Solutions to Assignment 4

1. a) $\mathbb{N}$ has a least element and $\mathbb{Z}$ doesn't:

$$\forall x \exists y \, (y < x) \quad \text{separates these structures.}$$

b) As mentioned in class and on the problem, $\mathcal{P}(\mathbb{N})$ has atoms $\{a\}$ for all $a \in \mathbb{N}$. The algebra $\mathcal{B}$ does not: arbitrary elements are unions of sets which look like

$$X = \bigcap_{i=1}^{n} U_{p_i} \cap \bigcap_{j=1}^{m} U_{q_j}^{c} \quad \text{for}$$

distinct primes $p_1 \cdots p_n, q_1 \cdots q_m$. If r is another prime then $U_r \cap X \subsetneq X$ and so X is not an atom.

$$\forall x \left(x \neq 0 \rightarrow \exists y \, (x \neq y \wedge y \neq 0 \wedge y \cap x = y) \right)$$

2. a) We first show that for any term τ,

$$f(\bar{\imath}(\tau)) = \overline{f(i)}(\tau)$$ (the interpretation of τ in M is sent to its interpretation in N by f).

By induction on terms: for a variable x.

$$f(\bar{\imath}(x)) = f(i(x)), \quad \overline{f(i)}(x) = f(i(x)).$$

Suppose g is an n-ary function and $\tau_1 \cdots \tau_n$ are terms

Then $f(\bar{\imath}(g(\tau_1, \dots \tau_n))) = f(g^M(\bar{\imath}(\tau_1), \dots \bar{\imath}(\tau_n))$

(since f is an embedding) $= g^N(f(\bar{\imath}(\tau_1)), \dots f(\bar{\imath}(\tau_n)))$

(by induction) $= g^N(\overline{f(i)}(\tau_1), \dots \overline{f(i)}(\tau_n))$

$= \overline{f(i)}(g(\tau_1, \dots \tau_n))$.

Now, for atomic formulas $R(\tau_1, \dots, \tau_n)$

$\mathcal{M} \models_i R(\tau_1, \dots \tau_n)$ iff $(\bar{\imath}(\tau_1), \dots \bar{\imath}(\tau_n)) \in R^M$

iff $(f(\bar{\imath}(\tau_1)), \dots f(\bar{\imath}(\tau_n))) \in R^N$

iff $(\overline{f(i)}(\tau_1), \dots \overline{f(i)}(\tau_n)) \in R^N$

iff $\mathcal{N} \models_{f(i)} R(\tau_1, \dots \tau_n)$.

If we know $\mathcal{M} \models_i \varphi$ iff $\mathcal{M} \models_{f(i)} \varphi$ then easily

$\mathcal{M} \models_i \neg\varphi$ iff $\mathcal{M} \models_{f(i)} \neg\varphi$. If additionally,

$\mathcal{M} \models_i \psi$ iff $\mathcal{M} \models_{f(i)} \psi$ then $\mathcal{M} \models_i \varphi \wedge \psi$ iff

$\mathcal{M} \models_i \varphi$ and $\mathcal{M} \models_i \psi$ iff $\mathcal{M} \models_{f(i)} \varphi$ and $\mathcal{M} \models_{f(i)} \psi$

iff $\mathcal{M} \models_{f(i)} \varphi \wedge \psi$.

b) An isomorphism is an embedding so by a) we know that quantifier-free formulas are preserved.

Suppose $f : \mathcal{M} \to \mathcal{N}$ is an isomorphism and

$\exists x \varphi$ is an existential formula. We want to show

$$\mathcal{M} \models_i \exists x \varphi \quad \text{iff} \quad \mathcal{N} \models_{f(i)} \exists x \varphi .$$

$\Rightarrow$ Choose $a \in M$ and $i' = i$ except $i'(x) = a$ s.t.

$\mathcal{M} \models_{i'} \varphi$. Then by induction $\mathcal{N} \models_{f(i')} \varphi$

$f(i) = f(i')$ except possibly at x so $\mathcal{N} \models_{f(i)} \exists x \varphi$.

$\Leftarrow$ Suppose $b \in N$ and j is an assignment of variables s.t. $j = f(i)$ except $j(x) = b$. Let $a = f^{-1}(b)$ and let $i' = i$ except $i'(x) = a$. Then $j = f(i')$.

If we assume that $\mathcal{N} \models_j \varphi$ then $\mathcal{N} \models_{f(i')} \varphi$ so by induction $\mathcal{M} \models_{i'} \varphi$ so $\mathcal{M} \models_i \exists x \varphi$.

3. Suppose $(\mathbb{R}, <) \cong (\mathbb{R} \setminus \{0\}, <)$ by a function ~~$f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$~~ $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

Consider $a_n = f(-\frac{1}{n})$ for $n > 0$ and
$b_n = f(\frac{1}{n})$ for $n > 0$.

Notice, for all $x \in \mathbb{R} \setminus \{0\}$, either $f(x) < a_n$ for some n

or $f(x) > b_n$ for some n. So the range of f is contained in $\{x \in \mathbb{R} : x < a_n \text{ for some } n\} \cup$
$\{x \in \mathbb{R} : x > b_n \text{ for some } n\}$.

But the a_n's form an increasing sequence in $\mathbb{R}$ bounded above by b_1 for instance.

So $\lim_{n\to\infty} a_n$ exists and is some $a \in \mathbb{R}$. By our first observation, $a \notin \text{range}(f)$ so f is not an isomorphism.

4. If $\mathcal{M}, \mathcal{N}$ are two countable dense linear orders with a left and right endpoint & then

$M = \{m_0, m_1\} \cup M'$ where m_0 is the left endpoint, m_1 is the right endpoint and M' is a ctble dense linear order without endpoints.

Similarly $N = \{n_0, n_1\} \cup N'$ with the same notation.

By Cantor's theorem $M' \cong N'$ as orders and if we extend this to m_0 and m_1 sending them to n_0 and n_1, we have an order isomorphism showing

$\mathcal{M} \cong \mathcal{N}$.