From last time: dividing

Definition

We say that a satisfiable partial type $p(x) \in S(B)$ divides over A if there is an A-indiscernible sequence $\langle B_i : i \in \mathbb{N} \rangle$ such that $B_0 = B$ and $\bigcup \{p_{B_i} : i \in \mathbb{N}\}$ is not satisfiable.

- For any theory, we will define $A \downarrow_C B$ to mean $t(A/BC)$ does not divide over C. We want to show that in a stable theory this yields a stationary independence relation.
- In all theories, this independence relation satisfies invariance, finite character and left transitivity (and the weakening direction of transitivity).
- In a stable theory, dividing satisfies local character.
- We still need to deal with extension, symmetry and stationarity.

A couple of facts about dividing

Proposition

$a \downarrow_A B$ and $b \downarrow_{Aa} Ba$ implies $ab \downarrow_A B$ in all theories.

Corollary

If $\varphi(x, a)$ k-divides over A (the witnessing unsatisfiable set is k-inconsistent) and $b \downarrow_A Aa$ then $\varphi(x, a)$ k-divides over Ab.

- The proof in both cases follows from the technical lemma proved last time which led to left transitivity.

Another version of dividing

- In our original definition of dividing, we worked with a partial type and so implicitly we were dealing with conditions. For convenience of notation, we are going to assume that those conditions are "equal to zero" conditions. So we will speak about a formula φ dividing and mean $\varphi = 0$ divides. There is no loss of generality in the same way we can convert arbitrary conditions to ones only involving 0.

Definition

We say that a formula $\varphi(x, b)$ (k, ϵ)-divides over A if there is an A-indiscernible sequence I with $b \in I$ such that for any k-element subset $S \subseteq I$, $\sup_x \max_{a \in S} \varphi(x, a) \geqslant \epsilon$.

Topology of dividing

Lemma

If $\varphi(x, a)$ divides over A then there is an δ such that if $d(\varphi(x, a), \psi(x, a)) \leqslant \delta$ then $\psi(x, a)$ divides over A.

- Assume that $\varphi(x, a)$ (k, ϵ)-divides over A.
- Let $\delta = \epsilon/2$. One can then check that $\psi(x, a)$ $(k, \delta/2)$-divides over A via the same indiscernible sequence which witneses the (k, ϵ)-dividing.

Corollary

The set of formulas over B which do not divide over A form a closed set.

Dividing rank

- Now fix a finite set of formulas Δ with a fixed type variable x and possibly different parameters variables, an integer $k \geqslant 2$ and $\epsilon > 0$.
- We define a rank $D(\cdot, \Delta, k, \epsilon)$ inductively as follows:
- $D(\varphi(x, a), \Delta, k, \epsilon) \geqslant 0$ if $\varphi(x, a)$ is consistent i.e. we can find some b such that $\varphi(b, a) = 0$.
- $D(\varphi(x, a), \Delta, k, \epsilon) \geqslant n + 1$ if there is a $\psi \in \Delta$ and a b such that $\max(\psi(x, b), \varphi(x, a))$ (k, ϵ)-divides over a and $D(\max(\psi(x, b), \varphi(x, a)), \Delta, k, \epsilon) \geqslant n$.
- We say the rank is -1 if φ is inconsistent and ∞ if it is greater than for all .

General facts about rank

- Expressing the fact that $D(\varphi(x, a), \Delta, k, \epsilon) \geqslant n$ is type-definable.
- Various monotonicity results are true between the D-ranks:

1. If $\varphi \models \psi$ then $D(\varphi, \Delta, k, \epsilon) \leqslant D(\psi, \Delta, k, \epsilon)$.
2. If $\Delta \subseteq \Delta'$ then $D(\varphi, \Delta, k, \epsilon) \leqslant D(\varphi, \Delta', k, \epsilon)$.
3. If $k \leqslant k'$ then $D(\varphi, \Delta, k, \epsilon) \leqslant D(\varphi, \Delta, k', \epsilon)$.
4. If $\epsilon \leqslant \epsilon'$ then $D(\varphi, \Delta, k, \epsilon') \leqslant D(\varphi, \Delta, k, \epsilon)$.

Important Lemma

Lemma

Fix formulas $\pi, \varphi_1, \dots, \varphi_n$ and Δ, k and ϵ. Moreover assume that $D(\pi, \Delta, k, \epsilon) < \infty$. Then

$$D(\pi \wedge \bigvee_{i \leqslant n} \varphi_i, \Delta, k, \epsilon) = \bigvee_{i \leqslant n} D(\pi \wedge \varphi_i, \Delta, k, \epsilon)$$

where $\bigvee$ stands for max and $\wedge$ stands for min.

- This lemma is proved by induction on the *D*-rank and hence the need to assume that the rank is finite.
- The point is that for any given indiscernible sequence, one of the φ's is going to be the reason for dividing since there will be infinitely many indiscernibles and only finitely many φ's.

Stability and rank

Proposition

If dividing has local character in T then for all finite Δ, k and ϵ there is N which depends on Δ, k and ϵ such that for all φ

$$D(\varphi, \Delta, k, \epsilon) \leqslant N$$

- If not then since Δ is finite, there must be one $\psi \in \Delta$ which is responsible for the rank increasing infinitely often.
- By compactness, for any κ we can construct a chain of (k, ϵ)-dividing via ψ of length κ^+ and produce a type which divides over every κ-sized subset of its domain.

Corollary

If T is stable then all the D-ranks are finite.

Extension

Theorem

Assume dividing has local character. If $A \subseteq B$ and $\pi(x)$ is any partial type over B which does not divide over A then $\pi(x)$ can be extended to a complete type over B which does not divide over A.

- Proof: For every formula $\varphi(x, b)$ with parameters in B which divides over A fix $\delta_\varphi > 0$ such that $\varphi(x, b) \leqslant \delta_\varphi$ also divides over A by our earlier lemma.
- Now consider the set of formulas $\Sigma = \pi$ together with

$$\{\varphi(x, a) \geqslant \delta_\varphi : \varphi(x, a) \text{ divides over } A\}$$

- Suppose that Σ was not satisfiable. Then we would have $\pi \models \bigvee_{i \leqslant n} \varphi_i(x, a_i) \leqslant \delta_{\varphi_i}$ for some formulas $\varphi_i(x, a_i)$ which (k_i, ϵ_i)-divide over A.
- Consider $\Delta = \{\varphi(x, y_i) : i \leqslant n\}$, $k = \max_i k_i$ and $\epsilon = \min_i \epsilon_i$.

Extension, cont'd

- Then we have

$$\begin{aligned} D(\pi, \Delta, k, \epsilon) &= D(\pi \wedge \bigvee_i \varphi_i(x, a_i), \Delta, k, \epsilon) \\ &= \bigvee_i D(\pi \wedge \varphi_i(x, a_i), \Delta, k, \epsilon) \\ &< D(\pi, \Delta, k, \epsilon) \end{aligned}$$

  which is a contradiction since $D(\pi, \Delta, k, \epsilon)$ is finite by local character.

- Extending Σ to any complete type over B and by construction it must not divide over A.

Corollary (To the above proof)

If $A \subseteq B$ and $p \in S(B)$ divides over A then for some Δ, k and ϵ, $D(p, \Delta, k, \epsilon) < D(p\restriction_A, \Delta, k, \epsilon)$.

Morley sequences

Definition

We say a sequence order by $(I, <)$, $\langle a_i : i \in I \rangle$ is a Morley sequence over A if it is an A-indiscernible sequence and for every $i \in I$, $a_i \downarrow_A a_{<i}$ where $a_{<i} = \{a_j : j < i\}$. If $p = tp(a_i/A)$ for any $i \in I$ then this sequence is called a Morley sequence in p.

Proposition

If dividing satisfies local character then for any order type I and any $p \in S(A)$ there is an I-indexed Morley sequence in p.

- The proof here is a combination of the extension theorem, compactness and the Erdos-Rado theorem.

Morley sequences, cont'd

- First of all, we can find a sequence $\langle a_i : i < \kappa \rangle$ with the property that each a_i realizes p and for each i, $a_i \downarrow_A a_{<i}$. This follows from the extension theorem.
- Now write down a collection of statements that express the fact that we have an I-indexed Morley sequence; note that dividing is a type-definable property.
- Apply compactness and choose a sequence long enough from the first point in order to satisfy the finitely many formulas you have to deal with at each stage; for this you use Erdos-Rado.

A surprising theorem

Theorem

Assuming that dividing has local character, $\varphi(x, a)$ divides over A iff for some Morley sequence over A, I, with $a \in I$, $\{\varphi(x, b) : b \in I\}$ is not satisfiable.

- Left to right is clear so let's prove the other direction. Fix I as on the right.
- Suppose that κ is the cardinal associated with local character for dividing in this theory. Let J be a Morley sequence in $tp(a/A)$ ordered in the reverse ordering of κ^+ with $a \in J$ and ordered n-tuples from J have the same type over A as ordered n-tuples in I for all n. We can do this much like the proof of the existence of Morley sequences.
- Suppose that $\{\varphi(x, b) : b \in J\}$ is satisfiable. Choose c with realizes this partial type.
- By the local character of dividing, there is an $\alpha < \kappa^+$ such that $c \downarrow_A a_{<\alpha}$.

A surprising theorem, cont'd

- We claim that $a_{<\alpha} \downarrow_A a_\alpha$. This follows from the consequences of left transitivity.
- Since $\varphi(x, a_\alpha)$ divides over A, it also divides over $Aa_{<\alpha}$. Again, this is a general fact about dividing.
- But c realizes $\varphi(x, a_\alpha)$ and c doesn't divide over $Aa_{<\alpha}$ which is a contradiction (to the existence of c).
- So $\{\varphi(x, b) : b \in J\}$ is not k-satisfiable for some k which is then also true with I replacing J.

Theorem

If dividing has local character, dividing is symmetric.

Proof of symmetry

- Suppose that $a \downarrow_C b$ and we wish to show $b \downarrow_C a$.
- Fix a formula $\varphi(x, a)$ satisfied by b; we want to show that it does not divide over C.
- Take a Morley sequence I in the type over a over C that is also a Morley sequence in the type of a over Cb. A few words ...
- Since b satisfies $\varphi(x, a)$, it also satisfies $\varphi(x, a')$ for all $a' \in I$.
- So $\varphi(x, a)$ does not divide over C and $b \downarrow_C a$.

Corollary

If dividing satisfies local character then for $A \subseteq B$ and $q \in S(B)$, q does not divide over A iff for all Δ, k and ϵ, $D(q, \Delta, k, \epsilon) = D(q\restriction_A, \Delta, k, \epsilon)$.

Definability

Definition

We say that a type $p \in S(A)$ is definable if for every $\varphi(x, y)$ there is a definable predicate $\psi(y)$ over A such that for every $a \in A$, $p^{\varphi(x,a)} = \psi(a)$. We often call this ψ the φ-definition for p and write $d_p\varphi$.

Theorem

If T is stable then

1. *every type over a model is definable,*
2. *every type over a model has a unique definable extension to any superset of that model, and*
3. *any type definable over a model does not divide over that model.*

Definability, proofs

- The proof of the first follows from the fact that in stable theories, all types over models are finitely determined but I want to leave this proof until later when we have a slightly better proof.
- The proof of the second is by compactness using the fact that the type is over a model and so approximately finitely satisfiable in that model.
- Suppose that $q \in S(A)$ is definable over M. Fix $\varphi(x, a) \in q$ and consider an M-indiscernible sequence I with $a \in I$.
- Then since the φ-definition is over M say ψ, we have $\psi(b) = 0$ for all $b \in I$.
- This means that $\{\varphi(x, b) : b \in I\}$ is contained in the definable extension of $q{\restriction}_M$ and is hence consistent.

Lemma

Suppose T is stable and that $p \in S(M)$. The following are equivalent:

1. $p \cup \varphi(x, a)$ *is contained in a definable extension of* p.
2. *If* $q = t(a/M)$ *then* $d_q\varphi \in p$.
3. $p \cup \varphi(x, a)$ *does not divide over* M.

Stationarity, cont'd

Theorem

If T is stable then if $p \in S(M)$ and $M \subseteq A$ then p has a unique non-dividing extension to $S(A)$.

- Proof: We know that p has a definable extension to $S(A)$ and that it does not divide over M.
- Moreover, it is unique among definable extensions.
- By the previous lemma, if we are looking at an extension of p which is not definable then if divides over M.
- So the unique, definable extension of p is in fact the unique non-dividing extension.