Assignment 3, Math 4GR3

Due Mar. 14, uploaded to Avenue

1. Let's finish the analysis of a group of order 12 started in class. Recall that we argued that if G has order 12 then it must have either a 2-Sylow normal subgroup N of size 4 or a 3-Sylow normal subgroup of size 3. Let H be a 3-Sylow (respectively 2-Sylow) subgroup in these two cases and note that $G = NH$. In both cases, H will act on N by conjugation so there is a homomorphism $\varphi\colon H \to Aut(N)$. In class we also said that if this homomorphism was trivial i.e. was identically equal to the identity element then $G \cong N \times H$ and so we would be looking at one of two possible abelian examples: $C_2 \times C_2 \times C_3$ and $C_4 \times C_3$. We concentrate then on the cases where φ is not trivial.

    (a) Case 1: In this case, let's suppose that H is of size 3 and N is normal of size 4. So $H \cong C_3$ and N is isomorphic to C_4 or $C_2 \times C_2$. Show that the automorphism group of C_4 is isomorphic to C_2 and conclude there is no non-trivial homomorphism from H to $Aut(N)$ in this case. Now we consider the automorphism group of $C_2 \times C_2$. Conclude that it is non-abelian of size 6 and hence is S_3. Describe a non-trivial homomorphism from C_3 to S_3 and argue, up to isomorphism, there is only one such. Use this information to write down a group of order 12.

    (b) Case 2: In this case, suppose that H has size 4 and N is normal of size 3. So H is isomorphic to $C_2 \times C_2$ or C_4 and $Aut(N)$ is isomorphic to C_2. In each of these cases, convince yourself that up to isomorphism there is only one non-trivial homomorphism from H to $Aut(N)$ and write down the two non-abelian groups of order 12 that arise.

2. Show that H_8 is not a semi-direct product. H_8 is the quaterion group and contains 8 elements: $\{\pm 1, \pm i, \pm j, \pm k\}$ and satisfies the following rules

   $(-1)^2 = 1, i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj$ and $ki = j = -ik$.

   Hint: If $H_8 \cong N \rtimes A$ then there is a normal subgroup of H_8, let's also call it N; how big is it? Show that if N were of size 2 then H_8 would be abelian (which it is not). Then argue that it can't be of size 4 by looking at elements of order 2.

3. Judson, chapter 17, # 18, 20, 21, 25