

Solutions to Assignment #1.

1. First we show that any maximal indep. set $A \subseteq X$ is a generating set i.e. $c|A| = X$.

Suppose not. Choose $a \in X - c|A|$. We claim that $A \cup \{a\}$ is indep. If $a' \in A$ then if $a' \in c|(A - \{a'\}) \cup \{a\}|$ then $a \in c|A|$ by exchange. So $A \cup \{a\}$ is indep. contradicting the max. of A .

Now suppose that $A = \{a_1, \dots, a_n\}$ is an indep. set and C is a generating set. We produce c_1, \dots, c_n s.t. for each i , $C_i = (C - \{c_1, \dots, c_{i-1}\}) \cup \{a_1, \dots, a_i\}$ is a generating set. The inductive assumption is that C_i is a generating set and consider $a_{i+1} \in c|C_i|$. If we choose $D \subseteq C_i$ minimal s.t. $a_{i+1} \in c|(D \cup \{a_1, \dots, a_i\})$ then since A is indep., $D \neq \emptyset$. Pick any $c_{i+1} \in D$. Then $a_{i+1} \in c|(D \cup \{a_1, \dots, a_i\}) = c|(D - \{c_{i+1}\}) \cup \{a_1, \dots, a_i\}$ and so $c_{i+1} \in c|(C_i - \{c_{i+1}\}) \cup \{a_1, \dots, a_{i+1}\}|$ which is enough to show that C_{i+1} is a generating set.

We conclude that $|A| \leq |C|$. Since the assumption for this question is that X is finitely generated we conclude that every maximal indep. set is finite. But then, since any maximal indep. set is a generating set, any two such must have the same size.

2. Suppose X and Y are two bases for an int. dim. F -v. space. Hence X and Y are both infinite. In particular, $|X| = |X^{<\omega}|$ where $X^{<\omega}$ is the set of finite sequences from X ; likewise for Y .

Now if $x \in X$ then there is a finite set $Y_x \subseteq Y$ so that $x \in \langle Y_x \rangle$; call the function $x \mapsto Y_x$, f .

Then for any given finite set $Y_0 \subseteq Y$, $f^{-1}(Y_0)$ is finite since $\langle Y_0 \rangle$ is fin. dim. This means f is a finite-to-one map giving us that $|X| \leq |Y^{<\omega}| = |Y|$.

Symmetry gives $|Y| \leq |X|$ so $|X| = |Y|$.

3. As F is an F -v.s.p., $V^* = \text{Hom}_F(V, F)$ and so is also an F -v.s.p.

To define $i: V \rightarrow V^{**}$, let $i(v) = ev_v$ i.e. evaluation at v . So $ev_v(f) = f(v)$ where $f \in V^*$. If $v \neq 0$ then let $f: V \rightarrow F$ be any functional s.t. $f(v) \neq 0$. Then $i(v) \neq 0$. So $\ker(i) = 0$ and i is 1-1.

If V is finite dimensional say with basis u_1, \dots, u_n , let $u_i^*: V \rightarrow F$ be defined as follows: If $v \in V$ then $v = c_1 u_1 + \dots + c_n u_n$ so let $u_i^*(v) = c_i$. Note $u_j^*(u_i) = 1$ if $i=j$ and 0 otherwise so u_1^*, \dots, u_n^* is lin. indep. If $f \in V^*$ then $f = f(u_1) u_1^* + \dots + f(u_n) u_n^*$ so it spans V^* as well. So $V \cong V^*$.

4. Define a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ as follows:

① if V is a fin. dim. v. space say of dim. n then fix a basis I_V and let $F(V) = n$.

② if V, W are fin. dim. v. spaces with bases I_V and I_W then ~~then their matrix representation~~ Let $f: V \rightarrow W$ be a linear transformation. Fix A , the $m \times n$ matrix representing f w.r.t the bases I_V and I_W .

Let $F(f) = A$. Note $A \in \text{hom}(n, m)$.

It is straightforward to see that F is a functor since all matrix representations involving V use the same basis I_V . For instance, id_V is sent to the identity matrix. Moreover if

$$V \xrightarrow{f} W \xrightarrow{g} X \quad \text{then automatically}$$

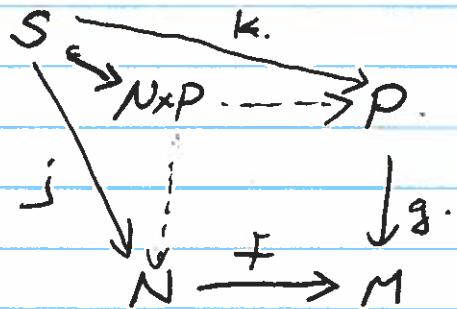
$$F(g)F(f) = F(gf).$$

Now to see the equivalence, if $V, W \in \mathcal{C}$ then $\text{hom}_{\mathcal{C}}(V, W) \cong \text{hom}(\dim(V), \dim(W))$ exactly by the matrix representations via I_V and I_W .

Moreover, $F(F^n) = n$ so we satisfy the two clauses
↑
and notation

of categorical equivalence.

5.



Let $S = \{(n, p) \in N \times P : f(n) = g(p)\}$:

$$f(n) = g(p)\}.$$

$$\text{and } j(n, p) = n, k(n, p) = p.$$

Clearly $gk = f j$.

Now suppose we have S', j' and k' as in the problem.

We define $h: S' \rightarrow S$ by $h(s') = (j'(s'), k'(s'))$.

Since $f j' = g k'$, $h(s') \in S$. By definition $j' h = s'$ and $k' h = k'$ and these equations force h to be unique.

Hence, S is the pullback of the given diagram.