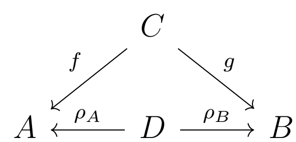
Assignment 1, Math 4GR3
Due Jan. 24, uploaded to Avenue

1. Suppose that A and B are two groups. In class we discussed the abstract notion of the product of two groups. That is, D together with two group homomorphisms $\rho_A\colon D \to A$ and $\rho_B\colon D \to B$ is a product of A and B if whenever C is a group and f and g are group homomorphisms are as pictured:



then there is a unique group homomorphism $h\colon C \to D$ such that $\rho_A \circ h = f$ and $\rho_B \circ h = g$. Show that if D', ρ'_A and ρ'_B is also a product of A and B then there is a unique isomorphism $i\colon D \to D'$ such that $\rho_A = \rho'_A \circ i$ and $\rho_B = \rho'_B \circ i$.

Solution: D, ρ_A and ρ_B is a product of A and B so in the definition, choose C to be D' together with the maps ρ'_A and ρ'_B. From the definition, we are given a map $h\colon D' \to D$ such that

$$\rho_A \circ h = \rho'_A \text{ and } \rho_B \circ h = \rho'_B.$$

LIkewise, if we apply the definition of product to D' and substitute D for C in the definition, we obtain a map $h'\colon D \to D'$ such that

$$\rho'_A \circ h' = \rho_A \text{ and } \rho'_B \circ h = \rho_B.$$

Now consider $h \circ h'$. We have

$$\rho_A \circ (h \circ h') = (\rho_A \circ h) \circ h' = \rho'_A \circ h' = \rho_A$$

and

$$\rho_B \circ (h \circ h') = (\rho_B \circ h) \circ h' = \rho'_B \circ h' = \rho_B.$$

But if D itself is substituted into the definition of product for C, the unique map must be the identity on D. But we just showed that $h \circ h'$ also works and so we conclude that $h \circ h' = id_D$. Similarly, we get that $\rho'_A \circ (h' \circ h) = \rho'_A$ and $\rho'_B \circ (h' \circ h) = \rho'_B$ and conclude that $h' \circ h = id_{D'}$. So h is an isomorphism and is the unique isomorphism we were looking for by the definition of product.

2. Show that if we have groups G_i for $i \leq n$ and normal subgroups N_i of G_i for $i \leq n$ then

$$G_1/N_1 \times G_2/N_2 \times \ldots G_n/N_n \cong (G_1 \times \ldots \times G_n)/(N_1 \times \ldots \times N_n).$$

Solution: We use the first isomorphism theorem. Suppose that $(g_1, \ldots, g_n) \in G_1 \times \ldots \times G_n$ and consider the homomorphism $\varphi\colon G_1 \times \ldots \times G_n \to G_1/N_1 \times G_2/N_2 \times \ldots G_n/N_n$ given by

$$\varphi(g_1, \ldots, g_n) = (g_1N_1, \ldots, g_nN_n).$$

φ is clearly onto $G_1/N_1 \times G_2/N_2 \times \ldots G_n/N_n$ and the kernel of φ is $N_1 \times \ldots \times N_n$ so we conclude that

$$(G_1 \times \ldots \times G_n)/(N_1 \times \ldots \times N_n) \cong G_1/N_1 \times G_2/N_2 \times \ldots G_n/N_n.$$

3. Prove that there is only one free abelian group up to isomorphism on n generators. That is, if F is a free abelian group on generators $x_1, \ldots, x_n$ and G is a free abelian group on generators $y_1, \ldots, y_n$ then there is a unique isomorphism $f\colon F \to G$ such that $f(x_i) = y_i$ for $i \leq n$.

Solution: Consider the map which sends x_i to y_i for all $i \leq n$. By the definition of being free abelian, there is a homomorphism f from F to G such that $f(x_i) = y_i$ for all $i \leq n$. Similarly, since G is free abelian on the generators y_i for $i \leq n$, there is a homomorphism g from G to F such that $g(y_i) = x_i$ for all $i \leq n$. If one considers $g \circ f$, we see that this map goes from F to F and fixes all the generators x_i. So $g \circ f = id_F$. Similarly, $f \circ g = id_G$. This shows that f is an isomorphism as required.

4. Show that if A and B are abelian groups, $\varphi_i\colon A \to B$ are group homomorphisms for $i \leq n$ and $m_1, \ldots, m_n \in Z$ then

$$m_1\varphi_1 + \ldots + m_n\varphi_n$$

is a group homomorphism from A to B.

Solution: As someone said in class, this is plug and chug. Suppose that $x, y \in A$ then

$$(m_1\varphi_1 + \ldots + m_n\varphi_n)(x - y) = m_1\varphi_1(x - y) + \ldots + m_n\varphi_n(x - y)$$

and this all equals

$$m_1\varphi_1(x) + \ldots + m_n\varphi_n(x) - m_1\varphi_1(y) - \ldots - m_n\varphi_n(y)$$

which is what we want to show.