Assignment 3, Math 4L3
Due Oct. 26, in class

1. For this exercise, a tile will mean a geometric figure constructed from finitely many unit squares such that any two unit squares either intersect on an entire edge or have empty intersection. We are given finitely many tiles Q. Consider the plane R^2 divided into unit squares with corners having integer coordinates. We say that we have a Q-tiling of the plane if it is possible to cover the plane with tiles congruent to tiles from Q so that no two tiles overlap (they meet only at edges). Let's prove that for a finite set of tiles Q, if every bounded subset of R^2 is contained in a finite Q-tiling then it is possible to Q-tile the entire plane. Here is a suggestion of how to do this using the compactness theorem of propositional logic.

   (a) As with the example in class, most of the work goes into choosing appropriately named propositional variables. I would suggest something like $p^d_{i,j}$ where i and j are integers and d is one of the unit squares in one of the tilings from Q. The intended meaning is that if $p^d_{i,j}$ is true then the unit square on the plane with bottom left corner (i, j) is covered by a copy of a Q-tile with the unit square associated to d.

   (b) Write down formulas that express that every unit square on the plane is covered by some Q-tile and there is at most one tile covering each square. Additionally express that this is a Q-tiling (think about what it would mean if a given square is covered and what that implies about other squares nearby).

   (c) Use the assumption about finite coverings and compactness to conclude that the entire plane has a Q-tiling.

2. Let's give a proof of the compactness theorem in the style that we used to prove the completeness theorem (as opposed to it being a corollary). Start with a set of formulas Γ that is finitely satisfiable; that is, every finite subset is satisfiable. As in the proof of completeness, start with Γ and build an increasing chain of sets of formulas Γ_n such that each Γ_n is finitely satisfiable. Use the same trick as with the completeness theorem to guarantee that every formula is considered at some stage. Define a truth assignment at the end which shows that the entire set is satisfiable.

3. Suppose that a term τ in a first order language L has n free variables. Show that in every L-structure $\mathcal{M}$, $\tau^{\mathcal{M}}$ defines a function from M^n to M. It is probably easiest to do this by induction on the formation of τ.