Assignment 4, Math 4GR3
Due Apr. 1, uploaded to Avenue

1. There is a PID which is not a Euclidean domain. An explicit example is $Z[\theta]$ where
$$\theta = \frac{1}{2} + \frac{\sqrt{-19}}{2}.$$
This requires some fussy work and turns out to be more than I want to put on an assignment. However, on the honour system, I ask you to look at some presentations of this online. Here is a write-up by Conan Wong who was at UBC:

www.m-hikari.com/imf/imf-2013/29-32-2013/wongIMF29-32-2013.pdf

He points to other sources in the literature. The harder part is showing that this example is a PID.

2. We will show that every field is contained in an algebraically closed field.

   (a) Fix a field F and consider the set A of all polynomials in $F[x]$ which are irreducible over F. Introduce a variable x_f for every $f \in A$ and let I be the ideal generated by all $f(x_f)$ in the ring $R = F[x_f : f \in A]$ which is the ring of polynomials with variables x_f for $f \in A$. Show that I is not equal to R and then take a maximal ideal $J \subset R$ which contains I (for purists, this step requires Zorn's Lemma but just assume it; the main point is that I is a proper ideal). Note that R/J is a field and show that every polynomial over F has a root in R/J.

   (b) Let $F_0 = F$. Assume we have defined F_n and let F_{n+1} be defined as in part (a) starting with F_n in place of F. Let $K = \bigcup_n F_n$. Show that K is algebraically closed.

3. Judson, chapter 18, # 5, 11, 19 ; chapter 22 # 8, 12, 21