Test Solutions.

1. Suppose that M is 1-generated say by $m \in M$. Let $\varphi : R \to M$ be defined by $\varphi(r) = rm$.

Then φ is surjective and $M \cong R/\ker(\varphi)$. But the kernel of φ is some left ideal I so $M \cong R/I$.

2. Let $Tor(M) = \{ m \in M : rm = 0 \text{ for some } r \neq 0 \}$. We need to show $Tor(M)$ is closed under + and multiplication by $r \in R$.

1) If $m_1, m_2 \in Tor(M)$ then there are $r_1, r_2 \neq 0$ so that $r_1 m_1 = 0$ and $r_2 m_2 = 0$. Since R is an integral domain $r_1 r_2 \neq 0$ and $r_1 r_2 (m_1 + m_2) = r_2 r_1 m_1 + r_1 r_2 m_2 = 0$ (R is commutative).

2) If $m \in Tor(M)$, $s \in R$ then for some $r \neq 0$, $rm = 0$. Then $srm = rsm = 0$ so $sm \in Tor(M)$.

3. First we show that eM is a submodule:
$e(m_1 + m_2) = em_1 + em_2$.
$e(rm) = r(em)$ since e commutes with everything in R.

So eM is a submodule. $(1-e)M$ is a submodule as well since $(1-e)(1-e) = 1 - e - e + e^2 = 1 - e$.

Now if $x \in eM \cap (1-e)M$ then $x = em_1$ for some m_1 so $ex = x$. But $x = (1-e)m_2$ for some m_2 and $ex = e(1-e)m_2) = 0$. So $x = 0$.

So $eM \cap (1-e)M = 0$.

Finally if $m \in M$ then $em \in eM$ and $(1-e)m \in (1-e)M$
so $m = em + (1-e)m$ and we have

$$M = eM \oplus (1-e)M.$$

4. Defined a bilinear map $\varphi : R/I \times N \to N/IN$
sending $(r+I, n) \longmapsto rn + IN$.

To see that this is well-defined, suppose that $r+I = r'+I$.

Then $r - r' \in I$ and for any $n \in N$, $(r-r')n \in IN$.
So $rn + IN = r'n + IN$.

Bilinear: $\varphi(r_1 + r_2 + I, n) = (r_1 + r_2)n + IN$
$= \varphi(r_1 + I, n) + \varphi(r_2 + I, n)$

$\varphi(r+I, n_1 + n_2) = r(n_1 + n_2) + IN$
$= \varphi(r+I, n_1) + \varphi(r+I, n_2)$

R-balanced: $\varphi((r+I)s, n) = rsn + IN$
$= \varphi(r+I, sn)$

So there is $\overline{\varphi} : R/I \otimes N \to N/IN$ and $\overline{\varphi}$ is surjective
since $\varphi(1+I, n) = n + IN$.

To see that $\overline{\varphi}$ is injective, we show that all elements of $R/I \otimes N$ have the form $(1+I) \otimes n$ for some $n \in N$.

A simple tensor looks like $(r+I) \otimes n = (1+I) \otimes rn$.
A sum of simple tensor then has the required form by linearity in the first coordinate.

But then $\overline{\varphi}(1+I, n) = 0$ iff $n \in IN$.
But $n \in IN$ means $n = \sum r_i n_i$ for some $r_i \in I$
and then $(1+I, n) = \sum (1+I, r_i n_i)$

$$= \sum (r_i + I, n_i) = 0.$$

So $\ker(\overline{\varphi}) = 0$ and $\overline{\varphi}$ is an isomorphism.

5. a) A module M is projective if $\mathrm{Hom}_R(M, -)$ takes short exact sequences to short exact sequences.

Equivalently, whenever $A \twoheadrightarrow B$, $h: M \to B$ there is $\bar{h}: M \to A$ as shown.

$$\begin{tikzcd} A \arrow[r, two heads] & B \\ & M \arrow[ul, dashed, "\bar{h}"] \arrow[u, "h"'] \end{tikzcd}$$

b) A module M is injective if $\mathrm{Hom}_R(-, M)$ takes short exact sequences to short exact sequences.

Equivalently, whenever $A \hookrightarrow B$ and $h: A \to M$ there is $\bar{h}: B \to M$ as shown.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "h"'] & B \arrow[dl, dashed, "\bar{h}"] \\ M & \end{tikzcd}$$

M is flat if $M \otimes_R -$ takes short exact sequences to short exact sequence.

Equivalently, whenever $A \xrightarrow{f} B$ is injective then

$1_M \otimes f : M \otimes_R A \to M \otimes_R B$ is injective.

b) Let's first show that free modules are flat:

A module free on generators X looks like $\coprod_X R$.

If $A \xrightarrow{f} B$ is 1-1 then we have $\coprod_X R \otimes_R A \cong \coprod_X A$ and $\coprod_X R \otimes_R B \cong \coprod_X B$ and $\coprod_X f : \coprod_X A \to \coprod_X B$ is 1-1.

Now if M is projective then M is the direct summand of a free modules so there is N s.t. $M \oplus N$ is free.

But $(M \oplus N) \otimes_R A \cong M \otimes_R A \oplus N \otimes_R A$ and

$(M \oplus N) \otimes_R B \cong M \otimes_R B \oplus N \otimes_R B$ so both

maps $1_M \otimes f$ and $1_N \otimes f$ are 1-1. Hence M is flat.