The omitting types theorem

Proposition

A type p is principal iff the logic and metric topologies agree locally at p.

Theorem

Suppose that L is a separable language, T is a complete theory in L and p is a finitely satisfiable type. Then there is a model which omits p iff p is not principal.

Proof: Suppose that p is not principal. We will construct a model of T using a Henkin construction.

The omitting types theorem, cont'd

- Since the language is separable, we can accomplish this Henkin construction in countably many steps. The key issue will be to guarantee that every constant not only doesn't realize p but stays some uniform distance away from potential realizations of p so that in the completion, p will not be realized.
- Since p is not principal, we know that there is some ϵ so that the ball of radius ϵ around p does not contain any open set from the logic topology. That is, for every formula φ and every r, if $O_{\varphi,r}$ is not empty then it contains q such that $d(p, q) \geqslant \epsilon$.

The omitting types theorem, cont'd

- If we take the q from the previous line, we get that the set of conditions

$$p(x) \cup q(y) \cup \{d(x,y) \leqslant \epsilon/2\}$$

is not satisfiable. So by compactness, there is some condition $\psi \leqslant s$ in q such that $p(x) \cup \{\psi \leqslant s, d(x,y) \leqslant \epsilon/2\}$ is not satisfiable.

- By approximate finite satisfiability, we even know that there is some n such that $p(x) \cup \{\psi < s + 1/n, d(x,y) \leqslant \epsilon/2\}$ is not satisfiable.
- Now the general set-up for the Henkin construction will have us looking at finitely many conditions $\varphi_i(c, \bar{c}) < r_i$ which are finitely satisfiable with T; here we have highlighted the constant c which we want to guarantee will not satisfy anything close to p.

The omitting types theorem, cont'd

- We consider the intersection of the basic open sets given by $\inf_{\bar{y}} \varphi_i(x, \bar{y}) < r_i$ and obtain some formula $\psi(x)$ and number s such that any type q in

$$\cap_i O_{\inf_{\bar{y}} \varphi_i, r_i} \cap O_{\psi, s}$$

  must satisfy $d(p, q) \geqslant \epsilon/2$.
- This proof would work if you try to omit countably many non-principal types.

Another characterization of definable zero sets

Theorem

Suppose that M is a metric structure and $Z \subseteq M^n$ is a closed subset. Then the following are equivalent:

1. *Z is a definable zero set.*
2. *For any definable predicate P with domain $M^n \times M^m$, $Q(x) = \inf\{P(x,z) : z \in Z\}$ is a definable predicate.*

- From bottom to top, just let $P(x,y)$ be $d(x,y)$.
- In the other direction, P is uniformly continuous so using MTFMS 2.10 again, we can find continuous α such that $|P(x,z) - P(y,z)| \leqslant \alpha(d(x,y))$ for all $x \in M^m$. Consider the formula $\inf_z(P(x,z) + \alpha(d(z,Z)))$. We claim this is Q.
- The conclusion here is that definable zero sets are exactly those sets which you can quantify over.
- Particularly useful examples of definable sets are ranges of terms.

Quantifier elimination

Definition

We say that a theory T has quantifier elimination if for any formula $\varphi(\bar{x})$ and $\epsilon > 0$ there is a quantifier-free formula $\psi(\bar{x})$ such that

$$\sup_{\bar{x}} |\varphi(\bar{x}) - \psi(\bar{x})| \leqslant \epsilon$$

holds in all models of T.

Theorem

Suppose that T is a complete theory in a separable language. T has quantifier elimination iff whenever M and N are separable models of T, A is a finitely generated substructure of both M and N and U is a non-principal ultrafilter on $\mathbb{N}$ then M embeds into N^U fixing A.

Proof of the theorem

- From left to right: Fix a countable dense subset of M, $\bar{m} = \langle b_i : i \in \mathbb{N} \rangle$ and consider the type $tp(\bar{m}/A)$.
- This type is finitely satisfiable in M and any finite approximation to it is approximated by a quantifier free formula by quantifier elimination.
- So this type is also finitely satisfied in N which means it is realized in N^U and we get an embedding of M into N over A.
- From right to left: It is enough to show that any inf formula is approximated by a quantifier-free formula. Fix any such $\varphi(\bar{x})$.
- Consider Σ_ϵ,

$$\{|\psi(\bar{x}) - \psi(\bar{y})| \leqslant 1/n : \psi \text{ is qff}, n \in \mathbb{N}\} \cup \{|\varphi(\bar{x}) - \varphi(\bar{y})| \geqslant \epsilon\}$$

Proof of the theorem, cont'd

- If Σ_ϵ is finitely satisfiable for some ϵ we contradict our assumption; here is how:
- If Σ_ϵ is not finitely satisfiable for any ϵ then the proof ends by the Stone-Weierstrass Theorem.
- In more detail: if we consider types with only conditions involving inf-formulas or qffs then this space is compact via the similarly restricted logic topology.
- The failure of finite satisfiability of Σ_ϵ for every ϵ tells us that the qffs determine any function defined by an inf formula from this type space to $\mathbb{R}$.

Example 1: Urysohn space

- For this example we will only consider metric spaces with metrics bounded by 1.
- We say that a separable metric space X is universal if every separable metric space can be embedded into X; it is homogeneous if whenever f is a finite isometry on X, it can be extended to an automorphism.
- The goal is to construct a separable metric space which is both universal and homogeneous; the construction is the analogue of the Fraïssé construction for metric structures.
- We start with the class $\mathcal{C}$ of finite metric spaces whose metrics take rational values in $[0, 1]$.
- We describe free amalgamation for this class: Suppose that $A, B, C \in \mathcal{C}$ and $A \subseteq B, C$. The underlying set of the free amalgamation $B * C$ is $B \sqcup_A C$, the disjoint union of B and C over A.

Example 1: Urysohn space, cont'd

- To define the metric on the free amalgamation, we need only define the distance from elements of $B \setminus A$ to $C \setminus A$. Suppose b and c are in those sets respectively. Define $d(b, c)$ by

$$\min_{a \in A}(d(b, a) + d(a, c))$$

- Exercise: check that this defines a metric on the free amalgamation and that $B * C$ is in $\mathcal{C}$.
- We now construct a separable space $\mathcal{U}$, Urysohn space, by induction. I leave the details to you. The key point is that up to isomorphism, $\mathcal{C}$ contains only countably many objects.
- $\mathcal{U}$ is the completion of the metric space built as a countable union of a chain $X_0 \subseteq X_1 \subseteq X_2 \ldots$ such that each $X_i \in \mathcal{C}$ and for every finite $F \subseteq X_i$ and every $G \in \mathcal{C}$ such that $F \subseteq G$ there is $j > i$ such that $F * G$ embeds into X_j over F.

Example 1: Urysohn space, cont'd

- $\mathcal{U}$ as described is a metric structure in the language with only one sort and whose only relation symbol is the metric symbol.
- For every possible finite metric configuration $\bar{r} = r_{ij}$ for $1 \leqslant i, j \leqslant n$ there is a formula, $C_{\bar{r}}(\bar{x})$, the configuration formula for $\bar{r}$ written as

$$max_{i,j}|d(x_i, x_j) - r_{ij}|$$

  which measures how far a tuple $\bar{x}$ is from realizing the given configuration.
- Claim: Given a configuration $\bar{r}$ and a one-point extension $\bar{s}$, for every $\epsilon > 0$ there is a $\delta > 0$ such that if in $\mathcal{U}$, $C_{\bar{r}}(\bar{a}) < \delta$ then $inf_y C_{\bar{s}}(\bar{a}, y) \leqslant \epsilon$.

The theory of Urysohn space

- We can write the last claim in continuous logic as

$$\sup_{\bar{x}} min\{\delta \dot{-} C_{\bar{r}}(\bar{x}), \inf_{y} C_{\bar{s}}(\bar{x}, y) \dot{-} \epsilon\}$$

- Claim: If the value of these sentences are 0 in metric structures then they are elementarily equivalent. In fact, if these sentences are 0 in two separable structures then those structures are isomorphic.