Solutions to Mid-term test, Math 4GR3

1. Find the smallest number n such that there are exactly 14 non-isomorphic abelian groups of size n. (As a warm up, ask yourself how many non-isomorphic abelian groups there are of size $p^2, p^3, \ldots$ for a prime p.)

   Solution: If we consider an abelian group G of size p^n for a prime p then the fundamental theorem of finite abelian groups says that this group is isomorphic to one of the form, for some k,

$$C_{p^{n_1}} \times C_{p^{n_2}} \times \ldots \times C_{p^{n_k}}$$

   with $n_1 \leq n_2 \leq \ldots n_k$ and the isomorphism type is determined uniquely the numbers $n_1, n_2, \ldots n_k$. We will have $n_1 + \ldots n_k = n$. The number of ways that a number n can be written, up to reordering, as the sum of positive integers, is called the partition number of n, $P(n)$. By what was just said, the number of non-isomorphic abelian groups of size p^n is $P(n)$. There is no closed formula for the partition numbers but in this question we only need to look at small values: $2 = 2 = 1 + 1$ so $P(2) = 2$. $P(3) = 3$ since $3 = 2+1 = 2+ 1+ 1$. Likewise, $P(4) = 5$ since $4 = 3+1 = 2+2 = 2+1+1=1+1+1+1$. $P(5) = 7$ since $5 = 4+1=3+2=3+1+1=2+2+1=2+1+1+1=1+1+1+1+1$. We also need to notice that $P(6) = 11$ and $P(7) = 15$.

   By a version of the fundamental theorem, we know that for any finite abelian group G there are primes $p_1, \ldots, p_k$ such that $G \cong G_1 \times \ldots \times G_k$ for abelian groups G_i which are p_i-groups. Since we are looking for 14 non-isomorphic abelian groups, we could have a group of size p^a for some a with partition number 14 or of size $p^a q^b$ where $P(a) \times P(b) = 14$. There is no number with partition number 14 and so we are in the second case. To minimize things we want $p = 2$ with $a = 5$ and $q = 3$ with $b = 2$. This gives us groups of size $2^5 3^2 = 288$ which is the minimal n.

2. Show that every group of order p^n where p is a prime has a composition series of length $n + 1$ and all quotients are isomorphic to C_p.

   Proof Every finite group has some composition series. Suppose that G is a group of size p^n with a composition series $G = G_k, G_{k-1}, G_{k-2}, \ldots, G_0 = \{e\}$. The quotient G_{i+1}/G_i is simple and also a p-group. We know that the centre of a p-group is a normal subgroup and is not trivial. So the only simple p-group up to isomorphism is C_p and each quotient in this composition series is isomorphic to C_p. The length of this series is $n+1$ since $|G| = (|G_k|/|G_{k-1}|) \cdot |G_{k-1}/G_{k-2}| \cdot \ldots |G_1|/|G_0| = p^n$.

3. Show that every group of size $1225 = 5^2 7^2$ is abelian.

   Proof The number of 5-Sylow subgroups of G of size 1225, n_5, is congruent to 1 modulo 5 and divides 49 so $n_5 = 1$. Let H be the 5-Sylow subgroup and notice that it is normal in G. Similarly, the number of 7-Sylow subgroups of G, n_7 is congruent to 1 modulo 7 and divides 25 so $n_7 = 1$. Let K be the 7-Sylow subgroup and again notice that it is normal. Since the sizes of H and K are co-prime, $H \cap K = \{e\}$. This means that $G = HK$. To see that the subgroups H and K commute with each other, choose $h \in H$ and $k \in K$ and compute $hkh^{-1}k^{-1}$. If we write it as $(hkh^{-1})k^{-1}$ we see that this element is in K by the normality of K in G. Similarly, if we write it as $h(kh^{-1}k^{-1})$ then by the normality of H in G, this element is in H. So $hkh^{-1}k^{-1} = e$ or $hk = kh$.

   We also know that groups of size p^2 when p is a prime are abelian so both H and K are abelian. Putting this all together we get that $G \cong H \times K$ which is abelian.

4. Show there is no simple group of order 48.

   Proof Here is the proof I had in mind; I will also comment on a common proof that many people wrote up afterwards.

   If we look at the number of 2-Sylow subgroups of a group G of size 48, n_2 then n_2 is conjugate to 1 modulo 2 and divides 3. So n_2 is 1 or 3. If it is 1 then our group is not simple. If it is 3 then there are 3 2-Sylow subgroups $X = \{H_1, H_2, H_3\}$ and G acts on this set by conjugation.This action induces a homomorphism φ from G to S_X, the permutation group on X. The image of φ is not trivial since by the second Sylow theorem, any 2 2-Sylow subgroups are conjugate. But by the first isomorphism theorem, $G/ker(\varphi) \cong Im(\varphi)$. $|G| = 48$ and $1 < |Im(\varphi)| \leq 6$ so $ker(\varphi)$ is not G or $\{e\}$ and is a normal subgroup of G. So G is not simple.

   Here is another proof which involves a little more counting: Suppose we are in the case where $n_2 = 3$ and we fix H and K two distinct 2-Sylow subgroups. By a lemma from Judson, $|HK| = |H||K|/|H \cap K|$.

Now we know $|H| = |K| = 16$ and $|HK| \leq 48$ so $|H \cap K| \geq 16^2/48$. $H \neq K$ and $H \cap K$ is a subgroup so $|H \cap K| = 8$. So $H \cap K$ has index 2 in both H and K which means that H and K are contained in the normalizer of $H \cap K$. Since $H \neq K$, this means that $N(H \cap K)$ has a subgroup of size 16 and has more than 16 elements. This means that $N(H \cap K) = G$ and we have $H \cap K$ is normal in G so G is not simple. The only thing missing is a proof that if a subgroup has index 2 then it is a normal subgroup.

Lemma: If $H \subset G$ is an index 2 subgroup then H is normal in G. **Proof** If $h \in H$ then $hHh^{-1} = H$. If $g \in G$ but not in H then gH is disjoint from H. Similarly, Hg is disjoint from H. But since H has index 2 in G, these two sets must be equal i.e. $gH = Hg$ or equivalently, $gHg^{-1} = H$. So H is normal in G.

This is a nice proof which uses some tricky counting. If you are going to use proofs like this, please cite where you saw them.