A topology on the type space

We fix a language L and a complete theory T in this language. Equivalently we fix a metric structure M for the language L and let $T = Th(M)$. For a tuple of sorts $\bar{s}$ from L and matching variables $\bar{x}$ we define the set $S_{\bar{s}}(T)$ to be all complete types in the variables $\bar{x}$ realized in models of T.
We put a topology on $S_{\bar{s}}(T)$ by letting the basic open sets be defined as follows: for every formula $\varphi(\bar{x})$ and real number r, let

$$O_{\varphi,r} = \{p \in S_{\bar{s}}(T) : p^{\varphi} < r\}$$

This is called the logic topology on the type space.

Fact

The logic topology on $S_{\bar{s}}(T)$ is compact.

A metric on the type space

- Define a metric on $S_{\bar{s}}(T)$ as follows: for $p, q \in S_{\bar{s}}(T)$, $d(p, q)$ is defined to be the infinum of $d^M(\bar{a}, \bar{b})$ where M ranges over all models of T, $\bar{a} \in M$ is a realization of p and $\bar{b} \in M$ is a realization of q. d is computed as the maximum of the values d_s as s ranges over the sorts in $\bar{s}$.
- Claim: d defines a metric on $S_{\bar{s}}(T)$.
- Notice that $d(p, q)$ is always realized - this follows by compactness.
- The only issue is the triangle inequality - another use of compactness.

Proposition

The metric topology on $S_{\bar{s}}(T)$ refines the logic topology.

Mysterious question

- When do the metric and logic topologies on $S_{\bar{s}}(T)$ coincide locally?
- Unravelling this a little bit, one sees that we are asking when the distance to a type is in some way defined by conditions at least approximately.

Zero sets and distance predicates

- A zero set is the set of realizations of a type i.e. if p is a type and M is an L-structure, we call the set of tuples $\bar{a} \in M$ which satisfy all the conditions in p the zero set of p.
- This looks like strange terminology - let me explain.
- If M is a metric space and X is a non-empty closed subset we call $P(x) = d(x, X) = \inf\{d(x, y) : y \in X\}$ a distance predicate for X.
- We call the zero set X in M of some type p a definable zero set or principal if the distance predicate for X is a definable predicate (in M).

Zero sets and distance predicates, cont'd

- If $P(x) = d(x, p(M))$ is a definable predicate in M and $M \prec N$ then what does $P(x)$ define in N?
- We know $(M, P) \prec (N, P^N)$ since P is a definable predicate. The issue is: does $P^N = d(x, p(N))$ in N?
- This is really two questions:
  1. Is P^N the distance function to its zero set? and
  2. Is its zero set $p(N)$?
- The answer to the second question is: yes. Proof:

Zero sets and distance predicates, cont'd

Theorem (MTFMS, 9.12)

Suppose that (M, F) *is a metric structure which satisfies*

$$\sup_x \inf_y \max(|F(y)|, |F(x) - d(x,y)|) = 0$$

and

$$\sup_x |F(x) - \inf_y (F(y) + d(x,y))| = 0$$

Then if $D = \{x \in M : F(x) = 0\}$ *then* $F(x) = d(x, D)$ *for all* $x \in M$*.*

Corollary

The notion of a type being principal does not depend on the structure in which it is defined.

Definable sets

Proposition (Mysterious answer)

A type p is principal iff the logic and metric topologies agree locally at p.

Proposition (MTFMS, 9.19)

The following are equivalent:

1. *p is principal.*
2. *There are formulas φ_m and numbers $\delta_m > 0$ such that for every m, $p^{\varphi_m} = 0$*

$$\text{if } \text{``}\varphi(\bar{x}) \leq \delta_m\text{''} \text{ is in } q \text{ then } d(p, q) \leq \frac{1}{m}$$

A useful lemma

Lemma (MTFMS, 2.10)

Suppose that $F, G : X \to [0,1]$ *are functions such that*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ (F(x) \leq \delta \implies G(x) \leq \epsilon)$$

Then there exists an increasing, continuous function $\alpha : [0,1] \to [0,1]$ *such that* $\alpha(0) = 0$ *and*

$$\forall x \in X \ (G(x) \leq \alpha(F(x)))$$

The omitting types theorem

Theorem

Suppose that L is a separable language, T is a complete theory in L and p is a finitely satisfiable type. Then there is a model which omits p iff p is not principal.

Proof:

- If p is principal we must see that every model of T realizes p. So fix a model M of T and since p is finitely satisfiable it is realized in M^U for any non-principal ultrafilter U. So we have the situation that if P is the definable predicate for $d(x, p(M))$ then $(M, P) \prec (M^U, P)$.
- But then $\inf_x P(x) = 0$ in M^U and so for some δ less than the bound on the metric in the sort of x, for all $a \in M$, $d(a, p(M)) \leq \delta$. So this means $p(M)$ is non-empty.
- Now suppose that p is not principal. We will construct a model of T using a Henkin construction.

The omitting types theorem, cont'd

- Since the language is separable, we can accomplish this Henkin construction in countably many steps. The key issue will be to guarantee that every constant not only doesn't realize p but stays some uniform distance away from potential realizations of p so that in the completion, p will not be realized.
- Since p is not principal, we know that there is some ϵ so that the ball of radius ϵ around p does not contain any open set from the logic topology. That is, for every formula φ and every r, if $O_{\varphi,r}$ is not empty then it contains q such that $d(p, q) \geq \epsilon$.

The omitting types theorem, cont'd

- If we take the q from the previous line, we get that the set of conditions

$$p(x) \cup q(y) \cup \{d(x,y)\} \leq \epsilon/2$$

  is not satisfiable. So by compactness, there is some condition $\psi \leq s$ in q such that $p(x) \cup \{\psi \leq s, d(x,y) \leq \epsilon/2\}$ is not satisfiable.
- By approximate finite satisfiability, we even know that there is some n such that $p(x) \cup \{\psi < s + 1/n, d(x,y) \leq \epsilon/2\}$ is not satisfiable.
- Now the general set-up for the Henkin construction will have us looking at finitely many conditions $\varphi_i(c, \bar{c}) < r_i$ which are finitely satisfiable with T; here we have highlighted the constant c which we want to guarantee will not satisfy anything close to p.

The omitting types theorem, cont'd

- We consider the intersection of the basic open sets given by $\inf_{\bar{y}} \varphi_i(x, \bar{y}) < r_i$ and obtain some formula $\psi(x)$ and number s such that any type q in

$$\cap_i O_{\inf_{\bar{y}} \varphi_i, r_i} \cap O_{\psi, s}$$

  must satisfy $d(p, q) \geq \epsilon/2$.
- This proof would work if you try to omit countably many non-principal types.

Another characterization of definable zero sets

Theorem

Suppose that M is a metric structure and $Z \subseteq M^n$ is a closed subset. Then the following are equivalent:

1. *Z is a definable zero set.*
2. *For any definable predicate P with domain $M^n \times M^m$, $Q(x) = \inf\{P(x,z) : z \in Z\}$ is a definable predicate.*

- From bottom to top, just let $P(x, y)$ be $d(x, y)$.
- In the other direction, P is uniformly continuous so using MTFMS 2.10 again, we can find continuous α such that $|P(x,z) - P(y,z)| \leq \alpha(d(x,y))$ for all $x \in M^m$. Consider the formula $\inf_z(P(x,z) + \alpha(d(z,Z)))$. We claim this is Q.
- The conclusion here is that definable zero sets are exactly those sets which you can quantify over.
- Particularly useful examples of definable sets are ranges of terms.