Solutions to Assignment #3.

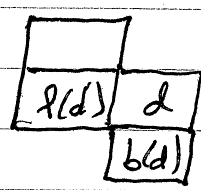
1. As suggested, introduce propositional variables $p^d_{i,j}$ where $i,j \in \mathbb{Z}$ and d is the name of a unit square on one of the tiles in Q.

The intended meaning is "if $p^d_{i,j}$ is true then the unit square on the plane with lower left corner (i,j) is covered by the square d using the appropriate tile from Q".

Now to ease notation, for a fixed tile in Q and unit square d, we define 4 partial functions:

$u(d)$ is the unit square above d if it exists – u for up.
$b(d)$ ———— below d ———— – b for below
$\ell(d)$ ———— to the left of d if it exists
$r(d)$ ———— to the right of d ————.

Ex:



← tile from Q.

Here $u(d)$ and $r(d)$ are not defined.

We consider the set Γ of formulas containing:

① $\bigvee_{d \in D} p^d_{i,j}$ for all $i,j \in \mathbb{Z}$. D is the set of names for unit squares on tiles in Q.

This says every square on the plane is covered by something.

2 $\quad \neg\left(p^{d}_{i,j} \wedge p^{d'}_{i,j}\right)$ for all $i,j \in \mathbb{Z}$, $d, d' \in D$
$d \neq d'$.

This says at most one thing covers each unit square on the plane.

3 $\quad p^{d}_{i,j} \rightarrow p^{u(d)}_{i+1,j}$ for all $i,j \in \mathbb{Z}$, $d \in D$ if $u(d)$ is defined

$p^{d}_{i,j} \rightarrow p^{b(d)}_{i-1,j}$ for all $i,j \in \mathbb{Z}$, $d \in D$ if $b(d)$ is defined.

$p^{d}_{i,j} \rightarrow p^{l(d)}_{i,j-1}$ for all $i,j \in \mathbb{Z}$, $d \in D$ if $l(d)$ is defined

$p^{d}_{i,j} \rightarrow p^{r(d)}_{i,j+1}$ for all $i,j \in \mathbb{Z}$, $d \in D$ if $r(d)$ is defined

These formulas tell us that if the square labelled d covers the ij unit square then the neighbouring squares must also be covered by the correct squares from the tile d is part of.

Altogether these form the set Γ. We would like to see that Γ is finitely satisfiable. Choose $\Gamma_0 \subseteq \Gamma$ finite. Now Γ_0 mentions only finitely many pairs (i,j). By making Γ_0 larger we can assume that Γ_0 contains all formulas from 1, 2 and 3 which mention these finitely many (i,j)'s.

By assumption, there is a Q-tiling that covers the finitely many unit squares mentioned in Γ_0.

By interpreting p_{ij}^{d} to be true if this covering covers the i,j square with the square labelled d, we conclude that Γ_0 is satisfiable.

By compactness now, Γ is satisfiable and so the entire plane can be $\mathcal{Q}$-tiled.

2. Assume that Γ is finitely satisfiable i.e. every finite subset of Γ is satisfiable. Construct a sequence $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ of finitely satisfiable sets of formulas with the following property:

Fix an enumeration of all formulas $\varphi_0, \varphi_1, \varphi_2, \ldots$

$$\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\} \quad \text{or} \quad \Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}.$$

Notice that if we can do this we will have shown that Γ is satisfiable. Why? Let $\bar{\Gamma} = \bigcup_n \Gamma_n$.

Claim: $\bar{\Gamma}$ is finitely satisfiable.

Choose any finite $\Delta \subseteq \bar{\Gamma}$. Then $\Delta \subseteq \Gamma_n$ for some n so Δ is satisfiable.

Now define a truth assignment for $\bar{\Gamma}$:

$$v(p) = \begin{cases} T & \text{if } p \in \bar{\Gamma} \\ F & \text{otherwise.} \end{cases}$$

Since $\bar{\Gamma}$ is finitely satisfiable, at most one of p or $\neg p$ is in $\bar{\Gamma}$ and by the way Γ_n is built, at least one is in $\bar{\Gamma}$ so v is well-defined.

Now show $v(\varphi) = T$ iff $\varphi \in \bar{\Gamma}$ by induction on φ. Propositional variables are handled above. We do the cases for $\neg$ and $\wedge$ since they are a complete set of connectives.

Case 1: $v(\neg\varphi) = T$ iff $v(\varphi) = F$
iff $\varphi \notin \bar{\Gamma}$ by induction
iff $\neg\varphi \in \bar{\Gamma}$ by def^n of the Γ_n's.

Case 2: $v(\varphi \wedge \psi) = T$ iff $v(\varphi) = v(\psi) = T$
iff $\varphi, \psi \in \bar{\Gamma}$
iff $\varphi \wedge \psi \in \bar{\Gamma}$

To justify that last line, if $\varphi \wedge \psi \in \bar{\Gamma}$ but $\neg\varphi \in \bar{\Gamma}$ or $\neg\psi \in \bar{\Gamma}$ then $\bar{\Gamma}$ is not finitely satisfiable. Similarly, if $\varphi, \psi \in \bar{\Gamma}$ but $\neg(\varphi \wedge \psi) \in \bar{\Gamma}$.

We conclude that $\bar{\Gamma}$ and hence Γ is satisfiable. This leaves us to show that if Γ_n is finitely satisfiable then one of $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is finitely sat. Suppose not. Then there is a finite $S_1 \subseteq \Gamma_n$ s.t. $S_1 \cup \{\varphi_n\}$ is not satisfiable. There is also $S_2 \subseteq \Gamma_n$ finite s.t. $S_2 \cup \{\neg\varphi_n\}$ is not satisfiable. But $S_1 \cup S_2$ is finite and satisfiable say by some truth assignment v.

But $v(\varphi_n) = T$ or $v(\neg\varphi_n) = T$ which is a contradiction. So at least one of $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is fin. sat. and we are done.

3. We need to show that for any term τ, the interpretation $\bar{i}(\tau)$ in any structure $\mathcal{M}$ only depends on the assignment of variables for the free variables in τ.

Said another way, we need to see that if i and i' are two assignments of variables for a structure $\mathcal{M}$ that agree on the free variables in τ then $\bar{i}(\tau) = \bar{i}'(\tau)$.

We do this by induction on the formation of τ:

① if τ is a variable x_k and i, i' are assignments s.t. $i(x_k) = i'(x_k)$ then $\bar{i}(x_k) = i(x_k) = i'(x_k) = \bar{i}'(x_k)$.

② assume i, i' are assignments that agree on the free variables in $F(\tau_1, \ldots \tau_n)$. Then

$$\bar{i}(F(\tau_1 \cdots \tau_n)) = F^{\mathcal{M}}(\bar{i}(\tau_1), \cdots \bar{i}(\tau_n))$$

$$= F^{\mathcal{M}}(\bar{i}'(\tau_1), \cdots \bar{i}'(\tau_n)) \quad (*)$$

$$= \bar{i}'(F(\tau_1, \cdots \tau_n))$$

The justification for $(*)$ is that i, i' agree on the free variables in $\tau_1, \cdots \tau_n$ and so by induction $\bar{i}(\tau_j) = \bar{i}'(\tau_j)$ for all $j \leq n$.