Assignment 2, Math 4GR3

Due Feb. 14, uploaded to Avenue (Happy Valentines Day!)

1. Prove the following version of the second isomorphism theorem: Suppose G is a group with two normal subgroups H and K. Then

$$HK/K \cong H/(H \cap K).$$

Notice that it is enough that H normalizes K; that is, for every $h \in H$, $hKh^{-1} = K$.

Proof: As we showed in class, if H normalizes K then the smallest subgroup generated by H and K is $\{hk : h \in H, k \in K\}$. We then define a map from H to HK/K by

$$h \mapsto hK.$$

This map is onto by the first sentence and its kernel is all $h \in H \cap K$. So by the first isomorphism theorem, $HK/K \cong H/(H \cap K)$.

2. (The third isomorphism theorem) Suppose that A and N are normal subgroups of G and that $A \subset N$. Prove that

$$(G/A)/(N/A) \cong G/N.$$

Proof: Define a map from G/A to G/N by sending gA to gN. Since $A \subset N$, this is well-defined and is onto. The kernel of this map is then all gA such that $g \in N$. That is, the kernel is N/A and so by the first isomorphism theorem again, $(G/A)/(N/A) \cong G/N$.

3. (# 11, chapter 13) Prove that if N is a normal subgroup of G and both N and G/N have composition series, then so does G.

Proof: Fix a composition series G_i for N and H_j for G/N. By the correspondence theorem, each H_j is of the form K_j/N for some subgroup K_j of G. We know that H_j is normal in H_{j+1}. Let's show that K_j is normal in K_{j+1}. If $g \in K_{j+1}$ then $gK_jg^{-1}/N = K_j/N$ which means $gK_jg^{-1} = K_j$ and so we get K_j is normal in K_{j+1}. Moreover, by the third isomorphism theorem,

$$(K_{j+1}/N)/(K_j/N) \cong K_{j+1}/K_j$$

and so consecutive quotients of K_j's are simple. Concatenating the series K_j with G_i, we get a composition series of G.

4. (#4, chapter 14) Suppose that G is the additive group of the reals and X is R^2, the plane. We will consider the plane in polar coordinates (r, α) which represents the point at distance r from the origin making an angle α with the x-axis. We define an action as follows:

$$(\theta, (r, \alpha)) \mapsto (r, \theta + \alpha).$$

To see that this is a group action, notice that 0 is the identify of R and adding 0 to the angle doesn't change it. If we apply θ_1 and then θ_2, this is the same as applying $\theta_1 + \theta_2$. These two facts demonstrate that this is a group action.

To see the orbit of any point on the plane, notice that the radius doesn't change under the action. So if $P = (r, \alpha)$ then the orbit of P is just the circle of radius r. The stabilizer of P is all multiples of 2π which is a subgroup of the additive reals.

5. (# 20, chapter 14) Show that G acts faithfully on X iff no two elements of G have the same action on all elements of X.

Proof: Suppose that g and h both have the same action on every element of X i.e. $gx = hx$ for all $x \in X$. Then $g^{-1}hx = x$ for all $x \in X$. But if G acts faithfully then $g^{-1}h = e$. That is, $g = h$. This show left to right. In the other direction, if G is not faithful then there is some $g \neq e$ which fixes every $x \in X$. But then g and e are two elements with the same action on X.

6. (# 23, chapter 14) Show that if $|G| = p^n$ for some prime p for some non-abelian group G then $Z(G)$ has few than p^{n-1} elements.

Proof If $Z(G) = G$ then G is abelian. Let's show that if $|Z(G)| = p^{n-1}$ then G is also abelian. We know $Z(G)$ is normal in G and if it has p^{n-1} elements then $G/Z(G) \cong C_p$. Pick $a \in G$ so that $aZ(G)$ is a generator of $G/Z(G)$. Every element of G then looks like $a^k b$ for some $k \in N$ and some $b \in Z(G)$. But then for $k, m \in N$ and $b, c \in Z(G)$ we have

$$a^k b a^m c = a^k a^m bc = a^m a^k cb = a^m c a^k b$$

which shows that G is abelian. So if G is not abelian then $|Z(G)| < p^{n-1}$.

7. (Bonus question from Dr. Cousins) The Schroeder-Bernstein theorem says that if X and Y are two sets and there is an injection from X to Y and also an injection from Y to X then there is a bijection between X and Y. Prove or disprove the same thing for groups. That is, if G and H are groups such that $g\colon G \to H$ is an injective group homomorphism and $h\colon H \to G$ is also an injective group homomorphism then G and H are isomorphic.

   Answer The Schroeder-Bernstein theorem is false for groups. The answer that most mathematician would give is that the free group on 2 generators contains a copy of the free group on countably many generators and these two groups can't be isomorphic. The problem with this answer is that you have to know what a free group is. Let me give you an abelian counter-example:

$$C_2 \oplus C_4 \oplus C_8 \oplus C_{16} \oplus \ldots$$

   and

$$C_4 \oplus C_8 \oplus C_{16} \oplus \ldots$$

   where these groups are ones where there are only finitely many non-zero entries in each infinite tuple. Both groups are countable. The second group embeds into the first by sending $(x_0, x_1, x_2, \ldots)$ to $(0, x_0, x_1, \ldots)$. The first embeds into the second by sending

$$(x_0, x_1, x_2, \ldots) \text{ to } (2x_0, 2x_1, 2x_2, \ldots)$$

   where the multiplication by 2 occurs in the image. Now consider only elements of order 2 in each group. In the second group, every such element is also divisible by 2. But in the first group $(1, 0, 0, 0, \ldots)$ is of order 2 but not divisible by 2. So these two groups are not isomorphic.