Assignment 3, Math 701
Due Nov. 20, in class

1. An R-module M is called irreducible if its only submodules are 0 and M. Show that for an irreducible R-module, $\hom_R(M, M)$ is a division ring with composition as multiplication.

2. **Examples are good.** We discussed projective, injective and flat modules. We saw that a projective module is necessarily flat. Logically this leaves us with 6 possibilities. Give examples of modules that are:

   (a) projective and injective,

   (b) projective but not injective,

   (c) not projective but injective and flat, (think about Q as an abelian group)

   (d) not flat but injective, (we saw an example in class)

   (e) not projective not injective but flat, (think about Z as an abelian group - it is projective but think about taking its direct sum with something that is not) and

   (f) not flat nor injective.

3. When teaching linear algebra, one always emphasizes the importance of the characteristic polynomial, the minimal polynomial and the eigenvalues and dimensions of the eigenspaces as invariants for a given linear operator on a finite-dimensional complex vector space. What is the smallest dimension in which these pieces of information do not characterize the linear operator up to similarity? That is, what is the smallest n for which there are two operators with the same characteristic polynomial, same minimal polynomial and the same eigenvalues and dimensions for the eigenspaces but the operators are not similar?

4. Which finitely generated modules over a principal ideal domain are projective? Which ones are injective? Which ones are flat?

5. Suppose that A is an $n \times n$ matrix over a field F and the characteristic polynomial of A splits completely over F. Further suppose that $F \subset K$, another field. Show that the Jordan canonical form of A is the same over F and K.