Introduction: why model theory?

- Here is a concrete example: Suppose V is an algebraic variety and f is an injective morphism from V to V.
  Claim: f is surjective.
- On the face of it, this doesn't look like logic - it looks like algebraic geometry. Let's give a proof.
- After unravelling the definitions, we can assume that V is the zero set of some finite collection of polynomials over C. Moreover, f is given by complex rational maps. This is to say everything can be expressed in the language of fields.
- Suppose we ask the same question over a finite field instead of the complex numbers. Are injective maps surjective? Yes, by the pigeonhole principle!

Introduction cont'd

- But this property ”injective implies surjective” also holds for unions of finite fields in this context. So the property in question holds for algebraically closed fields of finite characteristic.
- The limit of algebraically closed fields of arbitrarily large finite characteristic is an algebraically closed field of characteristic 0 - this is a use of either compactness or ultraproducts - so the same property holds for some algebraically closed field of characteristic 0.
- Finally, the complex numbers are an algebraically closed field of characteristic 0 and all such fields satisfy the same properties expressible in the language of fields. So all injective morphisms from a variety to itself are surjective.

Introduction cont'd

- What did we use here that was model theory?
- We identified a property that was expressible in a well-chosen language. Said another way, we found a language suitable for the interesting property.
- We were able to determine the properties that held in the relevant models in this language - we knew what the theory of algebraically closed fields looked like.
- We were able to conclude facts about one model (the complex numbers) by looking at other models. The techniques involved here - unions of chains, some combinatorial reasoning, compactness - are not difficult but need to be used in the right context.

Filters and Ultrafilters

Definition

If X is a set and $F \subseteq \mathcal{P}(X)$ then F is said to be a filter if

- $\emptyset \notin F$,
- if $A, B \in F$ then $A \cap B \in F$, and
- if $A \in F$ and $A \subseteq B \subseteq X$ then $B \in F$.

Lemma

$G \subseteq \mathcal{P}(X)$ is contained in a filter iff G has the finite intersection property i.e. for every finite $G_0 \subseteq G$, $\bigcap G_0 \neq \emptyset$.

Definition

An ultrafilter on X is a filter F such that for every $A \subseteq X$, either $A \in F$ or $X \setminus A \in F$.

Filters and Ultrafilters, cont'd

Lemma

- *If F is a filter on X then F is an ultrafilter iff it is a maximal filter.*
- *Any filter on X can be extended to an ultrafilter.*

Examples: Suppose that X is a set.

- If $a \in X$ then $U = \{A \in \mathcal{P}(X) : a \in A\}$ is an ultrafilter; ultrafilters of this kind are called principal.
- If X is infinite, the set of cofinite subsets of X is a filter called the Frechet filter on X; it is contained in all non-principal ultrafilters on X.
- Let $Y = \mathcal{P}_{fin}(X)$ be the set of finite subsets of X. For any finite subset A of X, let $O_A = \{B \in Y : A \subseteq B\}$. The set $F = \{O_A : A \in Y\}$ has the finite intersection property and is not contained in a principal ultrafilter.

Ultralimits

Now suppose U is an ultrafilter on a set I and $\bar{r} = \langle r_i : i \in I \rangle$ is an I-indexed family of real numbers. We define the ultralimit of $\bar{r}$ with respect to U as follows:

$$\lim_{i \to U} r_i = r \text{ iff for every } \epsilon > 0, \{i \in I : |r - r_i| < \epsilon\} \in U$$

Lemma

If $\bar{r}$ is bounded then

- $\lim_{i \to U} r_i$ *exists and is unique;*
- $\lim_{i \to U} r_i = \inf\{B : \{i \in I : r_i \leq B\} \in U\};$
- $\lim_{i \to U} r_i = \sup\{B : \{i \in I : r_i \geq B\} \in U\}$

Ultraproducts of metric spaces

Fix an index set I, an ultrafilter U and metric spaces (X_i, d_i) for $i \in I$ with a uniform bound on the metrics i.e. there is some B so that for all i and all $x, y \in X_i$, $d_i(x, y) \leq B$. Define d on $\prod_{i \in I} X_i$ as follows:

$$d(\bar{x}, \bar{y}) = \lim_{i \to U} d_i(x_i, y_i)$$

Lemma

d is a pseudo-metric on $\prod_{i \in I} X_i$.

Ultraproducts of metric spaces, cont'd

Definition

The ultraproduct of the X_i's with respect to U, $\prod_{i \in I} X_i / U$ is the metric space obtained by quotienting $\prod_{i \in I} X_i$ by d. If all the X_i's are equal to a fixed X we will often write X^U for this ultraproduct and call it the ultrapower.

Exercises

- Show that for any I and ultrafilter U on I, $[0,1]^U \cong [0,1]$. More generally, show that for a compact metric space X, $X^U \cong X$.
- Show that if each X_i is complete then $\prod_{i \in I} X_i/U$ is complete.
- Show for any metric spaces X_n for $n \in N$, $\prod_{n \in N} X_n/U$ is complete.
- Show that this definition of ultraproduct is the same as the discrete or set-theoretic ultraproduct i.e. suppose that X_i has the discrete metric and compute the ultraproduct.

Metric structures

- We want to add more structure to a (bounded) metric space; for now let's consider a single additional function f.
- So we will have a bounded metric space (X,d) and a function f say of one variable. We do want that the ultraproduct of these structures is still a structure of the same kind. So how do we define f on the ultrapower of X?
- f must be continuous!
- f must be uniformly continuous!
- There is nothing special about one variable; these arguments apply to functions of many variables.

Metric structures cont'd

- What about relations? Imagine that we have a one-variable relation R (taking values somewhere) on a metric space and we want to make sense of it in the ultrapower.
- Its range must be compact and R must be uniformly continuous.
- There is really no loss in assume that the range of R is $[0, 1]$ or some other compact interval in the reals.
- Again there is nothing special about one-variable; we can have relations of many variables.

The language of a metric structure

A language L will consist of

- a set S called sorts;
- $\mathcal{F}$, a family of function symbols. For each $f \in \mathcal{F}$ we specify the domain and range of f: $dom(f) = \prod_{i=1}^{n} s_i$ where $s_1, \ldots, s_n \in S$ and $rng(f) = s$ where $s \in S$. Moreover, we also specify a continuity modulus. That is, for each i we are given $\delta_i^f : [0, 1] \to [0, 1]$; and
- $\mathcal{R}$, a family of relation symbols. For each $R \in \mathcal{R}$ we are given the domain $dom(R) = \prod_{i=1}^{n} s_i$ where $s_1, \ldots, s_n \in S$ and the $rng(R) = K_R$ for some closed interval K_R. Moreover, for each i, we specify a continuity modulus $\delta_i^R : [0, 1] \to [0, 1]$.
- For each $s \in S$, we have one special relation symbol d_s with domain $s \times s$ and range of the form $[0, B_s]$. It's continuity moduli are the identity functions.

Definition of a metric structures

A metric structure M *interprets* a language L; it will consist of

- an S-indexed family of complete bounded metric spaces (X_s, d_s) for $s \in S$;
- a family of functions f^M for every $f \in \mathcal{F}$ such that $dom(f^M) = \prod_{i=1}^{n} X_{s_i}$ for the sequence of sorts corresponding to the domain of f and $rng(f^M) = X_s$ for the sort corresponding to the range of f. f^M is uniformly continuous as specified by the uniform continuity moduli associated to f; and
- a family of relations R^M for every $R \in \mathcal{R}$ such that $dom(R^M) = \prod_{i=1}^{n} X_{s_i}$ for the sequence of sorts corresponding to the domain of R and $rng(R^M) = K_R$ for the closed interval associate to R. R^M is uniformly continuous as specified by the uniform continuity moduli associated to R.

Examples of metric structures

Some simple examples:

- Any complete bounded metric space (X, d). This has the empty family of functions and relations although we often count the metric as a relation (why is it uniformly continuous?)
- Any ordinary first order structure M with some collection of functions and relations. To see this as a metric structure, we put the discrete metric on M to make it a bounded metric space. All functions become uniformly continuous. Relations which are usually thought of as subsets of M^n become $\{0, 1\}$-valued functions - again they are uniformly continuous.

Hilbert space

- A Hilbert space H is a complete complex inner product space; how can we see this as a metric structure?
- Let B_n be the ball of radius n centered at the origin in H; B_n is a bounded complete metric space with respect to the metric induced from the inner product.
- There are inclusion maps between B_n and B_m if $n \leq m$.
- 0 is a constant (our functions can have arity 0!) in B_1.
- For complex numbers λ and for every n, there is a unary function λ_n which is scalar multiplication by λ on B_n; this function has range in B_m where m is the least integer greater than or equal to $n|\lambda|$.
- The operation of addition has to be similarly divided up: for $m, n \in N$, there is an operation $+_{m,n}$ which takes $B_m \times B_n$ to B_{m+n}.

Hilbert space, cont'd

- The inner product is complex valued which is an additional issue. Besides dividing it up so that there is a relation defined on each product $B_m \times B_n$, we also have to separate this relation into its real and complex parts.
- So formally a Hilbert space can be thought of as a metric structure by considering
  - The family of bounded metric structures B_n for all $n \in N$;
  - the family of functions 0, λ_n for $\lambda \in C$ and $n \in N$ and $+_{m,n}$ for all $m, n \in N$; and
  - the family of relations $re(\langle -, - \rangle)_{m,n}$ and $im(\langle -, - \rangle)_{m,n}$ for $m, n \in N$.

A calculation

- Suppose that (X_i, d_i) are uniformly bounded metric spaces for all $i \in I$, U is an ultrafilter on I and f_i is an n-ary uniformly continuous relation with a fixed uniform continuity modulus for all $i \in I$ and range in K, a compact interval.
- Claim: Suppose that (Y, d, f) is the ultraproduct $\prod_{i \in I}(X_i, d_i, f_i)/U$ and $\bar{a}_2, \ldots, \bar{a}_n \in Y$ then

$$\sup_{x \in Y} f(x, \bar{a}_2, \ldots, \bar{a}_n) = \lim_{i \to U} \sup_{x \in X_i} f_i(x, a_2^i, \ldots, a_n^i)$$