A metric on formulas

Fix a language L and fix a tuple of variables $\bar{x}$ from a sequence of sorts $\bar{s}$. We define a pseudo-metric on the formulas with free variables $\bar{x}$ as follows: we define the distance between $\varphi(\bar{x})$ and $\psi(\bar{x})$ to be

$$\sup\{|\varphi^M(\bar{a}) - \psi^M(\bar{a})| : M, \text{ an } L\text{-structure, and } \bar{a} \in M\}$$

We will call this space $\mathcal{F}_{\bar{s}}$.
Exercise: Check that this is a pseudo-metric on the set of formulas in the free variables $\bar{x}$.

Density character

Definition

We say that the density character of a topological space X is the infinum of the cardinality of a dense subset of X. We will write $\chi(X)$ for the density character of X.

Note: An infinite separable space has countable density character.

Proposition

If L is countable i.e. there are only countably many relation and function symbols, then for any tuple of sorts $\bar{s}$, $\mathcal{F}_{\bar{s}}$ is separable.

Notation: $\chi(L)$ will mean $\sum_{\bar{s}} \chi(\mathcal{F}_{\bar{s}})$.

Exercise

In the Henkin construction, one could have worked with only a dense subset of formulas; convince yourself that the outcome could have been improved to say that if L was countable and Σ was a finitely satisfiable set then there is a separable model of Σ.

Elementary submodels

Definition

- Suppose that M and N are L-structures such that the universe of M is a complete subset of N. M is called a submodel if all functions and relations from L on M are the restriction of those from N i.e. for all functions $f \in L$, $f^M = f^N \cap M$ and for all relations $R \in L$, $R^M = R^N \cap M$. We write $M \subseteq N$.
- If $M \subseteq N$, it is called an elementary submodel if, for every L-formula $\varphi(\bar{x})$ and every $\bar{a} \in M$, $\varphi^M(\bar{a}) = \varphi^N(\bar{a})$. We write $M \prec N$.

Embeddings

- An embedding between metric structures is a map which preserves the functions and relations. An embedding is elementary if its image is an elementary submodel of the range.
- Notice that by Łoś Theorem, any metric structure M embeds elementarily into its ultrapower M^U for any ultrafilter U via the diagonal embedding.

Downward Löwenheim-Skolem

Proposition (Vaught's test)

If $M \subseteq N$ then M is an elementary submodel if for every formula $\varphi(x, \bar{y})$, $r \in R$ and $\bar{a} \in M$, if $\inf_x \varphi(x, \bar{a}) < r$ holds in N then there is $b \in M$ such that $\varphi(b, \bar{a}) < r$ holds in M.

Theorem

Suppose that N is an L-structure and $A \subseteq N$. Then there is an elementary submodel $M \subseteq N$ such that

1. *$A \subseteq M$ and*
2. *for every sort s,*

$$\chi(X_s^M) \leq \chi(L) + \chi(A \cap X_s^N)$$

Types

Definition

1. A condition is an expression of the form $\varphi(\bar{x}) \leq r$ or $\varphi(\bar{x}) \geq r$ for a formula φ and a real number r.
2. A type in the variables $\bar{x}$ is a set of conditions involving formulas with free variables $\bar{x}$.
3. A type p in the variables $\bar{x}$ is realized if there is a metric structure M and $\bar{a} \in M$ such that $\bar{a}$ satisfies every condition in p e.g. if $\varphi(\bar{x}) \leq r$ is in p then $\varphi^M(\bar{a}) \leq r$ holds.
4. If $\bar{a} \in M$ is a tuple from sorts $\bar{s}$ and $\bar{x}$ is a tuple of matching variables then we define $tp(\bar{a})$, the type of $\bar{a}$, to be the type in variable $\bar{x}$ given by

$$\{\varphi(\bar{x}) \leq r : \varphi^M(\bar{a}) \leq r\} \cup \{\varphi(\bar{x}) \geq r : \varphi^M(\bar{a}) \geq r\}$$

A type of this kind is called a complete type.

Facts about types

Fact

1. *A type is realized iff it is finitely satisfiable.*
2. *A type is complete iff it is maximal and finitely satisfiable.*

Fact

A complete type p in the variables $\bar{x}$ determines a function from formulas in the free variables $\bar{x}$ to the reals defined by

$$\varphi \mapsto p^{\varphi} := \varphi^{M}(\bar{a})$$

where M is a metric structure and $\bar{a} \in M$ realizes p. This is well-defined and does not depend on the choice of M or $\bar{a}$.

A topology on the type space

We fix a language L and a complete theory T in this language. Equivalently we fix a metric structure M for the language L and let $T = Th(M)$. For a tuple of sorts $\bar{s}$ from L and matching variables $\bar{x}$ we define the set $S_{\bar{s}}(T)$ to be all complete types in the variables $\bar{x}$ realized in models of T.
We put a topology on $S_{\bar{s}}(T)$ by letting the basic open sets be defined as follows: for every formula $\varphi(\bar{x})$ and real number r, let

$$O_{\varphi,r} = \{p \in S_{\bar{s}}(T) : p^{\varphi} < r\}$$

This is called the logic topology on the type space.

Fact

The logic topology on $S_{\bar{s}}(T)$ is compact.

What is a formula?

Proposition

If $\varphi(\bar{x})$ is a formula then the function f_φ from $S_{\bar{s}}(T)$ to R given by $p \mapsto p^\varphi$ is continuous with the logic topology on the domain.

Proposition

The following are equivalent:

1. *f is a continuous function from $S_{\bar{s}}(T)$ to R.*
2. *f is the uniform limit of functions of the form f_φ i.e. for every n there is a formula φ_n such that $|f(p) - p^{\varphi_n}| \leq 1/n$.*

Definition

A Cauchy sequence of formulas $\bar{\varphi}$ in $\mathcal{F}_{\bar{s}}$ will be called a definable predicate and interpreted in an L-structure M by

$$\bar{\varphi}^M(\bar{a}) = \lim_{n \to \infty} \varphi_n^M(\bar{a})$$

Definable predicates are essentially formulas

- Suppose that $P(\bar{x})$ is a definable predicate. There is a unique way of extending a model of T to interpret P.
- That is to say, the map sending M whose theory is T to $(M, P(M))$ is functorial so if $M \prec N$ then $(M, P(M)) \prec (N, P(N))$.
- Expanding a metric structure by a definable predicate is a conservative extension.

A metric on the type space

- Define a metric on $S_{\bar{s}}(T)$ as follows: for $p, q \in S_{\bar{s}}(T)$, $d(p, q)$ is defined to be the infinum of $d^M(\bar{a}, \bar{b})$ where M ranges over all models of T, $\bar{a} \in M$ is a realization of p and $\bar{b} \in M$ is a realization of q. d is computed as the maximum of the values d_s as s ranges over the sorts in $\bar{s}$.
- Claim: d defines a metric on $S_{\bar{s}}(T)$.
- Notice that $d(p, q)$ is always realized - this follows by compactness.
- The only issue is the triangle inequality - another use of compactness.

Proposition

The metric topology on $S_{\bar{s}}(T)$ refines the logic topology.

Mysterious question

- When do the metric and logic topologies on $S_{\bar{s}}(T)$ coincide locally?
- Unravelling this a little bit, one sees that we are asking when the distance to a type is in some way defined by conditions at least approximately.

Zero sets and distance predicates

- A zero set is the set of realizations of a type i.e. if p is a type and M is an L-structure, we call the set of tuples $\bar{a} \in M$ which satisfy all the conditions in p the zero set of p.
- This looks like strange terminology - let me explain.
- If M is a metric space and X is a closed subset we call $P(x) = d(x, X) = \inf\{d(x, y) : y \in X\}$ a distance predicate for X.
- We call the zero set X in M of some type p a definable set if the distance predicate for X is a definable predicate.

Definable sets

Proposition (Mysterious answer)

p is a definable set iff the logic and metric topologies agree locally at p.

Proposition (MTFMS, 9.19)

The following are equivalent:

1. *p is definable.*
2. *There are formulas φ_m and numbers $\delta_m > 0$ such that for every m, $p^{\varphi_m} = 0$*

$$\text{if ``}\varphi(\bar{x}) \leq \delta_m\text{'' is in } q \text{ then } d(p, q) \leq \frac{1}{m}$$

A useful lemma

Lemma (MTFMS, 2.10)

Suppose that $F, G : X \to [0,1]$ *are functions such that*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ (F(x) \leq \delta \implies G(x) \leq \epsilon)$$

Then there exists an increasing, continuous function $\alpha : [0,1] \to [0,1]$ *such that* $\alpha(0) = 0$ *and*

$$\forall x \in X \ (G(x) \leq \alpha(F(x)))$$