

# Test Solutions.

①

1. Suppose that  $M$  is 1-generated say by  $m \in M$ .  
Let  $\varphi: R \rightarrow M$  be defined by  $\varphi(r) = rm$ .

Then  $\varphi$  is surjective and  $M \cong R/\ker(\varphi)$ . But the kernel of  $\varphi$  is <sup>some</sup> left ideal  $I$  so  $M \cong R/I$ .

2. Let  $\text{Tor}(M) = \{m \in M : rm = 0 \text{ for some } r \neq 0\}$ .

We need to show  $\text{Tor}(M)$  is closed under  $+$  and multiplication by  $r \in R$ .

1) If  $m_1, m_2 \in \text{Tor}(M)$  then there are  $r_1, r_2 \neq 0$  so that  $r_1 m_1 = 0$  and  $r_2 m_2 = 0$ . Since  $R$  is an integral domain  $r_1, r_2 \neq 0$  and  $r_1 r_2 (m_1 + m_2) = r_2 r_1 m_1 + r_1 r_2 m_2 = 0$  ( $R$  is commutative).

2) If  $m \in \text{Tor}(M)$ ,  $s \in R$  then for some  $r \neq 0$ ,  $rm = 0$ . Then  $sr m = rsm = 0$  so  $sm \in \text{Tor}(M)$ .

3. First we show that  $eM$  is a submodule:

$$e(m_1 + m_2) = em_1 + em_2.$$

$$e(rm) = r(em) \text{ since } e \text{ commutes with everything in } R.$$

So  $eM$  is a submodule.  $(1-e)M$  is a submodule as well since  $(1-e)(1-e) = 1-e-e+e^2 = 1-e$ .

Now if  $x \in eM \cap (1-e)M$  then  $x = em_1$  <sup>for some  $m_1$</sup>  so  $ex = x$ .

But  $x = (1-e)m_2$  for some  $m_2$  and  $ex = e(1-e)m_2 = 0$

So  $x = 0$ .

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$$\text{So } eM \cap (1-e)M = 0.$$

Finally if  $m \in M$  then  $em \in eM$  and  $(1-e)m \in (1-e)M$   
so  $m = em + (1-e)m$  and we have

$$M = eM \oplus (1-e)M.$$

4. Defined a bilinear map  $\varphi: R/I \times N \rightarrow N/IN$   
sending  $(r+I, n) \mapsto rn + IN$ .

To see that this is well-defined, suppose that  $r+I = r'+I$ .

Then  $r-r' \in I$  and for any  $n \in N$ ,  $(r-r')n \in IN$ .

$$\text{So } rn + IN = r'n + IN.$$

$$\begin{aligned} \text{Bilinear: } \varphi(r_1+r_2+I, n) &= (r_1+r_2)n + IN \\ &= \varphi(r_1+I, n) + \varphi(r_2+I, n) \end{aligned}$$

$$\begin{aligned} \varphi(r+I, n_1+n_2) &= r(n_1+n_2) + IN \\ &= \varphi(r+I, n_1) + \varphi(r+I, n_2) \end{aligned}$$

$$\begin{aligned} \text{R-balanced: } \varphi((r+I)s, n) &= rsn + IN \\ &= \varphi(r+I, sn) \end{aligned}$$

So there is  $\bar{\varphi}: R/I \otimes N \rightarrow N/IN$  and  $\bar{\varphi}$  is surjective  
since  $\varphi(1+I, n) = n + IN$ .

To see that  $\bar{\varphi}$  is injective, we show that all elements

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of  $R/I \otimes N$  have the form  $(1+I) \otimes n$  for some  $n \in N$ .

A simple tensor looks like  $(r+I) \otimes n = (1+I) \otimes rn$ .

A sum of simple tensor then has the required form by linearity, in the first coordinate.

But then  $\bar{\varphi}(1+I, n) = 0$  iff  $n \in IN$ .

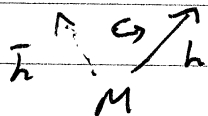
But  $n \in IN$  means  $n = \sum r_i n_i$  for some  $r_i \in I$   
and then  $(1+I, n) = \sum (1+I, r_i n_i)$

$$= \sum (r_i + I, n_i) = 0.$$

So  $\ker(\bar{\varphi}) = 0$  and  $\bar{\varphi}$  is an isomorphism.

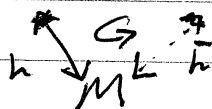
5. a) A module  $M$  is projective if  $\text{Hom}_R(M, -)$  takes short exact sequences to short exact sequences.

Equivalently, whenever  $A \twoheadrightarrow B$   
 $h: M \rightarrow B$  there is  
 $\bar{h}: M \rightarrow A$  as shown.



~~5~~ A module  $M$  is injective if  $\text{Hom}_R(-, M)$  takes short exact sequences to short exact sequences.

Equivalently, whenever  $A \hookrightarrow B$  and  $h: A \rightarrow M$   
there is  $\bar{h}: B \rightarrow M$   
as shown.



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$M$  is flat if  $M \otimes_R$  takes short exact sequences to short exact sequences.

Equivalently, whenever  $A \xrightarrow{f} B$  is injective then

$1_M \otimes f : M \otimes_R A \rightarrow M \otimes_R B$  is injective.

b) Let's first show that free modules are flat:

A module free on generators  $X$  looks like  $\coprod_X R$ .

If  $A \xrightarrow{f} B$  is 1-1 then we have  $\coprod_X R \otimes_R A \cong \coprod_X A$  and  $\coprod_X R \otimes_R B \cong \coprod_X B$  and  $\coprod_X f : \coprod_X A \rightarrow \coprod_X B$  is 1-1.

Now if  $M$  is projective then  $M$  is the direct summand of a free module so there is  $N$  s.t.  $M \oplus N$  is free.

But  $(M \oplus N) \otimes_R A \cong M \otimes_R A \oplus N \otimes_R A$  and

$(M \oplus N) \otimes_R B \cong M \otimes_R B \oplus N \otimes_R B$  so both

maps  $1_M \otimes f$  and  $1_N \otimes f$  are 1-1. Hence  $M$  is flat.