Solutions to the Midterm Test.

1 a) φ is a tautology and ψ is a contradiction.

b) $\Gamma \vdash (\varphi \to \psi)$, $\Gamma, \varphi \vdash \varphi$ so by Modus Ponens $\cancel{\Gamma \vdash}$ $\Gamma, \varphi \vdash \psi$.

c) If v satisfies all formulas in Γ and Δ then by assumption, v satisfies φ and so satisfies ψ.

2 a) A set of formulas Γ is satisfiable iff every finite subset of Γ is satisfiable.

b) By completeness, $\{\neg\varphi_1, \neg\varphi_2, \dots\}$ is not satisfiable so by compactness, for some N, $\{\neg\varphi_1, \dots \neg\varphi_N\}$ is not satisfiable. For any truth assignment v then, for some i, $v(\neg\varphi_i) = F$ so $v(\varphi_i) = T$. Hence v makes $\bigvee_{i=1}^{N} \varphi_i$ true. This holds for all v so $\bigvee_{i=1}^{N} \varphi_i$ is a tautology.

3. No. For instance, in one propositional variable p, one cannot obtain $\cancel{\neg p}$ the truth table for $\neg p$. To see this, notice that by induction on the formation of formulas starting from p and closing under $\wedge$ and $\vee$, if $v(p) = T$ then any formula φ formed from p, $\wedge$ and $\vee$ satisfies $v(\varphi) = T$.

4. a) For a language L we define terms inductively:

1) x_i is a term for all variables x_i

2) if f is an n-ary function symbol in L and $\tau_1, \dots \tau_n$ are terms then $f(\tau_1, \dots \tau_n)$ is a term.

b) By induction on terms: $FV(x_i) = \{x_i\}$ and $FV(f(\tau_1, \dots, \tau_n)) = \bigcup_{i=1}^{n} FV(\tau_i)$ so if $FV(\tau_i)$ is finite for all i then $FV(f(\tau_1, \dots \tau_n))$ is finite.

Alternatively, a term is a finite string and so only contains finitely many variables.