.

2 Compactness of normalized weak Jacobians

This section is devoted to the proof of Theorem 1.1(i). The main tool used to prove compactness of normalized weak Jacobians is the so called "vortex balls construction" taken from [17, Theorem 4.1].

Proposition 2.1. There exists a constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $u \in H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$ satisfying $E_\varepsilon(u, \mathcal{D}) \le \sqrt{\varepsilon}$, the following holds. For any $C_0\varepsilon^{1/4} < r < 1$, there exists a finite collection of disjoint closed balls $\mathcal{B}_r = \{\overline{B}(a_i, \rho_i)\}_{i\in I_r}$ such that (writing $B_i := \overline{B}(a_i, \rho_i)$),

(i) $r = \sum_{i\in I_r} \rho_i$;

(ii) $\left\{|1 - |u|| \ge \varepsilon^{1/8}\right\} \cap \mathcal{D}_\varepsilon \subset \mathcal{D}_\varepsilon \cap \bigcup_{i\in I_r} B_i =: V$;

(iii) setting $d_i := \deg(u, \partial B_i)$ if $B_i \subset \mathcal{D}_\varepsilon$ and $d_i := 0$ otherwise, we have

$$E_\varepsilon(u, V) \ge \pi D_r \left(\ln\left(\frac{r}{\varepsilon D_r}\right) - C_1 \right), \tag{2.1}$$

where $D_r := \sum_{i\in I_r} |d_i|$ is assumed to be positive;

(iv) $D_r \le C_2 |\ln\varepsilon|^{-1} E_\varepsilon(u, \mathcal{D})$;

here C_0, C_1, C_2 are universal constants. Moreover, if $C_0\varepsilon^{1/4} < r_1 < r_2 < 1$ and $\mathcal{B}_1$, $\mathcal{B}_2$ are the corresponding families of balls, then every ball of $\mathcal{B}_1$ is included in a ball of $\mathcal{B}_2$.

In the remainder of this section, we consider an arbitrary sequence $\varepsilon_n \to 0^+$. In the following lemma, we prove an upper bound on the Ginzburg-Landau energy for a sequence having an F_{ε_n}-energy uniformly bounded from above. This allows us to apply Proposition 2.1.

Lemma 2.1. Assume (H1) and (1.2). Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathcal{D};\mathbb{C})$ such that $\sup_n F_{\varepsilon_n}(u_n) < +\infty$. Then there exists a constant C independent of n such that $\|u_n\|_{L^4(\mathcal{D})} \le C$ and $E_{\varepsilon_n}(u_n, \mathcal{D}) \le C|\ln\varepsilon_n|^2$.

Proof. Observe that

$$\begin{aligned} E_{\varepsilon_n}(u_n, \mathcal{D}) &= \omega_n^2 F_{\varepsilon_n}(u_n) - \Omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx \\ &\leq \Omega_n \|\nabla \zeta\|_\infty \int_{\mathcal{D}} |u_n||\nabla u_n| dx + O(\omega_n^2) \\ &\leq \frac{1}{4} \int_{\mathcal{D}} |\nabla u_n|^2 dx + \Omega_n^2 \|\nabla \zeta\|_\infty^2 \int_{\mathcal{D}} |u_n|^2 dx + O(\omega_n^2). \end{aligned} \tag{2.2}$$

In particular,

$$\int_{\mathcal{D}} \big(|u_n|^4 - 2(1 + 2\varepsilon_n^2 \Omega_n^2 \|\nabla \zeta\|_\infty^2)|u_n|^2 + 1\big) dx \leq O(\varepsilon_n^2 \omega_n^2),$$

so that $\|u_n\|_{L^4(\mathcal{D})} \leq C$ for a constant C independent of n. The result follows easily by inserting this estimate into (2.2). □

The first step in proving compactness of the normalized Jacobians is to show that the *approximate total vorticity* is bounded by the excess rotation ω_n. We emphasize that here Σ can be any compact subset of $\mathcal{D}$.

Proposition 2.2. Assume (H1) and (1.2). Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$ such that $\sup_n F_{\varepsilon_n}(u_n) < +\infty$. Then there exist $r_n \to 0^+$ and a sequence of families of balls $\mathcal{B}_{r_n} = \{\overline{B}(a_i^n, \rho_{i,n})\}_{i\in I_{r_n}}$, as in Proposition 2.1, such that $D_{r_n} \leq C\omega_n$ for some constant $C > 0$ independent of n.

Proof. Let $r_n := |\ln \varepsilon_n|^{-4}$. In view of Lemma 2.1, we can apply Proposition 2.1 to u_n with $r = r_n$ and n large enough. For each such n, we denote the corresponding family of balls by $\mathcal{B}_{r_n} = \{\overline{B}(a_i^n, \rho_{i,n})\}_{i\in I_{r_n}}$. For convenience, we write

$$\begin{aligned} I_n &:= I_{r_n}, \\ B_i^n &:= \overline{B}(a_i^n, \rho_{i,n}), \\ d_{i,n} &:= \begin{cases} \deg(u, \partial B_i^n) & \text{if } \mathrm{B_i^n} \subset \mathcal{D}_{\varepsilon_n}, \\ 0 & \text{otherwise,} \end{cases}, \quad \text{and} \\ D_n &:= D_{r_n}. \end{aligned}$$

Step 1. We first claim that

$$\Omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx = -2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\, \zeta(a_i^n) + o(1) \quad \text{as } n \to +\infty\,. \tag{2.3}$$

Indeed, applying [17, Theorem 6.1] with the balls $\{B_i^n\}_{i\in I_n}$, we obtain the estimate

$$\Big\| \operatorname{curl} j(u_n) - 2\pi \sum_{i\in I_n} d_{i,n}\, \delta_{a_i^n} \Big\|_{(C_0^{0,1}(\mathcal{D}))^*} \leq C r_n \big(1 + E_{\varepsilon_n}(u_n, \mathcal{D})\big) \leq C |\ln \varepsilon_n|^{-2}\,, \tag{2.4}$$

using Lemma 2.1. Then (2.3) follows, since $\Omega_n = O(|\ln \varepsilon_n|)$ and $\zeta \in C_0^{0,1}(\mathcal{D})$.

Step 2. Without loss of generality, we may assume that $\omega_n = O(D_n)$; otherwise there is nothing to prove. In view of statement (iv) of Proposition 2.1 and Lemma 2.1, we have $D_n \le O(|\ln \varepsilon_n|)$. Next, combining the lower bound (2.1) with (2.3), we infer that

$$\begin{aligned} O(\omega_n^2) \ge \omega_n^2 F_{\varepsilon_n}(u_n) \ge \pi D_n\big(|\ln \varepsilon_n| - C \ln|\ln \varepsilon_n|\big) - 2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) \\ + \int_{\mathcal{D}\setminus\cup_{i\in I_n} B_i^n} |\nabla u_n|^2\,dx + o(1) \quad \text{as } n \to +\infty. \end{aligned} \tag{2.5}$$

Let us now fix a sequence $\eta_n \to 0^+$ such that

$$\max\{\omega_n, \ln|\ln \varepsilon_n|\} = o(\eta_n |\ln \varepsilon_n|) \quad \text{as } n \to +\infty\,. \tag{2.6}$$

We group the vortex balls into classes and make the following definitions:

$$\begin{aligned} D_n^* &:= \sum_{i\in I_n^*} d_{i,n}, & I_n^* &:= \big\{i \in I_n \,:\, d_{i,n} \ge 0 \text{ and } \zeta(a_i^n) > \zeta_{\max} - \eta_n\big\};\\ D_n^+ &:= \sum_{i\in I_n^+} d_{i,n}, & I_n^+ &:= \big\{i \in I_n \,:\, d_{i,n} \ge 0 \text{ and } \zeta(a_i^n) \le \zeta_{\max} - \eta_n\big\};\\ D_n^- &:= \sum_{i\in I_n^-} |d_{i,n}|, & I_n^- &:= \big\{i \in I_n \,:\, d_{i,n} < 0\big\}. \end{aligned}$$

Observe that $D_n = D_n^* + D_n^+ + D_n^-$. We claim that

$$D_n^+ \le C\,\frac{\max\{\omega_n, \ln|\ln \varepsilon_n|\}D_n}{\eta_n|\ln \varepsilon_n|}\,, \tag{2.7}$$

$$D_n^- \le C\,\frac{\max\{\omega_n, \ln|\ln \varepsilon_n|\}D_n}{|\ln \varepsilon_n|}\,, \tag{2.8}$$

$$D_n \le C\max\{\omega_n, \ln|\ln \varepsilon_n|\}\,, \tag{2.9}$$

for a constant $C > 0$ independent of n. In particular, if $\omega_n^2 = o(|\ln \varepsilon_n|)$, then we can choose η_n to satisfy, in addition, $\max\{\omega_n^2, (\ln|\ln \varepsilon_n|)^2\} = o(\eta_n|\ln \varepsilon_n|)$; consequently, (2.9) yields $D_n^+ = D_n^- = 0$ for sufficiently large n.

We evaluate the lower bound for each class of vortex ball separately. First, we use the explicit form of Ω_n (see (1.2)) and the bound $\zeta(x) \le \zeta_{\max}$ to obtain

$$\pi D_n^*|\ln \varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^*} d_{i,n}\zeta(a_i^n) \ge -2\pi\omega_n\zeta_{\max}D_n^*\,. \tag{2.10}$$

For negative degrees, we have the simple estimate

$$\pi D_n^-|\ln \varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^-} d_{i,n}\zeta(a_i^n) \ge \pi D_n^-|\ln \varepsilon_n|\,. \tag{2.11}$$

Then for the vortex balls staying away from Σ, we have

$$\begin{aligned}
&\pi D_n^+ |\ln\varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^+} d_{i,n}\zeta(a_i^n) \\
&\qquad \geq \left(\pi|\ln\varepsilon_n| - 2\pi\Omega_n\zeta_{\max}\right) D_n^+ + 2\pi\Omega_n \sum_{i\in I_n^+} d_{i,n}(\zeta_{\max} - \zeta(a_i^n)) \\
&\qquad \geq D_n^+ \left(-2\pi\omega_n\zeta_{\max} + 2\pi\Omega_n\eta_n\right) \\
&\qquad \geq C\Omega_n\eta_n D_n^+,
\end{aligned} \tag{2.12}$$

since $\omega_n = o(\eta_n\Omega_n)$. We now insert (2.10), (2.11) and (2.12) into (2.5) to obtain

$$\begin{aligned}
O(\omega_n^2) \geq\ & \pi D_n \left(|\ln\varepsilon_n| - C_1 \ln|\ln\varepsilon_n|\right) - 2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) + \\
& \int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx + o(1) \\
\geq\ & -\pi C_1 D_n \ln|\ln\varepsilon_n| - 2\pi\omega_n\zeta_{\max} D_n^* + \pi D_n^- |\ln\varepsilon_n| + C\eta_n\Omega_n D_n^+ \\
& + \int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx.
\end{aligned} \tag{2.13}$$

Rearranging terms, we obtain

$$\begin{aligned}
D_n^- |\ln\varepsilon_n| + D_n^+\eta_n\Omega_n &\leq C\left(\omega_n D_n^* + \ln|\ln\varepsilon_n| D_n + \omega_n^2\right) \\
&\leq C\max\{\omega_n, \ln|\ln\varepsilon_n|\} D_n,
\end{aligned}$$

which proves (2.7) and (2.8).

To prove (2.9), we argue as in [2, pg. 58–60] to obtain a constant $C' > 0$ (independent of n) such that

$$\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx \geq C' D_n^2 . \tag{2.14}$$

Accepting (2.14) for the moment, we return to the lower bound (2.13) to deduce

$$D_n^2 - C\left(\omega_n + \ln|\ln\varepsilon_n|\right) D_n \leq O(\omega_n^2),$$

so that $D_n \leq C\max\{\omega_n, \ln|\ln\varepsilon_n|\}$ and (2.9) is established.

It remains to show (2.14). To this end, we identify an annular band lying outside of Σ and use the fact that the total degree is approximately constant in that band. Since the boundary $\partial\mathcal{D}$ is assumed to be smooth, there exists $0 < \delta_0 < \operatorname{dist}(\Sigma, \partial\mathcal{D})/2$ such that $\varrho(x) := \operatorname{dist}(x, \partial\mathcal{D})$ is smooth in $\mathcal{D}_{\varepsilon_n}^{\delta_0}$,

$$\mathcal{D}_{\varepsilon_n}^{\delta_0} := \{x \in \mathcal{D} \,:\, \varepsilon_n < \varrho(x) \leq \delta_0\} .$$

The level sets

$$\mathcal{C}_t := \{x \in \mathcal{D} : \ \varrho(x) = t\}$$

are smooth and diffeomorphic to $\partial\mathcal{D}$ for all $t \in [0, \delta_0]$. Define the set $T_n \subset [0, \delta_0]$ by

$$T_n := \left\{t \in (\varepsilon_n, \delta_0] \ : \ \mathcal{C}_t \cap \cup_{i\in I_n} B_i^n = \varnothing\right\}.$$

By the choice of r_n and statement (i) of Proposition 2.1, T_n is a finite union of disjoint intervals and $\mathcal{L}^1((\varepsilon_n, \delta_0] \setminus T_n) \le O(|\ln \varepsilon_n|^{-4})$. From statement (ii) of Proposition 2.1, we can define the degree of u_n on $\mathcal{C}_t$ for every $t \in T_n$ by

$$D_n(t) := \deg\left(\frac{u_n}{|u_n|}, \mathcal{C}_t\right) = \frac{1}{2\pi}\int_{\mathcal{C}_t} \frac{u_n}{|u_n|} \wedge \nabla_\tau\left(\frac{u_n}{|u_n|}\right) d\mathcal{H}^1,$$

where ∇_τ denotes the tangential derivative along $\mathcal{C}_t$ oriented counterclockwise. Setting $I_n(t) := \{i \in I_n \ : \ \varrho(a_i^n) \ge \rho_{i,n} + t\}$ ($\rho_{i,n}$ being the radius of the ball B_i^n), we have

$$D_n(t) = \sum_{i\in I_n(t)} d_{i,n}.$$

Using (2.7) and (2.8), we infer that for sufficiently large n,

$$D_n(t) \ge D_n - 2D_n^- - D_n^+ \ge \frac{1}{2}D_n \quad \text{for every } t \in T_n.$$

Let $v_n := u_n/|u_n|$. In view of statement (ii) of Proposition 2.1, using the coarea formula and Jensen inequality, we obtain, for sufficiently large n,

$$\begin{aligned}\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx &\ge \int_{\mathcal{D}_{\varepsilon_n}^{\delta_0}\setminus\cup B_i^n} |u_n|^2|\nabla v_n|^2 dx \ge \frac{1}{2}\int_{\varepsilon_n}^{\delta_0}\left(\int_{\mathcal{C}_t} |\nabla v_n|^2 d\mathcal{H}^1\right)dt \\ &\ge \frac{1}{2}\int_{T_n}\left(\int_{\mathcal{C}_t} |v_n \wedge \nabla_\tau v_n|^2 d\mathcal{H}^1\right)dt \ge 2\pi^2 \int_{T_n} \frac{|D_n(t)|^2}{\mathcal{H}^1(\mathcal{C}_t)}\, dt, \\ &\ge C' D_n^2,\end{aligned}$$

which completes the proof of (2.14).

Step 3. If $\ln|\ln\varepsilon_n| \le o(\omega_n)$, the conclusion follows from (2.7), (2.8) and (2.9). If $\omega_n \le O(\ln|\ln\varepsilon_n|)$, we must refine our lower bound by enlarging the vortex balls. First, observe that in this regime, (2.7) and (2.8) ensure that

$$D_n^- = D_n^+ = 0 \tag{2.15}$$

for large n, so that each ball $B_i^n \subset \mathcal{D}_{\varepsilon_n}$ carries a nonnegative degree, and $D_n = D_n^*$.

We choose a new radius $s_n := e^{-\sqrt{\omega_n}}$; thus, for n sufficiently large, $s_n > r_n$. We now reapply Proposition 2.1 with $r = s_n$ to obtain a new family of larger balls

$\{\tilde{B}_j^n\}_{j\in J_n}$, each new ball $\tilde{B}_j^n$ containing one or more of the smaller balls $\{B_i^n\}$. By statement (ii) of Proposition 2.1 and (2.15), we have $D_{s_n} = D_n$. Using the lower bound (2.1) together with (2.3) and (2.15), we argue as in Step 2 to derive

$$\begin{aligned} O(\omega_n^2) \geq \omega_n^2 F_{\varepsilon_n}(u_n) &\geq \pi D_n \left(\ln\left(\frac{s_n}{\varepsilon_n D_n}\right) - C_1 \right) - 2\pi\Omega_n\zeta_{\max} D_n + C' D_n^2 \\ &\geq \pi D_n \left(\ln\left(\frac{s_n}{D_n}\right) - C_1 \right) - 2\pi\omega_n\zeta_{\max} D_n + C' D_n^2. \end{aligned} \tag{2.16}$$

Next, we distinguish two cases. First assume that $\ln D_n \leq O(\omega_n)$. In this case, (2.16) yields the inequality $D_n^2 - C\omega_n D_n \leq O(\omega_n^2)$ (with $C > 0$ independent of n) so that $D_n \leq O(\omega_n)$, as claimed. If $\omega_n = o(\ln D_n)$, we obtain the bound $D_n^2 - CD_n \ln D_n \leq O(\omega_n^2)$, which also yields $D_n \leq O(\omega_n)$. □

We are now ready to prove statement (i) of Theorem 1.1.

Theorem 2.1. Assume (H1) and (1.2). Let $\varepsilon_n \to 0^+$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathcal{D};\mathbb{C})$ such that $\sup_n F_{\varepsilon_n}(u_n) < +\infty$. Then there exist a subsequence (not relabelled) and a nonnegative Radon measure μ supported by Σ such that

$$\mu_n := \frac{1}{\omega_n} \operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } \left(C_0^{0,1}(\mathcal{D})\right)^*.$$

Proof. Step 1. We first assume that $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$. Using the notation of the previous proof, we apply Proposition 2.2 to obtain the family of vortex balls $\{B_i^n\}_{i\in I_n}$. Define the measure

$$\bar{\mu}_n := \frac{2\pi}{\omega_n} \sum_{i\in I_n} d_{i,n} \delta_{a_i^n}.$$

Since $D_n \leq O(\omega_n)$, we have $|\bar{\mu}_n|(\mathcal{D}) \leq C$ for a constant C independent of n. Therefore, up to a subsequence, $\bar{\mu}_n \rightharpoonup \mu$ weakly* as $n \to +\infty$ in the sense of measures on $\mathcal{D}$ for some finite Radon measure μ.

We claim that μ is nonnegative and supported by Σ. Indeed, decompose $\bar{\mu}_n$ by its Hahn decomposition; i.e., write $\bar{\mu}_n = \bar{\mu}_n^+ - \bar{\mu}_n^-$, where $\bar{\mu}_n^+$ and $-\bar{\mu}_n^-$ are the positive and the negative parts of $\bar{\mu}_n$, respectively. We then have

$$\bar{\mu}_n^-(\mathcal{D}) = \frac{2\pi D_n^-}{\omega_n} \to 0 \quad \text{as } n \to +\infty,$$

because of (2.8). The nonnegativity of μ follows.

Now consider the sequence of sets $\mathcal{V}_n := \{\zeta(x) \leq \zeta_{\max} - \eta_n\}$, where η_n is given by (2.6). In view of (2.7), we have

$$\bar{\mu}_n^+(\mathcal{V}_n) = \frac{2\pi D_n^+}{\omega_n} \to 0,$$

which clearly implies that $\operatorname{supp}\mu \subset \Sigma$.

By the compact embedding $(C_0^0(\mathcal{D}))^* \hookrightarrow (C_0^{0,1}(\mathcal{D}))^*$, we deduce that $\bar{\mu}_n \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$. On the other hand, (2.4) yields

$$\left\|\mu_n - \bar{\mu}_n\right\|_{(C_0^{01}(\mathcal{D}))^*} \underset{n\to+\infty}{\longrightarrow} 0.$$

The claim follows.

Step 2. We now consider the general case. In view of the strong continuity of the functional F_{ε_n} under strong H^1-convergence, we can find a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$ such that for every n,

$$\|u_n - \tilde{u}_n\|_{H^1(\mathcal{D})} \le \varepsilon_n \tag{2.17}$$

and

$$F_{\varepsilon_n}(\tilde{u}_n) \le F_{\varepsilon_n}(u_n) + 1, \tag{2.18}$$

so that $\sup_n F_{\varepsilon_n}(\tilde{u}_n) < +\infty$.

Given an arbitrary $\varphi \in C_0^{0,1}(\mathcal{D})$ satisfying $|\nabla\varphi| \le 1$, we obtain

$$\begin{aligned}\left|\int_{\mathcal{D}} \big(j(u_n) - j(\tilde{u}_n)\big)\cdot\nabla^\perp\varphi\,dx\right| \\ \le \|u_n - \tilde{u}_n\|_{L^2(\mathcal{D})}\|\nabla u_n\|_{L^2(\mathcal{D})} + \|\tilde{u}_n\|_{L^2(\mathcal{D})}\|\nabla(u_n - \tilde{u}_n)\|_{L^2(\mathcal{D})} \le C\varepsilon_n|\ln\varepsilon_n|\,,\end{aligned}$$

using (2.17) and (2.18) together with Lemma 2.1. As a consequence, setting $\tilde{\mu}_n := \omega_n^{-1}\operatorname{curl} j(\tilde{u}_n)$, we have

$$\left\|\mu_n - \tilde{\mu}_n\right\|_{(C_0^{01}(\mathcal{D}))^*} \underset{n\to+\infty}{\longrightarrow} 0\,. \tag{2.19}$$

Applying Step 1 to $\{\tilde{u}_n\}$, we have $\tilde{\mu}_n \to \mu$ in $(C_0^{0,1}(\mathcal{D}))^*$, up to a subsequence, for some nonnegative Radon measure μ supported by Σ. Then (2.19) yields $\mu_n \to \mu$ in $(C_0^{0,1}(\mathcal{D}))^*$. □

3 The Lower bound inequality

This section is devoted to the proof of statement (ii) of Theorem 1.1. We summarize it in the following result.

Theorem 3.1. Assume (H1) and (1.2). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence and $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ be such that

$$\mu_n = \frac{1}{\omega_n}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^* \tag{3.1}$$

for some nonnegative Radon measure μ supported by Σ. Then

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq I(\mu) - \zeta_{\max}\,\mu(\mathcal{D}). \tag{3.2}$$

In particular, if the left hand side of (3.2) is finite, then $\mu \in H^{-1}(\mathcal{D})$.

Proof. We continue to use the notation of the previous section. Without loss of generality, we assume that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) = \lim_{n\to+\infty} F_{\varepsilon_n}(u_n) < +\infty. \tag{3.3}$$

Moreover, by Step 2 in the proof of Theorem 2.1, we may also assume that $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$. We distinguish two cases.

Case 1: $\ln|\ln\varepsilon_n| \leq o(\omega_n)$. In this case, we consider the family of vortex balls $\{B_i^n\}_{i\in I_n}$ constructed in the proof of Proposition 2.2, and refer to it for notation. Arguing as in (2.3), we obtain

$$\begin{aligned}\Omega_n \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\,dx &= -\frac{\pi|\ln\varepsilon_n|}{\zeta_{\max}} \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) + \omega_n \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\,dx + o(1)\\ &\geq -\pi D_n|\ln\varepsilon_n| - \omega_n^2\langle\mu_n,\zeta\rangle + o(1) \quad \text{as } n\to\infty,\end{aligned} \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing $(C_0^{0,1})^* - C_0^{0,1}$.

Combining the lower bound (2.1) with (3.4), we infer that

$$\begin{aligned}\omega_n^2 F_{\varepsilon_n}(u_n) &\geq \pi D_n\left(\ln\left(\frac{r_n}{\varepsilon_n D_n}\right) - C_1\right) + \frac{1}{2}\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx + \Omega_n \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n) dx\\ &\geq \pi D_n\left(\ln\left(\frac{r_n}{D_n}\right) - C_1\right) + \frac{1}{2}\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx - \omega_n^2\langle\mu_n,\zeta\rangle + o(1).\end{aligned}$$

Since $r_n = |\ln\varepsilon_n|^{-4}$ and $D_n \leq O(\omega_n)$ by Proposition 2.2, dividing the previous inequality by ω_n^2 yields

$$\begin{aligned}F_{\varepsilon_n}(u_n) &\geq \frac{1}{2\omega_n^2}\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx - \langle\mu_n,\zeta\rangle + o(1)\\ &\geq \frac{1}{2\omega_n^2}\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx - \zeta_{\max}\mu(\mathcal{D}) + o(1).\end{aligned}$$

In the last inequality, we have used (3.1) and the fact that μ is supported by Σ. In view of statement (ii) of Proposition 2.1, we obtain the estimate

$$\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx \geq \frac{1}{1+\varepsilon_n^{1/4}} \int_{\mathcal{D}_{\varepsilon_n}\setminus\cup B_i^n} |j(u_n)|^2 dx.$$

Next, we define

$$\tilde{j}_n(x) := \begin{cases} \omega_n^{-1} j(u_n(x)) & \text{if } x \in \mathcal{D}_{\varepsilon_n} \setminus \bigcup_{i\in I_n} B_i^n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5}$$

so that by (3.3),

$$O(1) \geq F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |\tilde{j}_n(x)|^2 dx - \zeta_{\max}\mu(\mathcal{D}) + o(1). \tag{3.6}$$

Hence there exist a subsequence $\varepsilon_n \to 0$ (not relabelled) and $j_* \in L^2(\mathcal{D};\mathbb{R}^2)$ such that $\tilde{j}_n \rightharpoonup j_*$ weakly in $L^2(\mathcal{D};\mathbb{R}^2)$ as $n \to +\infty$. By lower semicontinuity, we have

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |j_*|^2 dx - \zeta_{\max}\mu(\mathcal{D}). \tag{3.7}$$

It remains to tie the limit j_* to the limit μ of the normalized weak Jacobians. To this end, fix $\varphi \in \mathscr{D}(\mathcal{D})$. Using Lemma 2.1, statement (i) of Proposition 2.1, and Hölder's inequality, we obtain

$$\begin{aligned} \left| \int_{\cup B_i^n} \nabla^\perp\varphi \cdot j(u_n)\, dx \right| &\leq \left(\mathcal{L}^2\big(\cup_{i\in I_n} B_i^n\big)\right)^{1/4} \|\nabla\varphi\|_\infty \|u_n\|_{L^4(\mathcal{D})} \|\nabla u_n\|_{L^2(\mathcal{D})} \\ &\leq C\, r_n^{1/2}\, |\ln\varepsilon_n| = o(1)\,. \end{aligned}$$

Since $\operatorname{supp}\varphi \subset \mathcal{D}_{\varepsilon_n}$ for sufficiently large n, we deduce that

$$\begin{aligned} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot j_*\, dx &= \lim_{n\to+\infty} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot \tilde{j}_n\, dx = \lim_{n\to+\infty} \omega_n^{-1} \int_{\mathcal{D}_{\varepsilon_n}\setminus\cup B_i^n} \nabla^\perp\varphi \cdot j(u_n)\, dx \\ &= \lim_{n\to+\infty} \omega_n^{-1} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot j(u_n)\, dx = -\lim_{n\to+\infty} \langle \mu_n, \varphi\rangle = -\int_{\mathcal{D}} \varphi\, d\mu\,. \end{aligned}$$

Consequently,

$$\operatorname{curl} j_* = \mu, \quad \text{in } \mathscr{D}'(\mathcal{D}). \tag{3.8}$$

In particular, $\mu \in H^{-1}(\mathcal{D})$ since $j_* \in L^2(\mathcal{D};\mathbb{R}^2)$.

Next, we introduce the unique solution $h_\mu \in H_0^1(\mathcal{D})$ of

$$\begin{cases} -\Delta h_\mu(x) = \mu(x) & \text{if } x \in H^{-1}(\mathcal{D}) \\ h_\mu(x) = 0 & \text{if } x \in \partial\mathcal{D}. \end{cases}$$

In view of (3.8) and the definition of h_μ, we have

$$\operatorname{curl}\big(j_* + \nabla^\perp h_\mu\big) = 0 \quad \text{in } H^{-1}(\mathcal{D}),$$

so that we can find $g_\mu \in H^1(\mathcal{D})$ satisfying $\nabla g_\mu = j_* + \nabla^\perp h_\mu$. Therefore,

$$\int_{\mathcal{D}} |j_*|^2 dx = \int_{\mathcal{D}} |\nabla h_\mu|^2 dx + \int_{\mathcal{D}} |\nabla g_\mu|^2 dx,$$

since an integration by parts yields $\int_{\mathcal{D}} \nabla^\perp h_\mu \cdot \nabla g_\mu = 0$ (using the fact that h_μ is constant on $\partial\mathcal{D}$). Returning to (3.7), we infer that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |\nabla h_\mu|^2 dx - \zeta_{\max}\mu(\mathcal{D}).$$

On the other hand, using the Green representation of h_μ, we have

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla h_\mu|^2 \, dx = \frac{1}{2}\iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, d\mu(x)\, d\mu(y) = I(\mu),$$

proving the theorem in this case.

Case 2: $\omega_n \leq O(\ln|\ln\varepsilon_n|)$. Consider the family of vortex balls $\{\tilde{B}_j^n\}_{j\in J_n}$ (of size $s_n = e^{-\sqrt{\omega_n}}$) constructed in Step 3 of the proof of Proposition 2.2. Recall that for large n, this family satisfies $D_{s_n} = D_n$. Combining the lower bound (2.1) for the family $\{\tilde{B}_j^n\}_{j\in J_n}$ with (3.4), we obtain

$$\omega_n^2 F_{\varepsilon_n}(u_n) \geq \pi D_n\left(\ln\left(\frac{s_n}{D_n}\right) - C_1\right) + \frac{1}{2}\int_{\mathcal{D}\setminus\cup\tilde{B}_j^n} |\nabla u_n|^2 dx - \omega_n^2\langle\mu_n,\zeta\rangle + o(1).$$

Arguing then as in (3.6), we infer that

$$F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |\hat{j}_n(x)|^2 dx - \zeta_{\max}\mu(\mathcal{D}) + o(1),$$

where

$$\hat{j}_n(x) := \begin{cases} \omega_n^{-1} j(u_n(x)) & \text{if } x \in \mathcal{D}_{\varepsilon_n} \setminus \bigcup_{j\in J_n} \tilde{B}_j^n, \\ 0 & \text{otherwise.} \end{cases}$$

As previously, up to a subsequence, we have $\hat{j}_n \rightharpoonup j_*$ weakly in $L^2(\mathcal{D};\mathbb{R}^2)$, and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |j_*(x)|^2 dx - \zeta_{\max}\mu(\mathcal{D}).$$

It now remains to show that curl $j_* = \mu$ in $\mathscr{D}'(\mathcal{D})$; then the proof can be completed as in Step 1.

We proceed as before, taking arbitrary $\varphi \in \mathscr{D}(\mathcal{D})$ and using the weak formulation of the Jacobians. The key observation is that the contribution of the vortex balls is negligible, provided we restrict our choice of test functions φ to functions *constant* in each vortex ball. Such a set of test functions exists because of [17,

Proposition 9.6]; i.e., given an arbitrary $\varphi \in \mathscr{D}(\mathcal{D})$, there exists a modified function $\tilde{\varphi}_n$ that is constant on each ball B_j^n and such that

$$\|\varphi - \tilde{\varphi}_n\|_{C^{0,\alpha}(\mathcal{D})} \leq C s_n^{1-\alpha}, \qquad \|\nabla\varphi - \nabla\tilde{\varphi}_n\|_{L^1(\mathcal{D})} \leq C s_n \tag{3.9}$$

for each $0 \leq \alpha \leq 1$. Moreover, for sufficiently large n, $\tilde{\varphi}_n$ has compact support in $\mathcal{D}$. From (3.9) and the L^2-boundedness of the normalized currents $\hat{j}_n$, we derive

$$\left| \int_{\mathcal{D}} (\nabla^\perp\varphi - \nabla^\perp\tilde{\varphi}_n) \cdot \hat{j}_n \, dx \right| \leq \|\hat{j}_n\|_{L^2(\mathcal{D})} \|\nabla\varphi - \nabla\tilde{\varphi}_n\|_{L^2(\mathcal{D})} \leq C s_n^{1/2}. \tag{3.10}$$

Then (3.9), (3.10), and the strong convergence of μ_n to μ yield

$$\begin{aligned} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot j_* \, dx &= \lim_{n\to+\infty} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot \hat{j}_n \, dx = \lim_{n\to+\infty} \int_{\mathcal{D}} \nabla^\perp\tilde{\varphi}_n \cdot \hat{j}_n \, dx \\ &= \lim_{n\to+\infty} \omega_n^{-1} \int_{\mathcal{D}} \nabla^\perp\tilde{\varphi}_n \cdot j(u_n) \, dx \\ &= \lim_{n\to+\infty} \left(- \int_{\mathcal{D}} \tilde{\varphi}_n d\mu + \langle \mu - \mu_n, \tilde{\varphi}_n \rangle \right) = - \int_{\mathcal{D}} \varphi \, d\mu. \end{aligned}$$

□

.

4 The upper bound inequality

Throughout this section, we use the following notation. For a nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ compactly supported in $\mathcal{D}$, we denote by $h_\mu \in H_0^1(\mathcal{D})$ the solution of

$$\begin{cases} -\Delta h_\mu(x) = \mu(x) & \text{if } x \in H^{-1}(\mathcal{D}), \\ h_\mu(x) = 0 & \text{if } x \in \partial\mathcal{D}. \end{cases} \tag{4.1}$$

Using the Green representation $h_\mu(x) = \int_{\mathcal{D}} G(x,y)\, d\mu(y)$, we have

$$I(\mu) = \frac{1}{2} \int_{\mathcal{D}} |\nabla h_\mu|^2 \, dx,$$

and the following elementary lemma holds.

Lemma 4.1. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative Radon measures in $H^{-1}(\mathcal{D})$ with compact support in $\mathcal{D}$. Assume that as $n \to +\infty$, $\mu_n \rightharpoonup \mu$ weak* as measures on $\mathcal{D}$ for some $\mu \in H^{-1}(\mathcal{D})$ compactly supported in $\mathcal{D}$. Then $I(\mu_n) \to I(\mu)$ if and only if $h_{\mu_n} \to h_\mu$ strongly in $H^1(\mathcal{D})$ as $n \to +\infty$.

With Lemma 4.1 in hand, can now start the proof of statement (iii) of Theorem 1.1 in the case that the measure μ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \Sigma$.

Proposition 4.1. Assume (H1), (H2), and (1.2). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. For every nonnegative Radon measure μ of the form

$$\mu = f(x)\,\mathcal{H}^1 \llcorner \Sigma, \tag{4.2}$$

$f \in L^\infty(\Sigma)$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ such that

$$\frac{1}{\omega_n}\,\mathrm{curl}\; j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^*$$

and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\mu(\mathcal{D}).$$

Proof. Without loss of generality, we may assume that $f \not\equiv 0$, since the case $\mu = 0$ is easily seen to be true. It is well known that a measure of the form (4.2) belongs to $H^{-1}(\mathcal{D})$; see e.g. [20, Theorem 4.7.5].

We recall that the Dirichlet Green's function G in $\mathcal{D}$ defined by (1.3) satisfies

(i) $G(x,y) \geq 0$ for every $x \in \overline{\mathcal{D}} \setminus \{y\}$ and $y \in \mathcal{D}$, and

(ii) for any compact set $K \subset\subset \mathcal{D}$, there exists a constant C_K such that

$$\left| G(x,y) + \frac{1}{2\pi}\ln|x-y| \right| \leq C_K \tag{4.3}$$

for all $y \in K$ and $x \in \overline{\mathcal{D}}$.

In the three first steps below, we assume that the density function f does not vanish on Σ, i.e., $f \geq \delta\ \mathcal{H}^1$ a.e. on Σ for some constant $\delta > 0$. The general case is considered in Step 4.

Step 1. In the spirit of [16], we construct a trial function using the Green's function $G(x,y)$. Let $\gamma : [0,\ell] \to \Sigma$ be an arclength parametrization of the curve Σ (so that $\ell = \mathcal{H}^1(\Sigma)$), and define for $t \in [0,\ell]$,

$$M(t) := \mu\big(\gamma([0,t])\big).$$

Then $M(\cdot)$ is strictly increasing, $M(0) = 0$, and $M(\ell) = \mu(\Sigma) = \mu(\mathcal{D})$. Moreover, $M(\cdot)$ is continuous since $\mu \in H^{-1}(\mathcal{D})$, and thus atomless.

Next, introduce for sufficiently large n,

$$D_n := \left\lfloor \frac{\omega_n \mu(\Sigma)}{2\pi} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ denotes the floor function. Since M is continuous and increasing, we can define for $k = 0, \ldots, D_n$,

$$t_{k,n} := M^{-1}\big(2\pi k \omega_n^{-1}\big).$$

Now, for $k = 0, \dots, D_n$, we set

$$a_k^n := \gamma(t_{k,n})$$

and claim that

$$C\omega_n^{-1} \le |a_k^n - a_{k-1}^n| \le 2\pi\delta^{-1}\omega_n^{-1} \tag{4.4}$$

for each $k \in \{1, \dots, D_n\}$ and for some constant $C > 0$ independent of n. For $k = 1, \dots, D_n$, write $\Sigma_k := \gamma([t_{k-1,n}, t_{k,n}])$, so that Σ_k is a smooth curve. Denote its endpoints by a_k^n and a_{k-1}^n. By construction,

$$2\pi\omega_n^{-1} = M(t_{k,n}) - M(t_{k-1,n}) = \mu\big(\Sigma_k\big) \ge \delta\mathcal{H}^1\big(\Sigma_k\big) \ge \delta|a_k^n - a_{k-1}^n|.$$

Now, since Σ is smooth and, for large n, $|a_k^n - a_{k-1}^n|$ is small, we conclude that

$$2\pi\omega_n^{-1} = \mu\big(\Sigma_k\big) \le \|f\|_{L^\infty(\Sigma)}\mathcal{H}^1\big(\Sigma_k\big) \le C\|f\|_{L^\infty(\Sigma)}|a_k^n - a_{k-1}^n|.$$

Define the family of measures

$$\bar\mu_n := \frac{2\pi}{\omega_n}\sum_{k=1}^{D_n}\delta_{a_k^n}.$$

We claim that $\bar\mu_n \rightharpoonup \mu$ weak* as measures on $\mathcal{D}$ and $\bar\mu_n(\mathcal{D}) \to \mu(\mathcal{D})$. To prove the weak* convergence of $\bar\mu_n$, we fix an arbitrary function $\psi \in C_0^0(\mathcal{D})$. Observe that by (4.4) and the smoothness of Σ, we have $\operatorname{diam}(\Sigma_k) \le C_0\omega_n^{-1}$ for a constant C_0 independent of k and n. Therefore, using $\mu(\Sigma_k) = 2\pi\omega_n^{-1}$, we obtain

$$\begin{aligned}\left|\int_{\mathcal{D}}\psi\,d\mu - \int_{\mathcal{D}}\psi\,d\bar\mu_n\right| &= \left|\sum_{k=1}^{D_n}\int_{\Sigma_k}\big(\psi(x) - \psi(a_k^n)\big)d\mu\right| + o(1)\\ &\le C\mu(\mathcal{D})\operatorname{osc}(\psi, C_0\omega_n^{-1}) + o(1),\end{aligned}$$

where

$$\operatorname{osc}(\psi, C_0\omega_n^{-1}) := \sup_{|x-y|\le C_0\omega_n^{-1}}|\psi(x) - \psi(y)| \underset{n\to+\infty}{\longrightarrow} 0,$$

proving the claim.

Next we regularize the measure $\bar\mu_n$. Let us define for $k = 1, \dots, D_n$,

$$f_{k,n}(x) := \frac{1}{\pi\varepsilon_n^2}\,\chi_{B_{\varepsilon_n}(a_k^n)},$$

where $\chi_{B_{\varepsilon_n}(a_k^n)}$ denotes the characteristic function of the ball $B_{\varepsilon_n}(a_k^n)$. Set

$$f_n(x) := \frac{2\pi}{\omega_n}\sum_{k=1}^{D_n} f_{k,n}(x) \quad\text{and}\quad \hat\mu_n := f_n(x)\mathcal{L}^2\llcorner\mathcal{D}. \tag{4.5}$$

Since $\varepsilon_n = o(\omega_n^{-1})$, the functions $f_{k,n}$ have disjoint supports for n large, by (4.4). As a consequence, $\hat{\mu}_n(\mathcal{D}) = \bar{\mu}_n(\mathcal{D}) \to \mu(\mathcal{D})$. Since $\bar{\mu}_n \rightharpoonup \mu$, one may also easily verify that $\hat{\mu}_n \rightharpoonup \mu$ weak* as measures on $\mathcal{D}$.

Step 2. We introduce

$$h_n := h_{\omega_n \hat{\mu}_n} = \omega_n h_{\hat{\mu}_n}.$$

Then, according to (4.1),

$$\int_{\mathcal{D}} |\nabla h_n|^2 dx = \omega_n^2 \iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, f_n(x) f_n(y)\, dxdy \tag{4.6}$$

$$= 4\pi^2 \sum_{i,j=1}^{D_n} \iint_{\mathcal{D}\times\mathcal{D}} G(x,y) f_{i,n}(x) f_{j,n}(y)\, dxdy. \tag{4.7}$$

We need to estimate the integral term in the right hand side of (4.6). We proceed as in [1], providing some details for the reader's convenience. Let $\mathcal{N}_0 \subset\subset \mathcal{D}$ be a small tubular neighborhood of Σ. Let $0 < \alpha < 1$ be given (small), and set $\Delta_\alpha := \{(x,y) \in \mathcal{D}\times\mathcal{D} : |x-y| < \alpha\}$. Since G is continuous on $(\mathcal{N}_0 \times \mathcal{N}_0) \setminus \Delta_\alpha$, and for n large, the support of $\hat{\mu}_n$ lies in $\mathcal{N}_0$, we have, by the weak* convergence of $\hat{\mu}_n$ to μ,

$$\begin{aligned} I_\alpha &:= \lim_{n\to+\infty} \iint_{(\mathcal{D}\times\mathcal{D})\setminus\Delta_\alpha} G(x,y)\, f_n(x) f_n(y) dxdy \\ &= \iint_{(\mathcal{N}_0\times\mathcal{N}_0)\setminus\Delta_\alpha} G(x,y) d\mu(x) d\mu(y) \le \iint_{\mathcal{D}\times\mathcal{D}} G(x,y) d\mu(x) d\mu(y) \\ &= 2I(\mu). \end{aligned} \tag{4.8}$$

Near the diagonal Δ_α, we split the sum into (4.7) in two terms. Using (4.3), we obtain the estimate

$$\begin{aligned} II_\alpha^n &:= 4\pi^2 \sum_{i=1}^{D_n} \iint_{\Delta_\alpha} G(x,y) f_{i,n}(x) f_{i,n}(y) dxdy \\ &= 4 \sum_{i=1}^{D_n} \iint_{B_1(0)\times B_1(0)} G\left(a_i^n + \varepsilon_n z_1, a_i^n + \varepsilon_n z_2\right) dz_1 dz_2 \\ &\le 4 \sum_{i=1}^{D_n} \iint_{B_1(0)\times B_1(0)} \left(\frac{1}{2\pi} \ln\left(\frac{1}{\varepsilon_n |z_1 - z_2|} \right) + C \right) dz_1 dz_2 \\ &\le 2\pi D_n |\ln \varepsilon_n| + O(\omega_n) \end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
III^n_\alpha &:= 4\pi^2 \sum_{0<|a^n_i-a^n_j|<\alpha} \iint_{\Delta_\alpha} G(x,y)\, f_{i,n}(x)\, f_{j,n}(y) dxdy \\
&\leq C \sum_{0<|a^n_i-a^n_j|<\alpha} \left| \ln |a^n_i - a^n_j| \right| .
\end{aligned} \tag{4.10}
$$

By the smoothness of Σ and (4.4), there exists a constant $c_0 > 0$ independent of n such that for every $i \neq j$ and every $(x, y) \in \Sigma_i \times \Sigma_j$,

$$|a^n_i - a^n_j| \geq c_0 |x - y|.$$

Since $\mu \in H^{-1}(\mathcal{D})$ is supported by Σ, the map $(x, y) \mapsto \left| \ln(c_0|x - y|) \right|$ belongs to $L^1(\Sigma \times \Sigma, \mu \otimes \mu)$. Therefore, by the Mean Value Theorem, for every $i \neq j$, we can find a pair $(x^n_i, y^n_j) \in \Sigma_i \times \Sigma_j$ such that

$$\frac{\omega_n^2}{4\pi^2} \iint_{\Sigma_i \times \Sigma_j} \left| \ln(c_0|x - y|) \right| d\mu(y) d\mu(x) = \left| \ln(c_0|x^n_i - y^n_j|) \right|,$$

noticing that $4\pi^2 \omega_n^{-2} = \mu(\Sigma_i)\mu(\Sigma_j)$. Applying the previous inequality to (x^n_i, y^n_j), we deduce from (4.10) that for n large,

$$
\begin{aligned}
III^n_\alpha &\leq C \sum_{0<|a^n_i-a^n_j|<\alpha} \left| \ln(c_0|x^n_i - y^n_j|) \right| \\
&\leq C\omega_n^2 \sum_{0<|a^n_i-a^n_j|<\alpha} \iint_{\Sigma_i \times \Sigma_j} \left| \ln(c_0|x - y|) \right| d\mu(y) d\mu(x) \\
&\leq C\omega_n^2 \iint_{(\Sigma\times\Sigma)\cap\Delta_{2\alpha}} \left| \ln(c_0|x - y|) \right| d\mu(y) d\mu(x).
\end{aligned} \tag{4.11}
$$

On the other hand, $\mu \in H^{-1}(\mathcal{D})$, so it is atomless. Thus $\mu \otimes \mu$ does not charge $\{x = y\} \cap \mathcal{D} \times \mathcal{D}$. Consequently, the integral term in the right hand side of (4.11) vanishes as $\alpha \to 0^+$, and

$$\lim_{\alpha\to 0^+} \limsup_{n\to+\infty} \omega_n^{-2} III^n_\alpha = 0. \tag{4.12}$$

Gathering (4.8), (4.9), and (4.12) yields

$$
\begin{aligned}
&\limsup_{n\to+\infty} \left(\iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, f_n(x) f_n(y)\, dxdy - 2\pi D_n |\ln \varepsilon_n|\, \omega_n^{-2} \right) \\
&\quad \leq \limsup_{\alpha\to 0^+} \left(I_\alpha + \limsup_{n\to+\infty} \left((II^n_\alpha - 2\pi D_n |\ln \varepsilon_n|)\omega_n^{-2} + \omega_n^{-2} III^n_\alpha \right) \right) \leq 2I(\mu)\,,
\end{aligned}
$$

and in view of (4.6), we conclude that

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla h_n|^2 dx \le \pi D_n |\ln \varepsilon_n| + \omega_n^2 I(\mu) + o(\omega_n^2). \tag{4.13}$$

Step 3. We now define a complex-valued order parameter u_n associated to h_n. We proceed as follows. Since

$$\operatorname{curl}\left(-\nabla^\perp h_n\right) = -\Delta h_n = \omega_n f_n \tag{4.14}$$

is supported by $\bigcup_k \overline{B}_{\varepsilon_n}(a_k^n)$, we may define a phase ϕ_n locally in $\mathcal{D}\setminus\bigcup_k \overline{B}_{\varepsilon_n}(a_k^n)$ by

$$\nabla\phi_n(x) = -\nabla^\perp h_n(x) \quad \text{for } x \in \mathcal{D}\setminus\bigcup_k \overline{B}_{\varepsilon_n}(a_k^n).$$

In fact, since the balls $B_{\varepsilon_n}(a_k^n)$ are pairwise disjoint (assuming n is sufficiently large) and the mass of $\omega_n f_n$ is quantized in each such ball, it is easy to show that ϕ_n is single valued modulo 2π; i.e., for any smooth Jordan curve Θ inside $\mathcal{D}\setminus\bigcup_k \overline{B}_{\varepsilon_n}(a_k^n)$,

$$\frac{1}{2\pi}\int_\Theta \nabla\phi_n\cdot\tau \in \mathbb{Z},$$

where $\tau:\Theta\to\mathbb{S}^1$ is any smooth vector field tangent to Θ. Hence $\exp(i\phi_n(x))$ is well defined for every $x\in\mathcal{D}\setminus\bigcup_k \overline{B}_{\varepsilon_n}(a_k^n)$.

Consider next a smooth cutoff function $\rho:\mathbb{R}\to[0,1]$ such that $\rho(t)\equiv 0$ for $t\le 1$ and $\rho(t)\equiv 1$ for $t\ge 2$. Define

$$\rho_n(x) := \begin{cases} \rho\left(\dfrac{|x-a_k^n|}{\varepsilon_n}\right) & \text{if } x\in B_{2\varepsilon_n}(a_k^n) \text{ for some } k=1,\dots,D_n,\\ 1 & \text{otherwise,}\end{cases} \tag{4.15}$$

and observe that

$$E_{\varepsilon_n}(\rho_n,\mathcal{D}) = O(\omega_n). \tag{4.16}$$

Then set

$$u_n(x) := \begin{cases} \rho_n(x)e^{i\phi_n(x)} & \text{if } x\in\mathcal{D}\setminus\bigcup_k \overline{B}_{\varepsilon_n}(a_k^n),\\ 0 & \text{otherwise.}\end{cases}$$

One can easily check that $u_n\in H^1(\mathcal{D};\mathbb{C})$. We claim that

$$\mu_n = \omega_n^{-1}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^*. \tag{4.17}$$

A simple computation gives

$$j(u_n) = \rho_n^2\nabla\phi_n = -\rho_n^2\nabla^\perp h_n \quad \text{a.e. in } \mathcal{D}.$$

Given $\varphi \in C_0^{0,1}(\mathcal{D})$ satisfying $|\nabla\varphi| \le 1$, we deduce from (4.14) and the previous indentity that

$$\begin{aligned}\langle \mu_n, \varphi\rangle &= \frac{1}{\omega_n}\int_{\mathcal{D}} \nabla\varphi\cdot\nabla h_n dx + \frac{1}{\omega_n}\int_{\mathcal{D}} (\rho_n^2-1)\nabla\varphi\cdot\nabla h_n dx\\ &= \langle \hat\mu_n, \varphi\rangle + \frac{1}{\omega_n}\int_{\cup_k B_{2\varepsilon_n}(a_k^n)} (\rho_n^2-1)\nabla\varphi\cdot\nabla h_n dx.\end{aligned} \tag{4.18}$$

In view of the compact embedding $(C_0^0(\mathcal{D}))^* \hookrightarrow (C_0^{0,1}(\mathcal{D}))^*$, $\hat\mu_n \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$. Hence we can estimate using (4.13) to obtain

$$\begin{aligned}|\langle \mu_n-\mu,\varphi\rangle| &\le \|\hat\mu_n-\mu\|_{(C_0^{0,1}(\mathcal{D}))^*} + \omega_n^{-1}\big(\mathcal{L}^2(\cup_k B_{2\varepsilon_n}(a_k^n))\big)^{1/2}\|\nabla h_n\|_{L^2(\mathcal{D})}\\ &\le \|\hat\mu_n-\mu\|_{(C_0^{0,1}(\mathcal{D}))^*} + C\varepsilon_n|\ln\varepsilon_n|^{1/2} \underset{n\to+\infty}{\longrightarrow} 0,\end{aligned} \tag{4.19}$$

proving (4.17).

We now compute the energy $F_{\varepsilon_n}(u_n)$. From (4.13) and (4.16), we infer that

$$\begin{aligned}E_{\varepsilon_n}(u_n,\mathcal{D}) &= E_{\varepsilon_n}(\rho_n,\mathcal{D}) + \frac{1}{2}\int_{\mathcal{D}\setminus\cup_k\overline{B}_{\varepsilon_n}(a_k^n)} \rho_n^2|\nabla\phi_n|^2 dx\\ &\le \frac{1}{2}\int_{\mathcal{D}} |\nabla h_n|^2 + O(\omega_n) \le \pi D_n|\ln\varepsilon_n| + \omega_n^2 I(\mu) + o(\omega_n^2).\end{aligned} \tag{4.20}$$

It remains to evaluate the interaction with the rotation potential. First, (4.17) yields

$$\begin{aligned}\Omega_n\int_{\mathcal{D}}\nabla^\perp\zeta\cdot j(u_n)\,dx &= \frac{|\ln\varepsilon_n|}{2\zeta_{\max}}\int_{\mathcal{D}}\nabla^\perp\zeta\cdot j(u_n)\,dx - \omega_n^2\langle\mu_n,\zeta\rangle\\ &= \frac{|\ln\varepsilon_n|}{2\zeta_{\max}}\int_{\mathcal{D}}\nabla^\perp\zeta\cdot j(u_n)\,dx - \zeta_{\max}\mu(\mathcal{D})\omega_n^2 + o(\omega_n^2)\,.\end{aligned}$$

Arguing as in (4.18)-(4.19), we obtain

$$\begin{aligned}\int_{\mathcal{D}}\nabla^\perp\zeta\cdot j(u_n)\,dx &= -\omega_n\langle\hat\mu_n,\zeta\rangle + O(\varepsilon_n\omega_n|\ln\varepsilon_n|^{1/2})\\ &= -2\pi\sum_{k=1}^{D_n}\frac{1}{\pi\varepsilon_n^2}\int_{B_{\varepsilon_n}(a_k^n)}\zeta(x)dx + o(\varepsilon_n|\ln\varepsilon_n|^{3/2})\\ &= -2\pi D_n\zeta_{\max} + o(\varepsilon_n|\ln\varepsilon_n|^{3/2})\end{aligned}$$

and, consequently,

$$\Omega_n\int_{\mathcal{D}}\nabla^\perp\zeta\cdot j(u_n)\,dx = -\pi D_n|\ln\varepsilon_n| - \zeta_{\max}\mu(\mathcal{D})\omega_n^2 + o(\omega_n^2)\,. \tag{4.21}$$

Combining (4.20) with (4.21) leads finally to

$$F_{\varepsilon_n}(u_n) \le I(\mu) - \zeta_{\max}\mu(\mathcal{D}) + o(1).$$

In view of Theorem 3.1, the proposition follows by taking the lim sup as $n \to +\infty$ in the previous inequality.

Step 4. We now consider the case that the density f is allowed to vanish. Let $\{\delta_k\} \subset \mathbb{R}$ be a sequence decreasing to 0 as $k \to +\infty$. Consider, for $k \in \mathbb{N}$, the measure

$$\mu_k := \mu + \delta_k \mathcal{H}^1 \llcorner \Sigma = \big(f(x) + \delta_k\big)\mathcal{H}^1 \llcorner \Sigma.$$

By the Monotone Convergence Theorem,

$$I(\mu_k) \underset{k\to+\infty}{\longrightarrow} I(\mu) \qquad \text{and} \qquad \mu_k(\mathcal{D}) \underset{k\to+\infty}{\longrightarrow} \mu(\mathcal{D}). \tag{4.22}$$

Obviously, μ_k also converges to μ strongly in $(C_0^{0,1}(\mathcal{D}))^*$. Applying Step 1 to Step 3, we find, for every $k \in \mathbb{N}$, a sequence $\{v_n^k\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ such that $\omega_n^{-1}\mathrm{curl}\, j(v_n^k) \to \mu_k$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$ and $F_{\varepsilon_n}(v_n^k) \to I(\mu_k) - \zeta_{\max}\mu_k(\mathcal{D})$ as $n \to +\infty$. Hence for every $k \in \mathbb{N}$, we can find $N_k \in \mathbb{N}$ such that for every $n \geq N_k$,

$$\|\omega_n^{-1}\mathrm{curl}\, j(v_n^k) - \mu\|_{(C_0^{0,1}(\mathcal{D}))^*} \leq 2^{-k} \quad \text{and} \quad \big|F_{\varepsilon_n}(v_n^k) - I(\mu) + \zeta_{\max}\mu(\mathcal{D})\big| \leq 2^{-k}.$$

Moreover, we can assume, without loss of generality, that the sequence of integers $\{N_k\}_{k\in\mathbb{N}}$ is strictly increasing. Therefore, for sufficiently large integral n, there exists a unique $k_n \in \mathbb{N}$ such that $N_{k_n} \leq n < N_{k_n+1}$, and $k_n \to +\infty$ as $n \to +\infty$. We may then define $u_n := v_n^{k_n}$. By construction, the sequence $\{u_n\}$ satisfies the required properties. □

To consider the case of a general measure μ in $H^{-1}(\mathcal{D})$, we need the following continuity lemma.

Lemma 4.2. Let μ be a nonnegative Radon measure in $H^{-1}(\mathcal{D})$ such that $\mathrm{supp}\,\mu \subset\subset \mathcal{D}$. For $\xi \in \mathbb{R}^2$, let $\tau_\xi\mu$ be the translated measure defined by

$$\tau_\xi\mu(B) = \mu(-\xi + B) \quad \text{for every Borel set } B \subset \mathbb{R}^2.$$

Then there exists $0 < \delta < \mathrm{dist}\,(\mathrm{supp}\,\mu, \partial\mathcal{D})$ such that $\tau_\xi\mu \in H^{-1}(\mathcal{D})$ for every $\xi \in B_\delta(0)$, and the mapping $\xi \mapsto h_{\tau_\xi\mu} \in H^1(\mathcal{D})$ is strongly continuous on $B_\delta(0)$.

Proof. For $\delta > 0$, set $\widetilde{\mathcal{D}}_\delta := \{x \in \mathbb{R}^2, \mathrm{dist}\,(x, \mathcal{D}) < 2\delta\}$. Then choose $\delta > 0$ such that $\partial\widetilde{\mathcal{D}}_\delta$ is smooth and $2\delta < \mathrm{dist}\,(\mathrm{supp}\,\mu, \partial\mathcal{D})$. For every $\xi \in B_\delta(0)$, $\mathcal{D} \subset\subset \xi + \widetilde{\mathcal{D}}_\delta$, $\mathrm{supp}\,\tau_\xi\mu = \xi + \mathrm{supp}\,\mu \subset \mathcal{D}$, and $\mathrm{dist}\,(\xi + \mathrm{supp}\,\mu, \partial\mathcal{D}) > \delta$.

Obviously, $\mu \in H^{-1}(\widetilde{\mathcal{D}}_\delta)$, and we can define $\bar{h} \in H^1(\widetilde{\mathcal{D}}_\delta)$ to be the unique solution of

$$\begin{cases} -\Delta\bar{h}(x) = \mu(x) & \text{if } x \in \widetilde{\mathcal{D}}_\delta, \\ \bar{h}(x) = 0 & \text{if } x \in \widetilde{\mathcal{D}}_\delta. \end{cases}$$

By our choice of δ, the function $\bar{h}$ is smooth in the δ-neighborhood of $\partial\mathcal{D}$. Next, for $\xi \in B_\delta(0)$, we denote by $\bar{h}_\xi \in H^1(\mathcal{D})$ the function $\bar{h}_\xi$ defined on $\mathcal{D}$ by $\bar{h}_\xi := \bar{h}(x-\xi)$. Observe that $\bar{h}_\xi \in H^1(\mathcal{D})$ and $-\Delta\bar{h}_\xi = \tau_\xi\mu$ in $\mathcal{D}$. Hence $\tau_\xi\mu \in H^{-1}(\mathcal{D})$.

Now consider a sequence $\{\xi_n\} \subset B_\delta(0)$ such that $\xi_n \to \xi \in B_\delta(0)$ as $n \to +\infty$. Set $h_n := h_{\tau_{\xi_n}\mu}$. We have $\Delta(\bar{h}_{\xi_n} - h_n) = 0$ in $\mathcal{D}$ and $(\bar{h}_{\xi_n} - h_n) = \bar{h}(x-\xi_n)$ on $\partial\mathcal{D}$. By standard elliptic estimates, $(\bar{h}_{\xi_n} - h_n)$ converges strongly in $H^1(\mathcal{D})$ to the harmonic function in $\mathcal{D}$ equal to $\bar{h}(x-\xi)$ on $\partial\mathcal{D}$, that is, $\bar{h}_\xi - h_{\tau_\xi\mu}$. On the other hand, $\bar{h}_{\xi_n} \to \bar{h}_\xi$ strongly in $H^1(\mathcal{D})$ by strong continuity of translations in H^1. Therefore, $h_n \to h_{\tau_\xi}$ strongly in $H^1(\mathcal{D})$. □

Theorem 4.1. Assume (H1), (H2), and (1.2). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. For every nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ supported by Σ, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{R}^2)$ such that

$$\frac{1}{\omega_n}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^*$$

and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\mu(\mathcal{D}).$$

Proof. We prove that for any nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ supported by Σ, there exists a sequence of nonnegative Radon measures $\{\mu_k\}_{k\in\mathbb{N}}$ of the form (4.2) such that $\mu_k \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$, $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$, and

$$I(\mu_k) \to I(\mu) \quad \text{as } k \to +\infty. \tag{4.23}$$

Assuming that such a sequence exists, Proposition 4.1 yields, for each k, a sequence $\{v_n^k\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{R}^2)$ such that as $n \to +\infty$, $\omega_n^{-1}\operatorname{curl} j(v_n^k) \to \mu_k$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$ and $F_{\varepsilon_n}(v_n^k) \to I(\mu_k) - \zeta_{\max}\mu_k(\mathcal{D})$. It then suffices to apply the diagonal argument used in the proof of Proposition 4.1, Step 4, to construct the required sequence.

Case 1: Σ is a segment in $\mathcal{D}$. In this case, without loss of generality, we may assume that $\Sigma = [a,b]\times\{0\} \subset\subset \mathcal{D}$ for some $a, b \in \mathbb{R}$, $a < b$. We assume, in addition, that $\Sigma' := \operatorname{supp}\mu \subset\subset]a,b[\times\{0\}$. We regularize the measure μ by the standard procedure. Consider a smooth function $\varrho \in C^\infty(\mathbb{R})$ such that $\varrho \geq 0$, $\operatorname{supp}\varrho \subset [-1,1]$, and $\int_{\mathbb{R}}\varrho = 1$. For a positive integer k and $x = (x_1,x_2) \in \mathbb{R}^2$, we introduce $\varrho_k(x) := k\varrho(kx_1)$ and define

$$g_k(x) := \int_{\Sigma'} \varrho_k(x-y)\,d\mu(y).$$

By construction, each g_k is nonnegative, smooth and, for k sufficiently large, supported by $[a, b] \times \mathbb{R}$. Next we define, for large k, the measure

$$\mu_k := g_k(x)\, \mathcal{H}^1 \llcorner \Sigma.$$

It is easy to verify that $\mu_k \in H^{-1}(\mathcal{D})$, $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$, and that as $k \to +\infty$, $\mu_k \rightharpoonup \mu$ on $\mathcal{D}$ weak* in the sense of measures. In particular, $\mu_k \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$.

We claim that (4.23) holds. Indeed, using Fubini's theorem, we first obtain

$$\begin{aligned} I(\mu_k) &= \frac{1}{2} \iint_{\Sigma\times\Sigma} G(z, z')\, d\mu_k(z) d\mu_k(z') \\ &= \frac{1}{2} \iint_{\Sigma'\times\Sigma'} \left(\iint_{(\Sigma\cap B_{\frac{1}{k}}(x))\times(\Sigma\cap B_{\frac{1}{k}}(y))} G(z, z')\varrho_k(z-x)\varrho_k(z'-y)\, d\mathcal{H}^1_z d\mathcal{H}^1_{z'} \right) d\mu(x) d\mu(y)\,. \end{aligned}$$

Next, we observe that for sufficiently large k, $\Sigma \cap B_{\frac{1}{k}}(x) = (x_1, 0) + J_k$ with $J_k := (\frac{-1}{k}, \frac{1}{k}) \times \{0\}$ for every $x = (x_1, x_2) \in \Sigma'$. Changing variables in (z, z') and using Fubini's theorem again, we obtain

$$\begin{aligned} I(\mu_k) &= \frac{1}{2} \iint_{\Sigma'\times\Sigma'} \left(\iint_{J_k\times J_k} G(x+\xi, y+\xi')\varrho_k(\xi)\varrho_k(\xi')\, d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'} \right) d\mu(x) d\mu(y) \\ &= \frac{1}{2} \iint_{J_k\times J_k} \left(\iint_{\Sigma'\times\Sigma'} G(x+\xi, y+\xi')\, d\mu(x) d\mu(y) \right) \varrho_k(\xi)\varrho_k(\xi') d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'} \\ &= \frac{1}{2} \iint_{J_k\times J_k} \left(\iint_{\mathcal{D}\times\mathcal{D}} G(x, y)\, d(\tau_\xi\mu)(x) d(\tau_{\xi'}\mu)(y) \right) \varrho_k(\xi)\varrho_k(\xi') d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'}. \end{aligned}$$

From the Green representation of $h_{\tau_\xi\mu}$, we infer that for every $(\xi, \xi') \in J_k \times J_k$,

$$\iint_{\mathcal{D}\times\mathcal{D}} G(x, y)\, d(\tau_\xi\mu)(x) d(\tau_{\xi'}\mu)(y) = \int_{\mathcal{D}} \left(\nabla h_{\tau_\xi\mu}\right) \cdot \left(\nabla h_{\tau_{\xi'}\mu}\right) dx.$$

Then from Lemma 4.2, we deduce that the function Θ defined by

$$\Theta : (\xi, \xi') \mapsto \int_{\mathcal{D}} \left(\nabla h_{\tau_\xi\mu}\right) \cdot \left(\nabla h_{\tau_{\xi'}\mu}\right) dx$$

is continuous on $B_\delta(0) \times B_\delta(0)$ for some $0 < \delta < \operatorname{dist}(\Sigma, \partial\mathcal{D})$. Therefore,

$$\lim_{k\to+\infty} I(\mu_k) = \lim_{k\to+\infty} \frac{1}{2} \iint_{J_k\times J_k} \Theta(\xi, \xi')\varrho_k(\xi)\varrho_k(\xi')\, d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'} = \frac{1}{2}\Theta(0, 0) = I(\mu),$$

proving (4.23).

Case 2: Σ is a smooth embedded arc. In this case, we further assume that there exists a C^1-diffeomorphism $\Phi : \mathcal{D} \to \mathcal{D}$ such that $\Phi(x) = x$ in a neighborhood of $\partial\mathcal{D}$ and $\bar{\Sigma} := \Phi(\Sigma)$ is a segment compactly included in $\mathcal{D}$. Let μ be a nonnegative

Radon measure in $H^{-1}(\mathcal{D})$ whose support is compactly included in the relative interior of Σ. Denote by $\bar\mu$ the push-forward of μ by Φ, i.e., $\bar\mu := \Phi_\# \mu$. Then $\operatorname{supp}\bar\mu$ is compactly included in the relative interior of $\bar\Sigma$ and $\bar\mu \in H^{-1}(\mathcal{D})$. Indeed, we easily verify that

$$\begin{aligned} I(\bar\mu) &= \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} G(x,y)\, d\bar\mu(x) d\bar\mu(y) \\ &= \frac{1}{2}\iint_{\Sigma\times\Sigma} G\big(\Phi(x),\Phi(y)\big)\, d\mu(x) d\mu(y) \\ &\le \frac{1}{4\pi}\iint_{\Sigma\times\Sigma} \ln\big|\Phi(x)-\Phi(y)\big|\, d\mu(x) d\mu(y) + C \\ &\le \frac{1}{4\pi}\iint_{\Sigma\times\Sigma} \ln|x-y|\, d\mu(x) d\mu(y) + C < +\infty\,, \end{aligned}$$

where we have used (4.3). The constant C depends only on Σ, $\mu(\Sigma)$, and $\|\nabla\Phi\|_{L^\infty(\mathcal{D})}$.

Therefore, we can apply Case 1 to $\bar\mu$ and find a sequence of measures $\{\bar\mu_k\}_{k\in\mathbb{N}}$ of the form (4.2) such that $\operatorname{supp}\bar\mu_k \subset \bar\Sigma$ and, as $k\to+\infty$, $\bar\mu_k(\mathcal{D}) \to \bar\mu(\mathcal{D})$, $\bar\mu_k \rightharpoonup \bar\mu$ weak* as measures on $\mathcal{D}$, and $I(\bar\mu_k) \to I(\bar\mu)$. For each integer k, we then set $\mu_k := (\Phi^{-1})_\#\bar\mu_k$. Observe that μ_k is of the form (4.2). To see this, set $\bar\mu_k = \bar g_k(x)\,\mathcal{H}^1 \llcorner \bar\Sigma$, where $\bar g_k \in C^0(\bar\Sigma)$. The area formula (see e.g. [5]) then yields

$$\mu_k = \bar g_k \circ \Phi(x) |\nabla_\tau \Phi(x)|\, \mathcal{H}^1 \llcorner \Sigma,$$

where $\nabla_\tau\Phi$ denotes the tangential gradient of Φ along Σ. Then one may check that $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$ and $\mu_k \rightharpoonup \mu$ weak* as measures on $\mathcal{D}$ as $k\to+\infty$.

We claim that (4.23) holds. First, write

$$\begin{aligned} I(\mu_k) &= \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} G\big(\Phi^{-1}(x),\Phi^{-1}(y)\big)\, d\bar\mu_k(x) d\bar\mu_k(y) \\ &= \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} S\big(\Phi^{-1}(x),\Phi^{-1}(y)\big)\, d\bar\mu_k(x) d\bar\mu_k(y) \\ &\qquad + \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma} \ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right) d\bar\mu_k(x) d\bar\mu_k(y) \\ &=: I_k + II_k, \end{aligned} \tag{4.24}$$

where S denotes the *regular part* of the Green function G, i.e.,

$$S(x,y) := G(x,y) + \frac{1}{2\pi}\ln|x-y|$$

(which is a locally smooth function on $\mathcal{D}\times\mathcal{D}$). Since $\bar\mu_k$ converges weak* as measures to $\bar\mu$, we have

$$\bar\mu_k \otimes \bar\mu_k \rightharpoonup \bar\mu\otimes\bar\mu \quad \text{weak* as measures on } \mathcal{D}\times\mathcal{D}, \tag{4.25}$$

and we deduce that

$$\begin{aligned}\lim_{k\to+\infty} I_k &= \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} S\big(\Phi^{-1}(x),\Phi^{-1}(y)\big)\,d\bar\mu(x)d\bar\mu(y)\\ &= \frac{1}{2}\iint_{\Sigma\times\Sigma} S(x,y)\,d\mu(x)d\mu(y).\end{aligned}\tag{4.26}$$

Next, we consider a decreasing sequence $\alpha_n \to 0$. For each integer n, we introduce a smooth cutoff function $\chi_n \in C^\infty(\overline{\mathcal{D}}\times\overline{\mathcal{D}})$ such that $0 \le \chi_n \le 1$, $\chi_n(x,y) = 0$ if $|\Phi^{-1}(x)-\Phi^{-1}(y)| \ge \alpha_n$, and $\chi_n(x,y) = 1$ if $|\Phi^{-1}(x)-\Phi^{-1}(y)| \le \alpha_{n+1}$. Observe that since $\bar\mu \in H^{-1}(\mathcal{D})$, the measure $\bar\mu$ has no atom, and hence $\bar\mu\otimes\bar\mu$ does not charge the diagonal $\{x=y\}\cap\mathcal{D}\times\mathcal{D}$. Consequently, $\chi_n \to 0$ $\bar\mu\otimes\bar\mu$ a.e. in $\mathcal{D}\times\mathcal{D}$. We then write

$$\begin{aligned}II_k &= \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right)d\bar\mu_k(x)d\bar\mu_k(y)\\ &\quad+\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}(1-\chi_n(x,y))\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right)d\bar\mu_k(x)d\bar\mu_k(y)\\ &:= III_k^n + IV_k^n.\end{aligned}\tag{4.27}$$

By (4.25) and the choice of χ_n, we have for every n,

$$\lim_{k\to+\infty} IV_k^n = \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}(1-\chi_n(x,y))\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right)d\bar\mu(x)d\bar\mu(y).$$

Next, observe that for every $(x,y)\in\mathcal{D}\times\mathcal{D}$, $x\neq y$,

$$\frac{C_1}{|x-y|} \le \frac{1}{\big|\Phi^{-1}(x)-\Phi^{-1}(y)\big|} \le \frac{C_2}{|x-y|}\tag{4.28}$$

for some constants $C_1>0$ and $C_2>0$ independent of x and y. Since $I(\bar\mu)<+\infty$, estimate (4.3) tells us that $\ln|x-y|\in L^1(\mathcal{D}\times\mathcal{D},\,\bar\mu\otimes\bar\mu)$. Therefore, we may apply the Dominated Convergence Theorem to obtain

$$\lim_{n\to+\infty}\lim_{k\to+\infty} IV_k^n = \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right)d\bar\mu(x)d\bar\mu(y).\tag{4.29}$$

We now analyze the term III_k^n. We first deduce from (4.28) that

$$\begin{aligned}&\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_1}{|x-y|}\right)d\bar\mu_k(x)d\bar\mu_k(y) \le III_k^n\\ &\qquad\le \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_2}{|x-y|}\right)d\bar\mu_k(x)d\bar\mu_k(y)\,.\end{aligned}\tag{4.30}$$

Since

$$G(x,y)-\frac{\chi_n(x,y)}{2\pi}\ln\left(\frac{C_i}{|x-y|}\right)=S(x,y)+\frac{1-\chi_n(x,y)}{2\pi}\ln|x-y|-\frac{\chi_n(x,y)}{2\pi}\ln(C_i)$$

is locally smooth in $\mathcal{D}\times\mathcal{D}$ and $I(\bar\mu_k)\to I(\bar\mu)$, we infer from (4.25) that

$$\lim_{k\to+\infty}\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_i}{|x-y|}\right)d\bar\mu_k(x)d\bar\mu_k(y)= \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_i}{|x-y|}\right)d\bar\mu(x)d\bar\mu(y),\quad i=1,2. \tag{4.31}$$

Using the facts that $\chi_n\to 0$ $\bar\mu\otimes\bar\mu$ a.e. and $\ln|x-y|\in L^1(\mathcal{D}\times\mathcal{D},\bar\mu\otimes\bar\mu)$, we infer, as previously, that

$$\lim_{n\to+\infty}\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_i}{|x-y|}\right)d\bar\mu(x)d\bar\mu(y)=0,\quad i=1,2. \tag{4.32}$$

Combining (4.30), (4.31), and (4.32), we obtain

$$\lim_{n\to+\infty}\liminf_{k\to+\infty}III_k^n=\lim_{n\to+\infty}\limsup_{k\to+\infty}III_k^n=0,$$

which yields, together with (4.27) and (4.29),

$$\begin{aligned}\lim_{k\to+\infty}II_k&=\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right)d\bar\mu(x)d\bar\mu(y)\\&=\frac{1}{4\pi}\iint_{\Sigma\times\Sigma}\ln\left(\frac{1}{|x-y|}\right)d\mu(x)d\mu(y).\end{aligned} \tag{4.33}$$

Gathering (4.24), (4.26), and (4.33), we obtain (4.23).

Case 3: general Σ. If Σ is an embedded arc, we may assume, without loss of generality, that $\Sigma\subset\Sigma'$ for some C^2–Jordan curve Σ' compactly included in $\mathcal{D}$. Hence it suffices to consider the case that Σ is a Jordan curve. We use the following lemma, whose proof is postponed.

Lemma 4.3. Let Σ be a C^2-Jordan curve. There exists $\delta_1>0$ such that for every $x_0\in\Sigma$, there exists a C^1-diffeomorphism $\Phi:\mathcal{D}\to\mathcal{D}$ satisfying $\Phi(x)=x$ in a neighborhood of $\partial\mathcal{D}$ and such that $\Phi(\Sigma\cap\overline{B}_{\delta_1}(x_0))$ is a segment compactly included in $\mathcal{D}$.

Let $\gamma:[0,1]\to\Sigma$ be a constant speed parametrization of Σ and N be a positive integer to be chosen shortly. Set $t_n=n/N$, $n=0,\dots,N$, and

$$\Sigma_n:=\gamma([t_{n-1},t_n]),\quad n=1,\dots,N.$$

We choose N in such that for $n = 1, \dots, N$, $\operatorname{diam}(\Sigma_n) \le \delta_1$, where the constant δ_1 is given by Lemma 4.3. We set $x_n = \gamma((t_{n-1} + t_n)/2)$ and apply Lemma 4.3 to each x_n to obtain a C^1-diffeomorphism $\Phi_n : \mathcal{D} \to \mathcal{D}$ such that $\Phi_n(\Sigma_n)$ is a segment compactly included in $\mathcal{D}$ and $\Phi_n(x) = x$ in a neighborhood of $\partial\mathcal{D}$.

Let μ be an arbitrary nonnegative Radon measure in $H^{-1}(\mathcal{D})$ supported by Σ. Consider a decreasing sequence $\alpha_k \to 0$ and define, for sufficiently large k,

$$\Sigma_n^k := \gamma([t_{n-1} + \alpha_k, t_n - \alpha_k]), \quad \mu_n^k := \mu \llcorner \Sigma_n^k, \quad n = 1, \dots, N.$$

Obviously, $\mu_n^k \in H^{-1}(\mathcal{D})$ with $\operatorname{supp} \mu_n^k \subset \Sigma_n^k$. Applying Case 2 to each n and k, we obtain a sequence of measures $\{\mu_{n,m}^k\}_{m\in\mathbb{N}}$ of the form (4.2) such that

$$\operatorname{supp} \mu_{n,m}^k \subset \Sigma_n, \quad \mu_{n,m}^k(\mathcal{D}) \to \mu_n^k(\mathcal{D}), \quad \mu_{n,m}^k \rightharpoonup \mu_n^k \text{ weak* as measures on } \mathcal{D},$$

and $I(\mu_{n,m}^k) \to I(\mu_n^k)$ as $m \to +\infty$. Define the measures

$$\mu_m^k := \sum_{n=1}^N \mu_{n,m}^k \quad \text{and} \quad \mu^k := \sum_{n=1}^N \mu_n^k = \mu \llcorner (\cup_n \Sigma_n^k).$$

Then as $m \to +\infty$, $\mu_m^k(\mathcal{D}) \to \mu^k(\mathcal{D})$ and $\mu_m^k \rightharpoonup \mu^k$ weak* as measures on $\mathcal{D}$. In addition, from Lemma 4.1, we infer that $h_{\mu_{n,m}^k} \to h_{\mu_n^k}$ strongly in $H^1(\mathcal{D})$ for all integers n and k. Hence

$$\begin{aligned} I(\mu_m^k) &= \frac{1}{2} \sum_{i,j=1}^N \iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, d\mu_{i,m}^k d\mu_{j,m}^k = \frac{1}{2} \sum_{i,j=1}^N \int_{\mathcal{D}} \nabla h_{\mu_{i,m}^k} \cdot \nabla h_{\mu_{j,m}^k}\, dx \\ &\xrightarrow[m\to+\infty]{} \frac{1}{2} \sum_{i,j=1}^N \int_{\mathcal{D}} \nabla h_{\mu_i^k} \cdot \nabla h_{\mu_j^k}\, dx = \frac{1}{2} \sum_{i,j=1}^N \iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, d\mu_i^k d\mu_j^k = I(\mu^k). \end{aligned}$$

Next, recall that μ is atomless. Hence by the Monotone Convergence Theorem, we have $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$ and $I(\mu^k) \to I(\mu)$ as $k \to +\infty$, as well as weak* convergence of μ^k to μ. Consequently,

$$\begin{aligned} \lim_{k\to+\infty} \lim_{m\to+\infty} |\mu_m^k(\mathcal{D}) - \mu(\mathcal{D})| &= \lim_{k\to+\infty} \lim_{m\to+\infty} |I(\mu_m^k) - I(\mu)| \\ &= \lim_{k\to+\infty} \lim_{m\to+\infty} \|\mu_m^k - \mu\|_{(C_0^{0,1}(\mathcal{D}))^*} = 0 \end{aligned}$$

(here we again use the compact embedding $(C_0^0(\mathcal{D}))^* \hookrightarrow (C_0^{0,1}(\mathcal{D}))^*$). Theorem 4.1 follows by choice of a suitable diagonal sequence $\mu_k = \mu_{m_k}^k$. □

Proof of Lemma 4.3. Since Σ is a C^2-Jordan curve, there exists $\delta_0 > 0$ such that for every $x_0 \in \Sigma$, $\Sigma \cap \overline{B}_{2\delta_0}(x_0)$ is the graph of a C^2-function and

$\overline{B}_{2\delta_0}(x_0) \subset \mathcal{D}$. Now fix $x_0 \in \Sigma$ and express each $x \in \mathcal{D}$ as $x = x_0 + s\tau + t\tau^\perp$, where τ denotes a unit tangent vector to Σ at x_0. Then there exist $0 > s_{\min} \geq -2\delta_0$, $0 < s_{\max} \leq 2\delta_0$, and a C^2-function $f : [s_{\min}, s_{\max}] \to \mathbb{R}$ satisfying $f(0) = f'(0) = 0$ such that $\Sigma \cap \overline{B}_{2\delta_0}(x_0) = \{x_0 + s\tau + f(s)\tau^\perp\,,\ s \in [s_{\min}, s_{\max}]\}$. Since Σ is C^2, there exists a constant $\kappa > 0$, depending only on Σ, such that $|f''(s)| \leq \kappa$ for every $s \in [s_{\min}, s_{\max}]$. Hence if necessary, we may choose δ_0 smaller (uniformly with respect to x_0) such that $|f'| \leq 1$. Then for any $\delta \leq 2\delta_0$, $s_{\min} \leq -\delta_0$, $s_{\max} \geq \delta_0$, and $\Sigma \cap \overline{B}_\delta(x_0)$ is still a connected arc.

Set $\delta_1 := \delta_0/(2+\kappa)$. We claim that $\Sigma \cap \overline{B}_{\delta_1}(x_0)$ satisfies the requirement. Indeed, we construct a C^1-diffeomorphism $\Phi : \mathcal{D} \to \mathcal{D}$ as follows. Consider a smooth cutoff function $\chi : \mathbb{R}^2 \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq \delta_1$, $\chi(x) = 0$ if $|x| \geq 2\delta_0$, and $|\nabla\chi| \leq \delta_0^{-1}$. Then set, for $x \in \mathcal{D}$,

$$\Phi(x) := x - \chi(x - x_0) f\big((x - x_0)\cdot\tau\big)\tau^\perp.$$

We leave it to the reader to verify that Φ is a C^1-diffeomorphism of $\mathcal{D}$ onto itself. Moreover, $\Phi(\Sigma \cap \overline{B}_{\delta_1}(x_0)) = \{x_0 + s\tau\,,\ -\delta_1 \leq s \leq \delta_1\}$ is a segment compactly included in $\mathcal{D}$. □

Proof of Corollary 1.1. Step 1. For any nonnegative Radon measure μ supported by Σ, we have

$$I(\mu) - \zeta_{\max}\mu(\mathcal{D}) \geq I_*(\mu(\mathcal{D}))^2 - \zeta_{\max}\mu(\mathcal{D}); \tag{4.34}$$

equality holds if and only if $\mu = \lambda\mu_*$ for some constant $\lambda \geq 0$. We recall that μ_* is the unique minimizer of I amongst all probability measures supported by Σ and that $I_* := I(\mu_*)$. The existence and uniqueness of μ_* is a classical result; we refer to [15] for further details. Optimizing (4.34) with respect to λ for measures of the form $\mu = \lambda\mu_*$, we obtain $\zeta_{\max}\mu_*/(2I_*)$ as the unique minimizer of $\mu \mapsto I(\mu) - \zeta_{\max}\mu(\mathcal{D})$ amongst all nonnegative Radon measures supported by Σ.

Step 2. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. The existence of a minimizer u_n of F_{ε_n} is a classical result and follows from standard arguments based on coercivity and lower semicontinuity properties of F_{ε_n}. We first observe that $F_{\varepsilon_n}(u_n) \leq F_{\varepsilon_n}(1) = 0$. Hence, by Theorem 1.1, there exists a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\frac{1}{\omega_{n_k}} j(u_{n_k}) \to \mu_0 \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^* \text{ as } k \to +\infty$$

for some nonnegative Radon measure $\mu_0 \in H^{-1}(\mathcal{D})$ supported by Σ. Moreover,

$$\liminf_{k\to+\infty} F_{\varepsilon_{n_k}}(u_{n_k}) \geq I(\mu_0) - \zeta_{\max}\mu_0(\mathcal{D})\,. \tag{4.35}$$

On the other hand, by Theorem 1.1, any nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ supported by Σ can be strongly approximated in $(C_0^{0,1}(\mathcal{D}))^*$ by some sequence $\{\omega_{n_k}^{-1} j(v_k)\}$, $\{v_k\} \subset H^1(\mathcal{D};\mathbb{C})$ satisfying

$$\lim_{k\to+\infty} F_{\varepsilon_{n_k}}(v_k) = I(\mu) - \zeta_{\max}\mu(\mathcal{D}).$$

Since $F_{\varepsilon_{n_k}}(u_{n_k}) \leq F_{\varepsilon_{n_k}}(v_k)$, we infer that μ_0 minimizes $\mu \mapsto I(\mu) - \zeta_{\max}\mu(\mathcal{D})$ amongst all nonnegative Radon measures supported by Σ. Consequently, $\mu_0 = \zeta_{\max}\mu_*/(2I_*)$ and the lim inf in (4.35) is actually a limit. The result along the full sequence $\{\varepsilon_n\}$ then follows from a standard argument on the uniqueness of the limit. □

5 Γ–convergence analysis for annular domains

In this section, we briefly show how to extend the above techniques to handle the case of a multiply connected domain. The method we outline here may be applied to a region with any finite number of holes (see [2]), but for simplicity, we restrict ourselves to domains that are homeomorphic to an annulus.

Let $\mathcal{D}$ be a simply connected domain in $\mathbb{R}^2$ with smooth boundary, and $\mathcal{B} \subset\subset \mathcal{D}$ a smooth, simply connected domain compactly contained in $\mathcal{D}$. Let $\mathcal{A} := \mathcal{D} \setminus \mathcal{B}$. For $u \in H^1(\mathcal{A};\mathbb{C})$, we define the functional

$$J_\varepsilon(u) := \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2 - \Omega_\varepsilon V(x)\cdot j(u) \right\} dx.$$

Here, the vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ is assumed given and (for simplicity) locally Lipschitz continuous. We are interested in the asymptotic behavior of J_ε as $\varepsilon \to 0$, with angular speed Ω_ε as in (1.2).

5.1 Asymptotic vorticity of the hole. For multiply connected domains, the highest order term in an expansion of the minimal energy is partially due to the turning of the phase of a minimizer around the holes. The first step in studying vortices in the interior {**of** $\mathcal{D}$**?**} is to identify the *asymptotic vorticity of the hole* and then to split the energy into contributions from the hole and from the interior. To this end, we first study the minimization of the functional J_ε over $\mathbb{S}^1$–valued maps. Observe that for $\mathbb{S}^1$–valued maps, the functional J_ε depends on ε only via the angular speed $\Omega = \Omega(\varepsilon)$, and not via ε itself; i.e., for every $u \in H^1(\mathcal{A};\mathbb{S}^1)$,

$$J_\varepsilon(u) = \mathcal{H}_\Omega(u) := \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega V(x)\cdot j(u) \right\} dx,$$

where $\Omega = \Omega_\varepsilon$. We are therefore interested in minimizing $\mathcal{H}_\Omega$ in the class $H^1(\mathcal{A};\mathbb{S}^1)$; here $\Omega > 0$ can be any positive parameter. It is well known that maps in $H^1(\mathcal{A};\mathbb{S}^1)$ are classified by their topological degree, i.e., their winding number around the hole $\mathcal{B}$. Hence, minimizing first in each homotopy class and then choosing the lowest energy level, one attains the minimum of $\mathcal{H}_\Omega$, i.e.,

$$\min \mathcal{H}_\Omega = \min_{d\in\mathbb{Z}} g(d,\Omega), \tag{5.1}$$

where

$$g(d,\Omega) := \min\left\{\mathcal{H}_\Omega(u) \,:\, u\in H^1(\mathcal{A};\mathbb{S}^1)\,,\ \deg u = d\right\}. \tag{5.2}$$

Concerning the minimization problem (5.2), we have the following result.

Proposition 5.1. For every $d\in\mathbb{Z}$, the minimization problem (5.2) admits a unique solution u_d up to a (complex) multiplicative constant of modulus one. Moreover,

$$g(d,\Omega) = \frac12\int_{\mathcal{A}}\left\{|\nabla\Phi_d|^2 - \Omega^2|V|^2\right\}dx, \tag{5.3}$$

where Φ_d is the unique solution of the linear equation

$$\begin{cases} -\Delta\Phi_d(x) = \Omega\,\mathrm{curl}\,V(x) & \text{if } x\in\mathcal{A},\\ \Phi_d(x) = 0 & \text{if } x\in\partial\mathcal{D},\\ \Phi_d(x) = \text{constant} & \text{if } x\in\partial\mathcal{B},\\ \displaystyle\int_{\partial\mathcal{B}}\frac{\partial\Phi_d}{\partial\nu} = 2\pi d - \Omega\int_{\partial\mathcal{B}} V\cdot\tau. \end{cases} \tag{5.4}$$

Proof. The arguments are in the same spirit as in [7, Chap. 1]; for the reader's convenience, we provide some details.

Step 1. We claim that for any $u\in H^1(\mathcal{A};\mathbb{S}^1)$, $\deg u = d$,

$$\mathcal{H}_\Omega(u) \ge \frac12\int_{\mathcal{A}}\left\{|\nabla\Phi_d|^2 - \Omega^2|V|^2\right\}dx.$$

Indeed, first observe that $\mathrm{curl}\, j(u) = 0$ since u is $\mathbb{S}^1$-valued. On the other hand, $\nabla^\perp\Phi_d + \Omega V$ is also curl-free, and

$$\int_{\partial\mathcal{B}}\left(j(u) - \nabla^\perp\Phi_d - \Omega V\right)\cdot\tau = 2\pi d - \int_{\partial\mathcal{B}}\frac{\partial\Phi_d}{\partial\nu} - \Omega\int_{\partial\mathcal{B}} V\cdot\tau = 0;$$

so we can find a scalar function $H \in H^1(\mathcal{A})$ such that $j(u) = \nabla H + \nabla^\perp \Phi_d + \Omega V$. Since u is $\mathbb{S}^1$-valued, we have $|\nabla u|^2 = |j(u)|^2$, and thus

$$\begin{aligned}\mathcal{H}_\Omega(u) &= \frac{1}{2}\int_{\mathcal{A}} \left\{ |j(u) - \Omega V|^2 - \Omega^2 |V|^2 \right\} dx \\ &= \frac{1}{2}\int_{\mathcal{A}} \left\{ |\nabla \Phi_d|^2 - \Omega^2 |V|^2 \right\} dx + \frac{1}{2}\int_{\mathcal{A}} |\nabla H|^2 dx + \int_{\mathcal{A}} \nabla^\perp \Phi_d \cdot \nabla H \, dx.\end{aligned}$$

Using the fact that the function Φ_d is constant on $\partial\mathcal{A}$, an integration by parts then yields $\int_{\mathcal{A}} \nabla^\perp \Phi_d \cdot \nabla H \, dx = 0$, and the claim follows.

Step 2. There exists $u_d \in H^1(\mathcal{A}; \mathbb{S}^1)$ such that $\deg u_d = d$ and $j(u_d) = \nabla^\perp \Phi_d + \Omega V$. Indeed, since $\operatorname{curl}(\nabla^\perp \Phi_d + \Omega V) = 0$ and

$$\frac{1}{2\pi}\int_{\partial\mathcal{B}} (\nabla^\perp \Phi_d + \Omega V) \cdot \tau = d \in \mathbb{Z},$$

we can define a scalar function ψ in $\mathcal{A}$ locally that satisfies $\nabla \psi = \nabla^\perp \Phi_d + \Omega V$. Then $u_d := \exp(i\psi)$ is well defined and has the required properties. Clearly the construction of u_d is unique modulo a constant phase. □

In order to solve problem (5.1), we now observe that it is possible to express (5.3) explicitly in terms of the integer d. For this purpose, we first introduce the solution ξ of the linear problem

$$\begin{cases} \Delta \xi(x) = 0 & \text{if } x \in \mathcal{A}, \\ \xi(x) = 0 & \text{if } x \in \partial\mathcal{D}, \\ \xi(x) = 1 & \text{if } x \in \partial\mathcal{B}. \end{cases} \tag{5.5}$$

The function ξ is smooth in $\overline{\mathcal{A}}$, and $0 \le \xi \le 1$ by the maximum principle. Moreover, the Dirichlet energy of ξ is the H^1-capacity of $\mathcal{B}$ inside $\mathcal{D}$, which we denote by $\operatorname{cap}(\mathcal{B})$, i.e.,

$$\operatorname{cap}(\mathcal{B}) := \int_{\mathcal{A}} |\nabla \xi|^2 \, dx = -\int_{\partial\mathcal{B}} \frac{\partial \xi}{\partial \nu} > 0. \tag{5.6}$$

Next, we consider the unique solution ζ of

$$\begin{cases} -\Delta \zeta(x) = \operatorname{curl} V(x) & \text{if } x \in \mathcal{A}, \\ \zeta(x) = 0 & \text{if } x \in \partial\mathcal{A}. \end{cases} \tag{5.7}$$

From the Lipschitz assumption on V and standard elliptic regularity, we infer that $\zeta \in C_0^{1,\alpha}(\overline{\mathcal{A}})$ for every $0 \le \alpha < 1$. We set

$$\gamma_V := \int_{\partial\mathcal{D}} \left\{ \frac{\partial \zeta}{\partial \nu} + V \cdot \tau \right\}.$$

Observing that (5.7) implies

$$\int_{\partial\mathcal{B}} \frac{\partial\zeta}{\partial\nu} = \gamma_V - \int_{\partial\mathcal{B}} V\cdot\tau,$$

we find that for every integer d, Φ_d, determined by (5.4), is explicitly given by

$$\Phi_d = \left(\frac{\gamma_V\Omega - 2\pi d}{\mathrm{cap}(\mathcal{B})}\right)\xi + \Omega\,\zeta. \tag{5.8}$$

Moreover, using (5.5), (5.6), and (5.7), we readily obtain that for every $d\in\mathbb{Z}$,

$$\frac{1}{2}\int_{\mathcal{A}}\left\{|\nabla\Phi_d|^2 - \Omega^2|V|^2\right\}dx = \frac{|\gamma_V\Omega - 2\pi d|^2}{2\mathrm{cap}(\mathcal{B})} - \frac{\Omega^2}{2}\int_{\mathcal{A}}\left\{|V|^2 - |\nabla\zeta|^2\right\}dx.$$

As a consequence, an integer d_Ω is a minimizer of $g(d,\Omega)$ in (5.1) if and only if d_Ω minimizes the function $d\in\mathbb{Z}\mapsto|\gamma_V\Omega - 2\pi d|$.

We may now state our result concerning problem (5.1).

Theorem 5.1. Up to multiplicative constants of modulus one, the minimization problem (5.1) admits exactly two solutions (of distinct topological degree) if $\gamma_V\Omega/\pi$ is an odd integer, and a unique solution otherwise. Moreover, if $d_\Omega\in\mathbb{Z}$ is a minimizer of $g(d,\Omega)$ in (5.1), then $d_\Omega\in\left\{\lfloor\frac{\gamma_V\Omega}{2\pi}\rfloor, \lfloor\frac{\gamma_V\Omega}{2\pi}\rfloor + 1\right\}$, where $\lfloor\cdot\rfloor$ denotes the floor function, and

$$\min\mathcal{H}_\Omega = -\frac{\Omega^2}{2}\int_{\mathcal{A}}\left\{|V|^2 - |\nabla\zeta|^2\right\}dx + O(1)\quad\text{as }\Omega\to+\infty. \tag{5.9}$$

5.2 The Γ–convergence result. To state the parallel Γ-convergence result for the annular domain case, we need more specific hypotheses on the potential V and the angular speed Ω_ε. In addition to the Lipschitz regularity, we assume that V satisfies the following assumptions:

(H1′) the solution ζ of (5.7) is such that

$$\zeta_{\max} := \max_{x\in\overline{\mathcal{A}}}\zeta(x) = \max_{x\in\overline{\mathcal{A}}}|\zeta(x)| > 0;$$

(H2′) the set $\Sigma := \{x\in\mathcal{A} : \zeta(x) = \zeta_{\max}\}\subset\subset\mathcal{A}$ is either a Jordan curve or a simple embedded arc of class C^2.

We note that in (H1′), the assumption that $\zeta_{\max}$ is achieved at positive values of ζ is not restrictive. Indeed, the complex conjugate of an admissible function replaces V in the energy with $-V$, and hence ζ with $-\zeta$.

As for the simply connected domain case, we assume that Ω_ε is near the critical value needed for the presence of vortices; i.e., exactly as in (1.2),

$$\Omega_\varepsilon = \frac{|\ln\varepsilon|}{2\zeta_{\max}} + \omega(\varepsilon) \tag{5.10}$$

for some positive function ω satisfying $\omega(\varepsilon) \to +\infty$, $\omega(\varepsilon) \le o(|\ln \varepsilon|)$, as $\varepsilon \to 0^+$.

For an arbitrary sequence $\varepsilon_n \to 0^+$, we denote by $u_n^\star$ a minimizer of $\mathcal{H}_{\Omega_n}$, i.e., a solution of (5.1); its corresponding topological degree is denoted by d_n. We also rewrite (5.8) in the form

$$\Phi_n := \Phi_{d_n} = \alpha_n \xi + \Omega_n \zeta \quad \text{with} \quad \alpha_n := \frac{\gamma_V \Omega_n - 2\pi d_n}{\operatorname{cap}(\mathcal{B})}.$$

We emphasize that $\alpha_n = O(1)$ as $n \to +\infty$ because of Theorem 5.1.

For $v \in H^1(\mathcal{A}; \mathbb{C})$, we now define

$$\overline{F}_{\varepsilon_n}(v) := \omega_n^{-2} \int_{\mathcal{A}} \left\{ \frac{|\nabla v|^2}{2} + \frac{(1-|v|^2)^2}{4\varepsilon_n^2} + \nabla^\perp \Phi_n \cdot j(v) \right\} dx.$$

The following proposition shows that$\overline{F}_{\varepsilon_n}(\bar{u}_n^\star u)$ captures the energy induced by interior vorticity of a given configuration u.

Proposition 5.2 (*Energy decomposition*). Assume that (5.10) holds. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence and $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{A}; \mathbb{C})$. Then

$$\sup_n \Omega_n^{-2} J_{\varepsilon_n}(u_n) < +\infty \quad \text{if and only if} \quad \sup_n \Omega_n^{-2} \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) < +\infty, \tag{5.11}$$

where $v_n = \bar{u}_n^\star u_n \in H^1(\mathcal{A}; \mathbb{C})$. Moreover, if one of the conditions in (5.11) holds, then

$$J_{\varepsilon_n}(u_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) + o(1) \quad \text{as } n \to +\infty. \tag{5.12}$$

Proof. Straightforward computations yield

$$\frac{|\nabla u_n|^2}{2} = |u_n|^2 \frac{|\nabla u_n^\star|^2}{2} + \frac{|\nabla v_n|^2}{2} + j(u_n^\star) \cdot j(v_n)$$

and

$$j(u_n) = |u_n|^2 j(u_n^\star) + j(v_n). \tag{5.13}$$

By the proof of Proposition 5.1, we have $j(u_n^\star) = \nabla^\perp \Phi_n + \Omega_n V$. Hence,

$$J_{\varepsilon_n}(u_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) + \int_{\mathcal{A}} (|u_n|^2 - 1) \left\{ \frac{|\nabla u_n^\star|^2}{2} - \Omega_n V \cdot j(u_n^\star) \right\} dx.$$

We now observe that

$$\|\nabla u_n^\star\|_{L^\infty(\mathcal{A})} = \|j(u_n^\star)\|_{L^\infty(\mathcal{A})} = \|\nabla^\perp \Phi_n + \Omega_n V\|_{L^\infty(\mathcal{A})} = O(\Omega_n).$$

Therefore,

$$\left| \int_{\mathcal{A}} (|u_n|^2 - 1) \left\{ \frac{|\nabla u_n^\star|^2}{2} - \Omega_n V \cdot j(u_n^\star) \right\} dx \right| \leq O\left(\varepsilon_n \Omega_n^2 \sqrt{E_{\varepsilon_n}(u_n, \mathcal{A})} \right).$$

Since $|u_n| = |v_n|$, we also have the same estimate as above with $E_{\varepsilon_n}(v_n, \mathcal{A})$ instead of $E_{\varepsilon_n}(u_n, \mathcal{A})$. Assuming that one of the conditions in (5.11) holds and arguing as in (2.2), we derive that either $\sqrt{E_{\varepsilon_n}(u_n, \mathcal{A})} \leq O(\Omega_n)$ or $\sqrt{E_{\varepsilon_n}(v_n, \mathcal{A})} \leq O(\Omega_n)$. Consequently, (5.12) holds, and the conclusion follows from (5.12) combined with (5.9). □

For a nonnegative Radon measure μ on $\mathcal{A}$, we define

$$\bar{I}(\mu) := \frac{1}{2} \iint_{\mathcal{A} \times \mathcal{A}} \bar{G}(x, y)\, d\mu(x)\, d\mu(y),$$

where $\bar{G}$ denotes the Dirichlet Green's function of the domain $\mathcal{A}$; i.e., for every $y \in \mathcal{A}$, $\bar{G}(\cdot, y)$ is the solution of

$$\begin{cases} -\Delta \bar{G}(x, y) = \delta_y(x) & \text{if } x \in \mathscr{D}'(\mathcal{A}), \\ \bar{G}(x, y) = 0 & \text{if } x \in \partial \mathcal{A}. \end{cases} \tag{5.14}$$

We can now state the Γ-convergence result for annular domains; it involves the family of *"reduced"* functionals $\{\overline{F}_\varepsilon\}_{\varepsilon > 0}$.

Theorem 5.2. Assume (H1′), (H2′), and (5.10). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. Then

(i) for any $\{v_n\}_{n \in \mathbb{N}} \subset H^1(\mathcal{D}; \mathbb{C})$ satisfying $\sup_n \overline{F}_{\varepsilon_n}(v_n) < +\infty$, there exist a subsequence (not relabelled) and a nonnegative Radon measure μ in $H^{-1}(\mathcal{A})$ supported by Σ such that

$$\frac{1}{\omega(\varepsilon_n)} \operatorname{curl} j(v_n) \underset{n \to +\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{A}))^*; \tag{5.15}$$

(ii) for any $\{v_n\}_{n \in \mathbb{N}} \subset H^1(\mathcal{D}; \mathbb{C})$ such that (5.15) holds for some nonnegative Radon measure μ in $H^{-1}(\mathcal{A})$ supported by Σ,

$$\liminf_{n \to +\infty} \overline{F}_{\varepsilon_n}(v_n) \geq \bar{I}(\mu) - \zeta_{\max}\, \mu(\mathcal{A});$$

(iii) for any nonnegative Radon measure μ in $H^{-1}(\mathcal{A})$ supported by Σ, there exists a sequence $\{v_n\}_{n \in \mathbb{N}} \subset H^1(\mathcal{A}; \mathbb{C})$ such that (5.15) holds and

$$\lim_{n \to +\infty} \overline{F}_{\varepsilon_n}(v_n) = \bar{I}(\mu) - \zeta_{\max}\, \mu(\mathcal{A}).$$

As in the simply connected case, this Γ-convergence result can lead to the asymptotic description of the vorticity in $\overline{F}_\varepsilon$-global minimizers. Theorem 5.2 combined with Proposition 5.2 also gives the asymptotic behavior of vorticity in J_ε-global minimizers. The key observation here is that minimizers for J_ε yield quasiminimizers for $\overline{F}_\varepsilon$, and vice versa.

Corollary 5.1. Assume (H1'), (H2'), and (5.10). Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. For each integer $n \in \mathbb{N}$, let $u_n \in H^1(\mathcal{A};\mathbb{C})$ be a minimizer of $J_{\varepsilon_n}(\cdot)$. Then

$$\frac{1}{\omega(\varepsilon_n)}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \frac{\zeta_{\max}}{2\bar{I}_*}\,\bar{\mu}_* \quad \text{strongly in } (C_0^{0,1}(\mathcal{A}))^*,$$

where $\bar{\mu}_*$ is the unique minimizer of $\bar{I}(\cdot)$ amongst all probability measures supported by Σ, and $\bar{I}_* := \bar{I}(\bar{\mu}_*)$. In addition,

$$J_{\varepsilon_n}(u_n) = -\frac{\Omega_n^2}{2}\int_{\mathcal{A}}\left\{|V|^2 - |\nabla\zeta|^2\right\}dx - \frac{\zeta_{\max}^2}{4\bar{I}_*}\omega_n^2 + o(\omega_n^2). \tag{5.16}$$

We conclude this subsection with an elementary example motivated by [1, 2, 3].

Example 5.1. Assume $\mathcal{D} = B_1(0)$, $\mathcal{B} = B_\rho(0)$ for some $0 < \rho < R$, and $V(x) = x^\perp$. Then the solution ζ of (5.7) is given by

$$\zeta(x) = -\frac{|x|^2}{2} + \frac{R^2-\rho^2}{2\ln(R/\rho)}\ln|x| + \frac{\rho^2\ln R - R^2\ln\rho}{2\ln(R/\rho)}.$$

In particular, the set Σ is given by the concentric circle $B_{r_*}(0)$,

$$r_* = \sqrt{\frac{R^2-\rho^2}{2\ln(R/\rho)}} \in (\rho, R).$$

Here, again the uniform measure $\mu_* = (2\pi r_*)^{-1} d\mathcal{H}^1 \llcorner \Sigma$ turns out to be the Green equilibrium measure for Σ in $\mathcal{A}$, i.e., $\bar{I}(\mu_*) = \bar{I}_*$. Indeed, it is easily verified that

$$h_*(x) = \begin{cases} \dfrac{\ln(R/r_*)}{2\pi(\ln(R/r_*)+\ln(r_*/\rho))}\ln(|x|/\rho) & \text{if } \rho \le |x| \le r_*, \\[2ex] \dfrac{\ln(r_*/\rho)}{2\pi(\ln(R/r_*)+\ln(r_*/\rho))}\ln(R/|x|) & \text{if } r_* \le |x| \le R, \end{cases}$$

solves $-\Delta h_* = \mu_*$ in $\mathcal{A}$ with $h_{*|\partial\mathcal{A}} = 0$. Hence $h_*(x) = \int_{\mathcal{A}} \bar{G}(x,y)\,d\mu_*(y)$, and, since h_* is constant on Σ, the conclusion follows from Theorem II.5.12 in [15].

5.3 Compactness of normalised weak Jacobians. In this subsection, we prove statement (i) of Theorem 5.2. We consider an arbitrary sequence $\varepsilon_n \to 0^+$. For any $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ satisfying $\sup_n \overline{F}_{\varepsilon_n}(v_n) < +\infty$, we first derive, exactly as in Lemma 2.1, the estimates

$$E_{\varepsilon_n}(v_n,\mathcal{A}) \le O(|\ln\varepsilon_n|^2) \quad \text{and} \quad \|v_n\|_{L^4(\mathcal{A})} \le O(1). \tag{5.17}$$

Hence, assuming, in addition, that $\{v_n\}_{n\in\mathbb{N}} \subset C^1(\mathcal{A})$, we can apply the vortex ball construction in Proposition 2.1 with $\mathcal{D}$ replaced with $\mathcal{A}$ and $\mathcal{D}_\varepsilon$ replaced with $\mathcal{A}_\varepsilon := \{x \in \mathcal{A} : \operatorname{dist}(x,\partial\mathcal{A}) > \varepsilon\}$. We again choose $r = r_n := |\ln\varepsilon_n|^{-4}$, thus obtaining a finite collection of disjoint closed balls $\{\overline{B}(a_i^n,\rho_{i,n})\}_{i\in I_n}$ (written $B_i^n := \overline{B}(a_i^n,\rho_{i,n})$) with associated degrees $d_{i,n}$ and total approximate vorticity

$$D_n := \sum_{i\in I_n} |d_{i,n}|,$$

as in Proposition 2.1.

The first difference between the multiply connected and simply connected cases is evident in an estimate analogous to (2.3); in the multiply connected case, we must take into account an additional contribution to the potential term coming from the boundary $\partial\mathcal{B}$. Since one or more of the vortex balls B_i^n may intersect $\partial\mathcal{A}_\varepsilon \setminus \partial\mathcal{D}_\varepsilon$, we need to perturb this boundary slightly. When calculating the boundary term, it is convenient to choose a level set of ξ. For $0 < s < t < 1$, define

$$\sigma_t := \{x \in \mathcal{A} : \xi(x) = t\}, \quad \text{and} \quad A_{s,t} := \{x \in \mathcal{A} : \ s < \xi < t\}. \tag{5.18}$$

It follows easily from the maximum principle and the Hopf Boundary Lemma that each σ_t is smooth and the family $\{\sigma_t\}_{0<t<1}$ forms a foliation of $\mathcal{A}$. Then for every $t \in (0,1)$, σ_t is diffeomorphic to $\partial\mathcal{B}$, and the set $A_{t,1}$ is a neighborhood of $\partial\mathcal{B}$ in $\mathcal{A}$. We now choose an appropriate level set of ξ. Define

$$J_n = \left\{t \in (0,1) \ : \ \sigma_t \cap \left(\cup_{i\in I_n} B_i^n\right) = \varnothing\right\}. \tag{5.19}$$

We note that the measure of the complement $(0,1) \setminus J_n$ is of the same order as $r_n = |\ln\varepsilon_n|^{-4}$. Hence we can find $t_n \in J_n$ such that the level curve $\gamma_n := \sigma_{t_n}$ satisfies

$$\varepsilon_n < \operatorname{dist}(\gamma_n,\partial\mathcal{B}) \le O(|\ln\varepsilon_n|^{-3})$$

and, consequently, $t_n = 1 + O(|\ln\varepsilon_n|^{-3})$.

Since v_n does not vanish on γ_n, this construction allows us to define the topological degree of v_n around γ_n, i.e.,

$$\delta_n := \deg\left(\frac{v_n}{|v_n|}, \gamma_n\right).$$

Then an approximate total vorticity in $\mathcal{A}$ of the configuration v_n is given by $|\delta_n| + D_n$.

We now state the following result, which parallels Proposition 2.2.

Proposition 5.3. Assume (H1′) and (5.10). Let $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})\cap C^1(\mathcal{A})$, and D_n and δ_n be as above. Then $|\delta_n| + D_n \le O(\omega_n)$.

Proof. Step 1. Arguing exactly as in the proof of (2.3), we first obtain

$$\Omega_n \int_{\mathcal{A}} \nabla^\perp \zeta \cdot j(v_n) = -2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) + o(1)\,. \tag{5.20}$$

Then, essentially following the proof of Lemma 3.4 in [2], we obtain

$$\int_{\mathcal{A}} \nabla^\perp \xi \cdot j(v_n) = -2\pi \sum_{i\in I_n} d_{i,n}\xi(a_i^n) - 2\pi t_n\delta_n + o(1)\,. \tag{5.21}$$

Here, the fact that γ_n is a level set of ξ is essential in obtaining the degree δ_n from the boundary term when integrating by parts.

Step 2. As in the proof of Proposition 2.2, we assume that $\omega_n \le O(D_n + |\delta_n|)$. From (5.17) and statement (iv) of Proposition 2.1, we have $D_n \le O(|\ln\varepsilon_n|)$. We then infer from (2.1), (5.20), and (5.21) that

$$\begin{aligned} O(\omega_n^2) &\ge \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) \\ &\ge \pi D_n\big(|\ln\varepsilon_n| - C\ln|\ln\varepsilon_n|\big) - 2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) - 2\pi|\alpha_n|D_n \\ &\quad - 2\pi|\alpha_n||\delta_n| + \int_{\mathcal{A}\setminus\cup_{i\in I_n}B_i^n} |\nabla v_n|^2\,dx + o(1), \end{aligned} \tag{5.22}$$

where we have used the fact that $0 \le \xi \le 1$ and $0 < t_n < 1$.

Next, as in (2.6), we consider a sequence $\eta_n \to 0$ and group the vortex balls into different classes as in the proof of Proposition 2.2 (and following the same notation). Exactly as in (2.10) and (2.12), we obtain

$$\pi D_n^*|\ln\varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^*} d_{i,n}\zeta(a_i^n) \ge -2\pi\omega_n\zeta_{\max}D_n^*, \tag{5.23}$$

and

$$\pi D_n^+|\ln\varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^+} d_{i,n}\zeta(a_i^n) \ge C\Omega_n\eta_n D_n^+. \tag{5.24}$$

For negative degrees, we observe that $\{\zeta \le 0\} \cap \Sigma = \varnothing$, since $\zeta_{\max} = |\zeta|_{\max} > 0$. Hence we can estimate as for the class I_n^+, obtaining

$$\begin{aligned} \pi D_n^-|\ln\varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^-} d_{i,n}\zeta(a_i^n) &\ge \pi D_n^-|\ln\varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^-,\,\zeta(a_i^n)\le 0} d_{i,n}\zeta(a_i^n) \\ &\ge C\Omega_n\eta_n D_n^-\,. \end{aligned} \tag{5.25}$$

Inserting (5.23), (5.24) and (5.25) into (5.22) yields

$$O(\omega_n^2) \geq -\pi C_1 D_n \ln|\ln\varepsilon_n| - 2\pi\omega_n\zeta_{\max}D_n + C\eta_n\Omega_n(D_n^+ + D_n^-) - 2\pi|\alpha_n||\delta_n| + \frac{1}{2}\int_{\mathcal{A}\setminus\cup B_i^n}|\nabla v_n|^2 dx. \tag{5.26}$$

Using the fact that $|\alpha_n| = O(1)$, we easily deduce the estimate

$$D_n^+ + D_n^- \leq C\,\frac{\max\{\omega_n, \ln|\ln\varepsilon_n|\}(D_n + |\delta_n|)}{\eta_n|\ln\varepsilon_n|} \tag{5.27}$$

for a constant $C > 0$ independent of n.

We claim that

$$\int_{\mathcal{A}\setminus\cup B_i^n}|\nabla v_n|^2 dx \geq C(D_n + |\delta_n|)^2. \tag{5.28}$$

Assuming (5.28), we infer from (5.26) that

$$(D_n + |\delta_n|)^2 - C\max\{\omega_n, \ln|\ln\varepsilon_n|\}(D_n + |\delta_n|) \leq O(\omega_n^2),$$

which clearly implies

$$D_n + |\delta_n| \leq O(\max\{\omega_n, \ln|\ln\varepsilon_n|\}). \tag{5.29}$$

To prove (5.28), we introduce

$$t_0 := 1/2\min\{t \in (0,1) : \sigma_t \cap \Sigma \neq \varnothing\}, \quad t_1 := 1/2(1+\max\{t \in (0,1) : \sigma_t \cap \Sigma \neq \varnothing\}),$$

and

$$J_n^1 := \left\{t \in (\varepsilon_n, t_0) \,:\, \sigma_t \cap \left(\bigcup_{i\in I_n} B_i^n\right) = \varnothing\right\},$$
$$J_n^2 := \left\{t \in (t_1, t_n) \,:\, \sigma_t \cap \left(\bigcup_{i\in I_n} B_i^n\right) = \varnothing\right\}.$$

Then $\operatorname{dist}(\sigma_t, \Sigma) \geq C > 0$ for every $t \in J_n^1 \cup J_n^2$ and $\min(\mathcal{L}^1(J_n^1), \mathcal{L}^1(J_n^2)) \geq C > 0$ for a constant C independent of n. Next we consider

$$D_n(t) := \deg\left(\frac{v_n}{|v_n|}, \sigma_t\right).$$

For n sufficiently large and $2|\delta_n| \geq D_n$, we estimate, using (5.27),

$$|D_n(t)| \geq \big||\delta_n| - D_n^+ - D_n^-\big| \geq \frac{1}{2}|\delta_n| \geq C(D_n + |\delta_n|) \quad \text{for every } t \in J_n^2. \tag{5.30}$$

For n sufficiently large and $2|\delta_n| < D_n$, we have, in view of (5.27),

$$|D_n(t)| \geq \left|D_n - D_n^+ - 2D_n^- - |\delta_n|\right| \geq \frac{1}{4} D_n \geq C(D_n + |\delta_n|) \quad \text{for every } t \in J_n^1. \tag{5.31}$$

Set $\tilde{v}_n := v_n/|v_n|$. Using statement (ii) of Proposition 2.1, the coarea formula, and Jensen's inequality, we conclude from (5.30) and (5.31) that

$$\begin{aligned}
\int_{\mathcal{A}\setminus\cup B_i^n} |\nabla v_n|^2 \, dx &\geq C \int_{\mathcal{A}\setminus\cup B_i^n} |\nabla \tilde{v}_n|^2 |\nabla \xi| \, dx \geq C \int_{J_n^1 \cup J_n^2} \left(\int_{\sigma_t} |\nabla \tilde{v}_n|^2 \right) dt \\
&\geq C \int_{J_n^1 \cup J_n^2} \left(\int_{\sigma_t} |\tilde{v}_n \wedge \nabla_\tau \tilde{v}_n|^2 \, d\mathcal{H}^1 \right) dt \\
&\geq C \int_{J_n^1 \cup J_n^2} \frac{|D_n(t)|^2}{\mathcal{H}^1(\sigma_t)} \, dt \geq C(D_n + |\delta_n|)^2,
\end{aligned}$$

proving (5.28). (Here, we have also used the fact that $|\nabla \xi|$ does not vanish in $\overline{\mathcal{A}}$).

Step 3. If $\ln|\ln \varepsilon_n| = o(\omega_n)$, the conclusion follows from (5.27) and (5.29). If $\omega_n = O(\ln|\ln \varepsilon_n|)$, we refine the lower bound by enlarging the vortex balls as in Step 3 of the proof of Proposition 2.2. The result here follows by same arguments used in Proposition 2.2 with minor modifications, so we omit the details. □

Proof of Theorem 5.2(i). In view of Proposition 5.3 and estimate (5.27), we can follow the proof of Theorem 2.1, considering first a sequence

$$\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{A})$$

and then the general case. □

5.4 The lower bound inequality. Proof of Theorem 5.2(ii). We use the notation of Subsection 5.3. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) = \lim_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) < +\infty\,. \tag{5.32}$$

As in the proof of Theorem 3.1, we also assume that $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$. We consider only the case $\ln|\ln \varepsilon_n| = o(\omega_n)$, since the other cases follow as in the proof of Theorem 3.1.

By Proposition 5.3 and (5.21), we have

$$\left| \alpha_n \int_{\mathcal{A}} \nabla^\perp \xi \cdot j(v_n) \, dx \right| = O(\omega_n).$$

We can then argue exactly as in (3.6) to obtain

$$\overline{F}_{\varepsilon_n}(v_n) \geq \frac{1}{2\omega_n^2} \int_{\mathcal{A}_{\varepsilon_n}\setminus\cup B_i^n} |j(v_n)|^2 - \zeta_{\max}\mu(\mathcal{A}) + o(1).$$

Setting $\tilde{j}_n(x)$ as in (3.5) (with $\mathcal{A}_{\varepsilon_n}$ in place of $\mathcal{D}_{\varepsilon_n}$), up to a subsequence, we have $\tilde{j}_n \rightharpoonup j_*$ weakly in $L^2(\mathcal{A};\mathbb{R}^2)$ as $n \to +\infty$. By lower semicontinuity, we obtain

$$\liminf_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) \geq \frac{1}{2}\int_{\mathcal{A}} |j_*|^2 - \zeta_{\max}\mu(\mathcal{A}). \tag{5.33}$$

In addition, arguing as in the proof of (3.8), we deduce that

$$\operatorname{curl} j_* = \mu \quad \text{in } \mathscr{D}'(\mathcal{A}),$$

and thus $\mu \in H^{-1}(\mathcal{D})$, since $j_* \in L^2(\mathcal{A};\mathbb{R}^2)$.

Next we introduce $h_\mu \in H^1_0(\mathcal{A})$, the unique solution of

$$\begin{cases} -\Delta h_\mu(x) = \mu(x) & \text{if } x \in H^{-1}(\mathcal{A}), \\ h_\mu(x) = 0 & \text{if } x \in \partial\mathcal{A}, \end{cases}$$

and set

$$\bar{h}_\mu := h_\mu + \frac{1}{\operatorname{cap}(\mathcal{B})}\left(\int_{\partial\mathcal{B}} j_* \cdot \tau + \frac{\partial h_\mu}{\partial \nu}\right)\xi.$$

By construction, $\operatorname{curl}(j_* + \nabla^\perp \bar{h}_\mu) = 0$ in $H^{-1}(\mathcal{A})$ and $\int_{\partial\mathcal{B}}(j_* + \nabla^\perp \bar{h}_\mu)\cdot\tau = 0$. Hence there exists $g \in H^1(\mathcal{A})$ such that $j_* + \nabla^\perp \bar{h}_\mu = \nabla g$. Arguing as in the proof of Proposition 5.1, we have

$$\int_{\mathcal{A}} |j_*|^2 = \int_{\mathcal{A}} |\nabla \bar{h}_\mu|^2 + \int_{\mathcal{A}} |\nabla g|^2 \geq \int_{\mathcal{A}} |\nabla \bar{h}_\mu|^2. \tag{5.34}$$

Using (5.5) and $h_{\mu\,|\partial\mathcal{A}} = 0$, we then obtain $\int_{\mathcal{A}} \nabla h_\mu \cdot \nabla \xi = 0$, so that

$$\int_{\mathcal{A}} |\nabla \bar{h}_\mu|^2 = \int_{\mathcal{A}} |\nabla h_\mu|^2 + \frac{1}{\operatorname{cap}(\mathcal{B})}\left(\int_{\partial\mathcal{B}} j_* \cdot \tau + \frac{\partial h_\mu}{\partial \nu}\right)^2 \geq \int_{\mathcal{A}} |\nabla h_\mu|^2. \tag{5.35}$$

Finally, using the Green representation of h_μ, we obtain

$$\frac{1}{2}\int_{\mathcal{A}} |\nabla h_\mu|^2\, dx = \frac{1}{2}\iint_{\mathcal{A}\times\mathcal{A}} G(x,y)\, d\mu(x) d\mu(y) \tag{5.36}$$

and the conclusion follows by gathering (5.33), (5.34), (5.35), and (5.36). □

5.5 The upper bound inequality. Proof of Theorem 5.2(iii). We present here the proof in the case that the measure $\mu \in H^{-1}(\mathcal{A})$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \Sigma$, more precisely, for μ of the form (4.2) with a nonvanishing density function f. The general case follows by approximation, as in Section 4.

For such a measure μ, we first proceed exactly as in Step 1 of the proof of Proposition 4.1; we adopt the same notation here. For each integer n, we consider the function f_n as defined in (4.5) and set $\hat{\mu}_n := f_n \mathcal{L}^2 \llcorner \mathcal{A}$. Then, as $n \to +\infty$, $\hat{\mu}_n \rightharpoonup \mu$ weak* as measures on $\mathcal{A}$. Next, we consider the solution h_n of

$$\begin{cases} -\Delta h_n(x) = \omega_n f_n(x) & \text{if } x \in \mathcal{A}, \\ h_n(x) = 0 & \text{if } x \in \partial\mathcal{A}. \end{cases}$$

Arguing exactly as in Step 2 of the proof of Proposition 4.1, we obtain

$$\frac{1}{2}\int_{\mathcal{A}} |\nabla h_n|^2 \, dx \leq \pi D_n |\ln \varepsilon_n| + \omega_n^2 \bar{I}(\mu) + o(\omega_n^2).$$

Next, we introduce

$$\bar{h}_n := h_n + \frac{\kappa_n}{\mathrm{cap}(\mathcal{B})}\xi, \quad \kappa_n := \int_{\partial\mathcal{B}} \frac{\partial h_n}{\partial \nu} - 2\pi \left\lfloor \frac{1}{2\pi}\int_{\partial\mathcal{B}} \frac{\partial h_n}{\partial \nu} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Noticing that $\kappa_n = O(1)$, we obtain

$$\frac{1}{2}\int_{\mathcal{A}} |\nabla \bar{h}_n|^2 \, dx = \frac{1}{2}\int_{\mathcal{A}} |\nabla h_n|^2 \, dx + \frac{\kappa_n^2}{2\mathrm{cap}(\mathcal{B})^2}\int_{\mathcal{A}} |\nabla \xi|^2 \, dx \tag{5.37}$$

$$\leq \pi D_n |\ln \varepsilon_n| + \omega_n^2 \bar{I}(\mu) + o(\omega_n^2). \tag{5.38}$$

In view of (5.6), we have

$$\frac{1}{2\pi}\int_{\partial\mathcal{B}} \frac{\partial \bar{h}_n}{\partial \nu} = \left\lfloor \frac{1}{2\pi}\int_{\partial\mathcal{B}} \frac{\partial h_n}{\partial \nu} \right\rfloor \in \mathbb{Z},$$

and since ξ is harmonic in $\mathcal{A}$,

$$\frac{1}{2\pi}\int_{\partial B_{\varepsilon_n}(a_k^n)} \frac{\partial \bar{h}_n}{\partial \nu} = \frac{1}{2\pi}\int_{\partial B_{\varepsilon_n}(a_k^n)} \frac{\partial h_n}{\partial \nu} = -1 \quad \text{for every } k = 1, \dots, D_n.$$

Hence, for any smooth Jordan curve Θ inside $\mathcal{A} \setminus \bigcup_k \overline{B}_{\varepsilon_n}(a_k^n)$,

$$\frac{1}{2\pi}\int_{\Theta} \nabla^\perp \bar{h}_n \cdot \tau \in \mathbb{Z},$$

where $\tau : \Theta \to \mathbb{S}^1$ is any smooth vector field tangent to Θ. Consequently, we may define a phase ϕ_n locally in $\mathcal{A} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)$ by

$$\nabla \phi_n(x) = -\nabla^\perp \bar{h}_n(x), \quad x \in \mathcal{A} \setminus \bigcup_k \overline{B}_{\varepsilon_n}(a_k^n).$$

Then the map $\exp(i\phi_n(x))$ is well defined for every $x \in \mathcal{A} \setminus \bigcup_k \overline{B}_{\varepsilon_n}(a_k^n)$.

Finally, we consider a profile function ρ_n as defined in (4.15) and set

$$v_n(x) := \begin{cases} \rho_n(x)e^{i\phi_n(x)} & \text{if } x \in \mathcal{A} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n), \\ 0 & \text{otherwise.} \end{cases}$$

Since $-\Delta \bar{h}_n = \omega_n \hat{\mu}_n$ in $\mathcal{A}$, using (5.37), we may proceed as in the proof of Proposition 4.1, Step 3 to prove

$$\omega_n^{-1}\mathrm{curl}\, j(v_n) \to \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{A}))^* \text{ as } n \to +\infty$$

and

$$E_{\varepsilon_n}(v_n, \mathcal{A}) \le \pi D_n |\ln \varepsilon_n| + \omega_n^2 \bar{I}(\mu) + o(\omega_n^2). \tag{5.39}$$

To evaluate the rotational part of the energy, we first argue as in the derivation of (4.21) to obtain

$$\Omega_n \int_{\mathcal{A}} \nabla \zeta^\perp \cdot j(v_n)\, dx = -\pi D_n |\ln \varepsilon_n| - \zeta_{\max}\mu(\mathcal{A})\omega_n^2 + o(\omega_n^2). \tag{5.40}$$

Similarly, we obtain

$$\alpha_n \int_{\mathcal{A}} \nabla^\perp \xi \cdot j(v_n)\, dx = -\alpha_n \int_{\mathcal{A}} \nabla \bar{h}_n \cdot \nabla \xi\, dx + o(1) = -\alpha_n \kappa_n + o(1) = O(1), \tag{5.41}$$

since $\alpha_n = O(1)$ and $\kappa_n = O(1)$. The conclusion then follows by combining (5.39), (5.40) and (5.41). □

5.6 Application to J_ε-minimizers. Proof of Corollary 5.1. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. As in the proof of Corollary 1.1, the existence of minimizers for J_{ε_n} and $\overline{F}_{\varepsilon_n}$ is a classical result, and so we omit the details. For each $n \in \mathbb{N}$, let $u_n \in H^1(\mathcal{A}; \mathbb{C})$ be a minimizer of J_{ε_n}. Since $J_{\varepsilon_n}(u_n) \le \min \mathcal{H}_{\Omega_n} = O(\Omega_n^2)$, we can apply Proposition 5.2 to the sequence $\{u_n\}_{n\in\mathbb{N}}$ and infer that

$$\min \mathcal{H}_{\Omega_n} \ge J_{\varepsilon_n}(u_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) + o(1), \tag{5.42}$$

where $v_n := \bar{u}_n^\star u_n$. On the other hand, for any minimizer $\tilde{v}_n \in H^1(\mathcal{A}; \mathbb{C})$ of $\overline{F}_{\varepsilon_n}$, we have $\overline{F}_{\varepsilon_n}(\tilde{v}_n) \le \overline{F}_{\varepsilon_n}(1) = 0$. Hence Proposition 5.2 yields

$$J_{\varepsilon_n}(u_n) \le J_{\varepsilon_n}(u_n^\star \tilde{v}_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \min \overline{F}_{\varepsilon_n} + o(1). \tag{5.43}$$

Combining (5.42) with (5.43), we infer that $\omega_n^2 \overline{F}_{\varepsilon_n}(v_n) \le o(1)$ and that $\{v_n\}$ is a sequence of quasi-minimizers for $\{\overline{F}_{\varepsilon_n}\}$, i.e.,

$$\overline{F}_{\varepsilon_n}(v_n) = \min \overline{F}_{\varepsilon_n} + o(1) \tag{5.44}$$

as $n \to +\infty$. By Theorem 5.2, there exists a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\omega_{n_k}^{-1} j(v_{n_k}) \to \mu_0$$

strongly in $(C_0^{0,1}(\mathcal{A}))^*$ for some nonnegative Radon measure $\mu_0 \in H^{-1}(\mathcal{A})$ supported by Σ. Using (5.44) together with satements (ii) and (iii) of Theorem 5.2, we deduce, as in the proof of Corollary 1.1, that μ_0 minimizes $\mu \mapsto \bar{I}(\mu) - \zeta_{\max}\mu(\mathcal{A})$ amongst all nonnegative Radon measures supported by Σ. The minimizer is again unique and given by

$$\mu_0 = \frac{\zeta_{\max}}{2\bar{I}_*}\,\bar{\mu}_*,$$

which implies the convergence of $\omega_n^{-1} j(v_n)$ along the full sequence (recall that $\bar{I}_* = \bar{I}(\mu_*)$ and that $\bar{\mu}_*$ is the minimizer of $\bar{I}$ amongst all probability measures supported by Σ; see [15]). In addition,

$$\lim_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) = \bar{I}(\mu_0) - \zeta_{\max}\mu_0(\mathcal{A}) = -\frac{\zeta_{\max}^2}{4\bar{I}_*},$$

which, combined with (5.42) and (5.9), yields (5.16).

Next it remains to prove that $\omega_n^{-1}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu_0$ strongly in $(C_0^{0,1}(\mathcal{A}))^*$. Indeed, given an arbitrary function $\varphi \in C_0^{0,1}(\mathcal{A})$, we have

$$\begin{aligned}\frac{1}{\omega_n}\int_{\mathcal{A}} j(u_n)\cdot\nabla^\perp\varphi &= \frac{1}{\omega_n}\int_{\mathcal{A}} j(v_n)\cdot\nabla^\perp\varphi + \frac{1}{\omega_n}\int_{\mathcal{A}} |v_n|^2 j(u_n^\star)\cdot\nabla^\perp\varphi\\ &= \langle\mu_0,\varphi\rangle + \frac{1}{\omega_n}\int_{\mathcal{A}} (|v_n|^2-1) j(u_n^\star)\cdot\nabla^\perp\varphi + o(1),\end{aligned}$$

where we have used (5.13) and the fact that $\operatorname{curl} j(u_n^\star) = 0$ in $\mathcal{A}$. Arguing as in the proof of Proposition 5.2, we estimate

$$\left|\frac{1}{\omega_n}\int_{\mathcal{A}} (|v_n|^2-1) j(u_n^\star)\cdot\nabla^\perp\varphi\right| \le C\frac{\varepsilon_n}{\omega_n}\|j(u_n^\star)\|_{L^\infty(\mathcal{A})}\sqrt{E_{\varepsilon_n}(v_n)} \le C\frac{\varepsilon_n\Omega_n^2}{\omega_n} = o(1),$$

where the constant C only depends on φ. □

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