Ehrenfeucht-Fraïssé games revised

- Fix $\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x})$ atomic formulas in the variables $x_1, \ldots, x_n$ and $\epsilon > 0$
- The EF-game of length n with respect to this data is played as follows:
- Player 1 chooses either $a_1 \in M$ or $b_1 \in N$ respecting the sort of x_1; player 2 chooses $b_2 \in N$ or $a_2 \in M$ respectively.
- Player 1 and Player 2 alternate in this manner until they have produced two sequences $a_1, \ldots, a_n \in M$ and $b_1, \ldots, b_n \in N$.
- Player 2 wins if for all i, $|\varphi_i(\bar{a}) - \varphi_i(\bar{b})| \leqslant \epsilon$.

Theorem

$M \equiv N$ iff Player 2 has a winning strategy for all EF-games.

Diagrams

Definition

- Suppose that M is a metric structure in a language L and that $A \subseteq M$. The language L_A is L together with a new constant symbol for each $a \in A$. M can be canonically expanded to a structure in this language by letting a name its constant.
- The atomic diagram of M, $Diag_{at}(M)$, is the theory in the language L_M containing the conditions $\varphi(\bar{a}) \leqslant r + 1/n$ for all $r \in R$, $n \in N$ and atomic formulas φ such that $\varphi^M(\bar{a}) \leqslant r$.
- The elementary diagram of M, $Diag_{el}(M)$, is the theory in the language L_M containing the conditions $\varphi(\bar{a}) \leqslant r + 1/n$ for all $r \in R$, $n \in N$ and any formulas φ such that $\varphi^M(\bar{a}) \leqslant r$.

Diagrams and types

Proposition

- $N \models Diag_{at}(M)$ *iff* M *embeds into* N.
- $N \models Diag_{el}(M)$ *iff* M *elementarily embeds into* N.

Type-space notation

Suppose M is a metric structure and $A \subseteq M$. Fix a tuple of sorts $\bar{s}$. Then $S_{\bar{s}}^{M}(A)$ is the collection of all complete types in some fixed tuple of variables from the sorts $\bar{s}$ in the language L_A which are approximately finitely satisfied in M. We will often omit the superscript and subscript if they are clear from context.

Saturated models

Definition

Fix an infinite cardinal κ.

- M is κ-saturated if for all $A \subseteq M$ such that $|A| < \kappa$ and $p \in S(A)$, p is realized in M.
- M is saturated if M is $\chi(M)$-saturated.
- M is κ-universal if whenever $N \equiv M$ and $\chi(N) < \kappa$ then N embeds into M elementarily.
- M is κ-homogeneous if whenever $\bar{a}$ and $\bar{b}$ are $< \kappa$-sequences of the same length and $(M, \bar{a}) \equiv (M, \bar{b})$ then for all $a \in M$, there is $b \in M$ such that $(M, \bar{a}, a) \equiv (M, \bar{b}, b)$.

Proposition

M is κ-saturated iff it is κ-universal and κ-homogeneous.

Saturated models, cont'd

Proposition

Given any κ and model M with $\kappa, \chi(M) \geqslant \chi(L)$, there is N, $M \prec N$ such that N is κ^+-saturated and $\chi(N) \leqslant \chi(M)^\kappa$.

- Sketch of proof: Start with M and form a chain of models M_α for $\alpha < \kappa^+$.
- Make sure that at each stage the density character is $\leqslant \chi(M)^\kappa$.
- This is possible because to start there will be at most $\chi(M)^\kappa$ many subsets of size κ to worry about and at most 2^κ many types over each set.
- By using elementary diagrams and downward Lowenheim-Skolem, we will be able to realize all these types without making the density character go above $\chi(M)^\kappa$.

Saturated models, cont'd

- Note that with a little help from cardinal arithmetic, we can have saturated models. For instance, if $2^{\aleph_0} = \aleph_1$, then any separable model can be extended to a saturated model of density character $\aleph_1$.

Theorem

If M_n for $n \in N$ are L-structures for a separable language L then $\prod_{n \in N} M_n/U$ is $\aleph_1$-saturated.

- Proof: Suppose that A is a countable subset of $\prod_{n \in N} M_n/U$ and $p \in S(A)$.
- Since L is separable, p is determined by countably many conditions $\varphi_i(\bar{x}) \leqslant r_i$ for $i \in N$.

Saturated models, cont'd

- Since this type is approximately finitely satisfied in $\prod_{n \in N} M_n/U$, we can fix $U_k \in U$ such that
  1. $U_1 \supseteq U_2 \supseteq U_3 \ldots$,
  2. $\bigcap_{k \in N} U_k = \emptyset$, and
  3. for every $j \in U_k$, M_j satisfies $\inf_{\bar{x}} \varphi_i(\bar{x}) \leqslant r_i$ for all $i \leqslant k$.
- Now define a tuple $\bar{a}_j$; if $j \notin U_1$ then define it arbitrarily. Otherwise, if $j \in U_k \backslash U_{k+1}$ then fix $\bar{b} \in M_j$ such that $\varphi_i(\bar{b}) \leqslant r_i + 1/k$ for all $i \leqslant k$ and let $\bar{a}_j = \bar{b}$.
- Exercise: $\bar{a}$ in $\prod_{n \in N} M_n/U$ realizes p. The point is that for every $j \in U_k$, $\varphi_i(\bar{a}_j) \leqslant r_i + 1/k$ for all $i \leqslant k$ (it could be better) and so $\varphi_i(\bar{a}) \leqslant r_i$ in $\prod_{n \in N} M_n/U$.

Atomic models

Definition

- A model M is atomic if all types realized in M are principal.
- A model M is prime if whenever $M \equiv N$ then M embeds elementarily into N.

Proposition

If L is a separable language and M is a prime model then M is atomic.

- Proof: Omitting types.

Theorem

If L is a separable language and M is a separable atomic L-structure then M is prime and unique up to isomorphism.

Proof of theorem

- First we will show that if $N \equiv M$ and N is separable and atomic then $M \cong N$.
- We construct two sequences

$$a^0_0, a^1_0 a^1_1, a^2_0 a^2_1 a^2_2, \ldots$$

in M and

$$b^0_0, b^1_0 b^1_1, b^2_0 b^2_1 b^2_2, \ldots$$

in N such that

1. all initials segments of the same length have the same type i.e. for any k there is a fixed type for $a^n_0 \ldots a^n_k$ and $b^n_0 \ldots b^n_k$ independent of $n \geqslant k$.
2. for every k, $\langle a^n_k : n \geqslant k \rangle$ and $\langle b^n_k : n \geqslant k \rangle$ form Cauchy sequences converging to $\mathbf{a_k}$ and $\mathbf{b_k}$ respectively.
3. $\{\mathbf{a_k} : \mathbf{k} \in \mathbf{N}\}$ and $\{\mathbf{b_k} : \mathbf{k} \in \mathbf{N}\}$ are dense in M and N respectively.

- If we can achieve this then the map sending $\mathbf{a_k}$ to $\mathbf{b_k}$ extends to an isomorphism from M to N.

Proof of theorem, cont'd

- To start, we enumerate countable dense subsets in M and N; call them $\langle c_k : k \in N \rangle$ and $\langle d_k : k \in N \rangle$.
- At stage 0, let $a_0^0 = c_0$. By atomicity, the type of c_0 is principal and hence realized in N by some b_0^0.
- In general, we alternate steps either choosing a c_k or d_k and we revisit each c_k and d_k infinitely often in the construction.
- Assume we have chosen $a_0^n \ldots a_n^n$ already and we consider whatever c_k is given to us at this stage.
- Let $p(x_0, \ldots, x_n)$ be the type of $a_0^n \ldots a_n^n$ and $q(x_0, \ldots, x_{n+1})$ be the type of $a_0^n \ldots a_n^n c_k$.
- Suppose that $d_q(x_0, \ldots, x_{n+1})$ is the distance function to the zero set of the type q - remember q is principal.
- So $d_q^M(a_0^n \ldots a_n^n, c_k) = 0$ which means $\inf_y d_q^M(a_0^n \ldots a_n^n, y) = 0$.

Proof of theorem, cont'd

- $b_0^n \ldots b_n^n$ satisfies p by assumption so $\inf_y d_q^M(b_0^n \ldots b_n^n, y) = 0$.
- This means we can find $b_0^{n+1} \ldots b_{n+1}^{n+1}$ realizing q and such that $d(b_i^n, b_i^{n+1}) \leqslant 1/2^n$ for $i = 0, \ldots, n$.
- This guarantees we have the required Cauchy sequences and we have the required density as well.
- If $\epsilon > 0$, choose N large enough so that $\sum_{n \geqslant N} 1/2^n < \epsilon$. Suppose we visit c_k at stage $t > N$.
- Then $\mathbf{a_t}$ is within ϵ of c_k and so the $\mathbf{a_k}$'s are dense in M. Similarly, the $\mathbf{b_k}$'s are dense in N.
- This shows $M \cong N$.
- To see that if M is separable and atomic then M is prime, we use the same argument but only in the forth direction.

Imaginaries: the discrete case, version 1

- Fix a complete theory T in a language L.
- Suppose that $E(\bar{x}, \bar{y})$ is an L-formula that defines an equivalence relation in models of T.
- Form a new language $L_E = L \cup \{S_E, \pi_E\}$ where S_E is a new sort and π_E is a new function symbols with domain the sorts of the variables $\bar{x}$ and range S_E.
- If $M \models T$ then we expand it to a model M_E of L_E by letting S_E be the equivalence classes of E in M and π_E the projection from appropriate tuples to their equivalence class. We let $T_E = Th(M_E)$.
- We consider the class of models of a first order theory as a category with the models as objects and elementary maps as the morphisms.

Imaginaries: the discrete case, version 1, cont'd

- There is a forgetful functor $F : Mod(T_E) \to Mod(T)$ which is just the reduct of the L_E structures to L. We also have the functor which sends M to M_E which goes in the other direction. This pair is an equivalence of categories; that is:
  1. $F(M_E) \cong M$ (in fact equals M),
  2. $F(N)_E \cong N$, and
  3. $F : Hom(N, N') \to Hom(F(N), F(N'))$ is a bijection for all $N, N' \in Mod(T_E)$.
- One says that T_E is a conservative extension of T.

Imaginaries: the discrete case, version 2

- Suppose that $\varphi(\bar{x}, \bar{y})$ is an L-formula and $\bar{x}$ and $\bar{y}$ needn't have equal length.
- Form a new language $L_\varphi = L \cup \{S_\varphi, \pi_\varphi\}$ where S_φ is a new sort and π_φ is a new function symbols with domain the sorts of the variables $\bar{y}$ and range S_φ.
- Consider the formula $E_\varphi(\bar{y}, \bar{y}') := \forall \bar{x}(\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{y}'))$; this is an equivalence relation in all L-structures.
- If $M \models T$ then we expand it to a model M_φ of L_φ by letting S_φ be the equivalence classes of E_φ in M and π_φ the projection from appropriate tuples to their equivalence class. We let $T_\varphi = Th(M_\varphi)$.
- The forgetful functor $F : Mod(T_\varphi) \to Mod(T)$ is an equivalence of categories and T_φ is a conservative extension of T.

Imaginaries: the discrete case, version 2, cont'd

- T_φ looks like a more general construction but it is not.
- What this construction does is create an element in S_φ for every definable set of the form $\varphi(M, \bar{a})$. This is often called adding *canonical parameters* for the following reason:
- Suppose that M is a saturated model of T. Then for all automorphisms f of M, f fixes $\varphi(M, \bar{a})$ setwise iff f fixes $\bar{a}/E_\varphi$ (f induces a unique automorphism of M_φ which extends f).
- Iterating either version of this construction over all possible formulas (or equivalence relations) leads to a theory called T^{eq} which is essentially closed under the addition of canonical parameters. It has a special place among the conservatives extensions of T; we will look at this next week.

Imaginaries: the continuous case, canonical parameters

- Fix a complete theory T in a continuous language L and fix a formula $\varphi(\bar{x}, \bar{y})$.
- Consider the formula $\rho_\varphi(\bar{y}, \bar{y}') := \sup_{\bar{x}} |\varphi(\bar{x}, \bar{y}) - \varphi(\bar{x}, \bar{y}')|$.
- ρ_φ defines a pseudo-metric on the product of the sorts corresponding to the $\bar{y}$ variables in all L-structures and $\rho_\varphi(\bar{y}, \bar{y}') = 0$ means $\varphi(\bar{x}, \bar{y})$ and $\varphi(\bar{x}, \bar{y}')$ define the same function of the $\bar{x}$-variables.
- We consider $L_\varphi = L \cup \{S_\varphi, d_\varphi, \pi_\varphi\}$ where S_φ is a new sort, d_φ is its metric symbol and π_φ is a function from the sorts of the $\bar{y}$ variables to S_φ. The uniform continuity modulus for π_φ is the same as the uniform continuity modulus for the $\bar{y}$ variables in φ.

Imaginaries: the continuous case, canonical parameters, cont'd

- If M is a model of T and $X(M)$ is the product of the sorts corresponding to the $\bar{y}$ variables the ρ_φ is a pseudo-metric on $X(M)$. We define an expansion M_φ of M to L_φ by letting $S_\varphi(M_\varphi) = X(M)/\rho_\varphi$ and d_φ is the induced metric; π_φ is the projection from $X(M)$ to $S_\varphi(M_\varphi)$.
- We let $T_\varphi = Th(M_\varphi)$ and again there is a forgetful function from $Mod(T_\varphi)$ to $Mod(T)$. The question is: if N is a model of T_φ and $M = F(N)$ then why is $N \cong M_\varphi$?
- T_φ knows the following information: for all $m, m' \in X(M)$,

$$d_\varphi(\pi_\varphi(m), \pi_\varphi(m')) = \rho_\varphi(m, m')$$

and that π_φ is surjective.

Imaginaries: the continuous case, canonical parameters, cont'd

- These facts guarantee that the map $i : S_\varphi(N) \to X(M)/\rho_\varphi$ given by

$$i(n) = \pi_\varphi^{M_\varphi}(m) \text{ for any } m \in X(M) \text{ such that } \pi_\varphi^N(m) = n$$

  is well-defined and a surjective isometry.
- The elements of the sort S_φ can be thought of as the canonical parameters asociated to the function $\varphi(\bar{x}, \bar{y})$ of the $\bar{x}$ variables when the $\bar{y}$ variables are fixed.

Imaginaries: the continuous case, products

- Fix a complete theory T in a continuous language L.
- Suppose $\bar{S} = \langle S_n : n \in N \rangle$ is a sequence of sorts from L. The goal is to create $\prod_{n \in N} S_n$ as a new sort.
- Take a model of T and let $X_{\bar{S}} = \prod_{n \in N} X_{S_n}(M)$. We need a metric on $X_{\bar{S}}$.
- Suppose d_i is the metric on S_i with bound B_i; let

$$d(\bar{x}, \bar{y}) = \sum_{i \in N} \frac{d_i(x_i, y_i)}{B_i 2^i}$$

  where $\bar{x}, \bar{y} \in X_{\bar{S}}(M)$.
- d is a metric on $X_{\bar{S}}(M)$ which is complete and bounded by 1.
- We have projection maps $\pi_i : X_{\bar{S}}(M) \to X_{S_i}(M)$ sending $\bar{x}$ to x_i.
- Notice that if $d(\bar{x}, \bar{y}) < \delta$ then $d_i(x_i, y_i) < B_i 2^i \delta$ so π_i is uniformly continuous.

Imaginaries: the continuous case, products cont'd

- Let $L_{\bar{S}} = L \cup \{S_{\bar{S}}, d_{\bar{S}}, \{\pi_i : i \in N\}\}$ where $S_{\bar{S}}$ is a new sort, $d_{\bar{S}}$ is its metric symbol and π_i is a function symbol with domain $S_{\bar{S}}$, range S_i and uniform continuity modulus given as above.
- The construction above shows how to take a model M of T and produce a model $M_{\bar{S}}$ of $L_{\bar{S}}$. Let $T_{\bar{S}} = Th(M_{\bar{S}})$.
- Once again we have a forgetful functor $F : Mod(T_{\bar{S}}) \to Mod(T)$ and we would like to see that it is an equivalence of categories.
- We need to see if $N \models T_{\bar{S}}$ and $M = F(N)$ then $M_{\bar{S}} \cong N$ fixing M.

Imaginaries: the continuous case, products cont'd

- For $n \in X_{\bar{S}}(N)$, let $\rho(n) = \langle \pi_i(n) : i \in N \rangle \in \prod_{i \in N} X_{\bar{S}_i}(M)$.
- If this map is a surjective isometry then it commutes with the π_i's and so is an isomorphism.
- Notice that follows from the theory $T_{\bar{S}}$ that for all $n, n' \in X_{\bar{S}}(N)$, and $k \in N$,

$$\left| d_{\bar{S}}(n, n') - \sum_{i \leqslant k} \frac{d_i(\pi_i(n), \pi_i(n'))}{B_i 2^i} \right| \leqslant \frac{1}{2^k}$$

which shows that ρ is an isometry.

Imaginaries: the continuous case, products cont'd

- It is also part of the theory that for any k

$$\sup_{x_1 \in S_1} \dots \sup_{x_k \in S_k} \inf_{y \in S_{\bar{S}}} \max\{d_i(x_i, \pi_i(y)) : i \leqslant k\}$$

evaluates to 0.
- By completeness of $X_{\bar{S}}(N)$, ρ is surjective.
- So $M_{\bar{S}} \cong N$ fixing M and $T_{\bar{S}}$ is a conservative extension of T.
- One issue is that the metric we defined is not canonical - there are other metrics we could have used. We will have to return to this.