Solutions to Assignment #1.

1. First we show that any maximal indep set $A \subseteq X$ is a generating set i.e. $cl(A) = X$.

Suppose not. Choose $a \in X \setminus cl(A)$. We claim that $A \cup \{a\}$ is indep. If $a' \in A$ then if $a' \in cl((A \setminus \{a'\}) \cup \{a\})$ then $a \in cl(A)$ by exchange. So $A \cup \{a\}$ is indep. contradicting the max. of A.

Now suppose that $A = \{a_1, \cdots a_n\}$ is an indep. set and C is a generating set. We produce $c_1, \cdots c_n$ s.t. for each i, $C_i = (C \setminus \{c_1 \cdots c_i\}) \cup \{a_1 \cdots a_i\}$ is a generating set. The inductive assumption is that C_i is a generating set and consider $a_{i+1} \in cl(C_i)$. If we choose $D \subseteq C_i$ minimal s.t. $a_{i+1} \in cl(D \cup \{a_1 \cdots a_i\})$ then since A is indep, $D \neq \phi$. Pick any $c_{i+1} \in D$. Then $a_{i+1} \in cl(D \cup \{a_1 \cdots a_i\}) \setminus cl((D \setminus \{c_{i+1}\}) \cup \{a_1 \cdots a_i\})$ and so $c_{i+1} \in cl(C_i \setminus \{c_{i+1}\} \cup \{a_1 \cdots a_{i+1}\})$ which is enough to show that C_{i+1} is a generating set.

We conclude that $|A| \leq |C|$. Since the assumption for this question is that X is finitely generated we conclude that every maximal indep. set is finite. But then, since any maximal indep. set is a generating set, any two such must have the same size.

2. Suppose X and Y are two bases for an inf. dim. F-v.space. Hence X and Y are both infinite. In particular, $|X| = |X^{<\omega}|$ where $X^{<\omega}$ is the set of finite sequences from X; likewise for Y.

Now if $x \in X$ then there is a finite set $Y_x \subseteq Y$ so that $x \in \langle Y_x \rangle$; call the function $x \mapsto Y_x$, f.

Then for any given finite set $Y_0 \subseteq Y$, $f^{-1}(Y_0)$ is finite since $\langle Y_0 \rangle$ is fin. dim. This means f is a finite-to-one map giving us that $|X| \leq |Y^{<\omega}| = |Y|$.

Symmetry gives $|Y| \leq |X|$ so $|X| = |Y|$.

3. As F is an F-v.sp., $V^* = Hom_F(V, F)$ and so is also an F-v.sp.

To define $i : V \to V^{**}$, let $i(v) = ev_v$ i.e. evaluation at v. So $ev_v(f) = f(v)$ where $f \in V^*$. If $v \neq 0$ then let $f : V \to F$ be any functional s.t. $f(v) \neq 0$. Then $i(v) \neq 0$. So $\ker(i) = 0$ and i is 1-1.

If V is finite dimensional say with basis $u_1 \cdots u_n$, let $u_i^* : V \to F$ be defined as follows: If $v \in V$ then $v = c_v^1 u_1 + \cdots + c_v^n u_n$ so let $u_i^*(v) = c_v^i$.
Note $u_i^*(u_j) = 1$ if $i = j$ and 0 otherwise so $u_1^* \cdots u_n^*$ is lin. indep. If $f \in V^*$ then $f = f(u_1)u_1^* + \cdots + f(u_n)u_n^*$ so it spans V^* as well. So $V \cong V^*$

4. Define a functor $F: \mathcal{C} \to \mathcal{D}$ as follows:

① if V is a fin. dim. v. space say of dim. n then fix a basis I_V and let $F(V) = n$.

② if V, W are fin. dim v. spaces of dim n and m with bases I_V and I_W ~~then Let A be the mxn matrix~~ Let $f: V \to W$ be a linear transformation. Fix A, the $m \times n$ matrix representing f w.r.t the bases I_V and I_W.

Let $F(f) = A$. Note $A \in \text{hom}(n, m)$.

It is straightforward to see that F is a functor since all matrix representations involving V use the same basis I_V. For instance, id_V is sent to the identity matrix. Moreover if $V \xrightarrow{f} W \xrightarrow{g} X$ then automatically

$F(g)F(f) = F(gf)$.

Now to see the equivalence, if $V, W \in \mathcal{C}$ then $\text{hom}_{\mathcal{C}}(V, W) \cong \text{hom}_{\mathcal{D}}(\dim(V), \dim(W))$ exactly by the matrix representations via I_V and I_W.

Moreover, $F(F^n) = n$ so we satisfy the two clauses

bad notation

of categorical equivalence.

5.

$$\begin{tikzcd}
S \arrow[dr, hook] \arrow[drr, bend left, "k"] \arrow[ddr, "j"'] & & \\
& N\times P \arrow[r, dashed] \arrow[d, dashed] & P \arrow[d, "g"] \\
& N \arrow[r, "f"] & M
\end{tikzcd}$$

Let $S = \{ (n,p) \in N\times P : f(n) = g(p) \}$ and $j(n,p) = n$, $k(n,p) = p$.

Clearly $gk = fj$.

Now suppose we have S', j' and k' as in the problem.

We define $h : S' \to S$ by $h(s') = (j'(s'), k'(s'))$.

Since $fj' = gk'$, $h(s') \in S$. By definition $j \circ h = j'$ and $k \circ h = k'$ and these equations force h to be unique.

Hence, S is the pullback of the given diagram.